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q-Analogue of the Krawtchouk and Meixner Orthogonal Polynomials[1]

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Abstract

The comparative analysis of Krawtchouk polynomials on a uniform grid with Wigner D-functions for the $SU(2)$ group is done. As a result the partnership between corresponding properties of the polynomials and D-functions is established giving the group-theoretical interpretation of the Krawtchouk polynomials properties. In order to extend such an analysis on the quantum groups $SU_q(2)$ and $SU_q(1,1)$, q-analogues of Krawtchouk and Meixner polynomials of a discrete variable are studied in detail as solutions of the finite difference equation of hypergeometric type on the nonuniform grid $x(s) = q^{2s}$. However, on this grid there are two kinds of Krawtchouk and Meixner polynomials characterized by different weight functions in the orthogonality relation. As a first step to such analysis the simpler polynomials of the first kind are considered. The total set of characteristics of these polynomials (orthogonality condition, normalization factor, recurrent relation, the explicit analytic expression, the Rodrigues formula, the difference derivative formula, various particular cases and values) is calculated.

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1. Introduction

It is well-known which crucial rôle the Lie group representation theory plays in the special function theory. Group theory gives the possibility to unify the systematics of special functions and, moreover, it is by itself an effective instrument for the investigation of the origin and the properties of different special functions. In a very brilliant way, on the basis of group theory concepts, the theory of special functions is developed in the outstanding monography of N. Ya. Vilenkin [1], which, indeed, has an encyclopedic character.

No doubt, the theory of representation of quantum groups and quantum algebras worked out in recent years [2], supports a new stimulating impulse to the evolution of the theory of special functions.

The present work represents the definite part of the investigations, which has been undertaken by our theoretical group during the study of the connections between different constructions of the Wigner-Racah algebra for the quantum groups $SU_q(2)$ and $SU_q(1,1)$ and the orthogonal polynomials of a discrete variable. In particular, in the frame work of this program, the properties of the Clebsch-Gordan coefficients for the groups $SU_q(2)$ and $SU_q(1,1)$ and the q-Hahn polynomials on the exponential grid $x(s) = q^{2s}$ were considered in detail. In a similar way, the Racah coefficients for such groups have been connected with the Racah polynomials on a grid $x(s) = [s][s+1]$. The dual q-Hahn polynomials on the same grid are proportional to the Clebsch-Gordan coefficients for the quantum groups $SU_q(2)$, $SU_q(1,1)$, etc. Partially, these results are published in [3]. To continue this line it seems reasonable to investigate the interrelations between Wigner D-functions for quantum groups $SU_q(2)$ and $SU_q(1,1)$ and orthogonal polynomials of a discrete variable [4,5]. It is well-known that the usual Wigner D-functions for $SU(2)$ group can be expressed in terms of Krawtchouk polynomials on the uniform grid [4,5,6]. The Bargmann D-functions for the discrete series of the $SU(1,1)$ irreducible representation are proportional to the Meixner polynomials on the uniform grid [7]. The detailed comparative analysis of the $SU(2)$ D-functions and the usual Krawtchouk polynomials is given below in Section 2. The starting point of this analysis is the recurrent relation (2.6) which can be transformed into the second order finite difference equation for the Krawtchouk polynomials. The recurrent relation similar to (2.6) holds for the D-functions of the quantum groups $SU_q(2)$ and $SU_q(1,1)$. It can be also used for the investigation of the interrelation between quantum D-functions and corresponding orthogonal polynomials on the nonuniform grid. As it was established at first by T. H. Koornwinder [6,8,9] the $SU_q(2)$ D-functions can be expressed in terms of q-Krawtchouk polynomials. Similarly, the $SU_q(1,1)$ D-functions should be connected with q-Meixner polynomials at least for the discrete series of the irreducible representation [10]. In this connection a natural intention arises to investigate in detail the properties of the Krawtchouk and Meixner polynomials as solutions of the second order finite difference equation (3.1) on the exponential grid $x(s) = q^{2s}$ and to do a comparative analysis of the properties of these polynomials and quantum D-functions in a spirit of Section 2 of the present paper. However, it is well-known [4,5,11,17] that there are two kinds of the Krawtchouk (Meixner) q-polynomials characterized by different weight functions determining their orthogonality properties. Therefore, as a first step to our aim we discuss in Section 3 and 4 of this paper the properties of the Krawtchouk and Meixner q-polynomials of the first kind with simpler weight functions (3.3) and (4.1), respectively. The detailed analysis of the second kind Krawtchouk and Meixner q-polynomials and the discussion of their interrelations with quantum D-functions is postponed to a future paper. Therefore, we follow the method of the classical orthogonal polynomials of a discrete variable, as described in the monographs [4], i.e., we suppose them to be the solutions of an eigenvalue problem for the second order difference equation on nonuniform grids. Such a method is close to the quantum-mechanical viewpoint and permits to treat these orthogonal polynomials, in some way, as "wave functions", defined on a discrete manifold of some interval of the straight line.

We then can use our quantum-mechanical intuition, like the discrete WKB method and the asymptotical consideration discussed in [13], to understand the properties of orthogonal polynomials of a discrete variable. Applying the method explained in the monographs [4], in Sections 3 and 4 we shall find the explicit form and the main characteristics of the Krawtchouk and Meixner polynomials of the first kind on the exponential grid $x(s) = q^{2s}$. The detailed analysis of such polynomials of the second kind and their connections with the Wigner D-functions for the groups $SU_q(2)$ and $SU_q(1,1)$ are supposed to be developed in the following paper.

2. Wigner D-functions and Krawtchouk polynomials

The fact, the Wigner D-functions $d^j_{m,m'}(\beta)$ being the matrix elements of the irreducible representation D^j of the group $SU(2)$ and being connected with the Krawtchouk polynomials $k_n^{(p)}(s,N)$, which are orthogonal on the linear grid $x(s) = s$,

$$(-1)^{m-m'} d^j_{m,m'}(\beta) = \frac{\sqrt{\rho(s)}}{d_n}\, k_n^{(p)}(s,N), \tag{2.1}$$

where $n = j - m$, $s = j - m'$, $N = 2j$, $p = \sin^2(\beta/2)$ and β is the Euler angle, is well-known [4,8,14]. The weight function $\rho(s)$ giving the orthogonality condition

$$\sum_{s=0}^{N} k_n^{(p)}(s,N)\, k_{n'}^{(p)}(s,N)\, \rho(s) = \delta_{nn'}\, d_n^2 \tag{2.2}$$

is of the form

$$\rho(s) = \frac{N!\, p^s\, (1-p)^{N-s}}{\Gamma(s+1)\Gamma(N+1-s)}. \tag{2.3}$$

It is evident from eqs. (2.1) – (2.3), that the orthogonality relation is equivalent to the unitarity property of the D-functions

$$\sum_{m''=-j}^{j} d^j_{m,m''}(\beta)\, d^j_{m',m''}(\beta) = \delta_{mm'}, \tag{2.4}$$

and their weight function coincides with the squared simplest D-function $d^j_{j,m'}(\beta)$

$$\rho(s) = (d^j_{j,m'}(\beta))^2$$

which is well-known [15], and is equal to

$$(-1)^{j-m'} d^j_{j,m'}(\beta) = \left[\frac{(2j)!\, \sin^{2(j-m')}(\beta/2)\, \cos^{2(j+m')}(\beta/2)}{\Gamma(j-m'+1)\Gamma(j+m'+1)}\right]^{1/2}. \tag{2.5}$$

For the Wigner D-function $D^j_{m,m'}$ we have the Clebsch-Gordan expansion (m_1 and m_2 are fixed) [15]

$$\sum_{m'_1,m'_2} D^{j_1}_{m_1,m'_1}(\Omega)\ D^{j_2}_{m_2,m'_2}(\Omega)\ (j_1\ m'_1\ j_2\ m'_2|j\ m') = (j_1\ m_1\ j_2\ m_2|j\ m)\ D^j_{m,m'}(\Omega), \quad (2.6)$$

where Ω stands for the three Euler angles α, β, γ. Setting $\alpha = \gamma = 0$, we obtain $D^j_{m,m'}(\Omega) = d^j_{m,m'}(\beta)$. Let us suppose, that $j_1 = 1$, $m_1 = 0$, $j_2 = j$, $m_2 = m$ and insert them into relation (2.6). Thus, using the explicit expression for the Clebsch-Gordan coefficients $(1\ m'_1\ j\ m' - m'_1|j\ m')$, we get the three-term recurrent relation (TRR) for the D-functions

$$\begin{aligned} -\frac{1}{2}\sin\beta\sqrt{(j-m)(j+m+1)}\ d^j_{m+1,m'}(\beta)\ &+\ m\cos\beta\ d^j_{m,m'}(\beta) \\ -\frac{1}{2}\sin\beta\sqrt{(j+m)(j-m+1)}\ d^j_{m-1,m'}(\beta)\ &=\ m'\ d^j_{m,m'}(\beta) \end{aligned} \quad (2.7)$$

which is equivalent to the second order finite difference equation for the Krawtchouk polynomials

$$\sigma(s)\ \Delta\nabla k(s) + \tau(s)\ \Delta k(s) + \lambda k(s) = 0. \quad (2.8)$$

Here we used the notations $\Delta f(s) = f(s+1) - f(s)$ and $\nabla f(s) = f(s) - f(s-1)$. $\sigma(s)$ is a polynomial not higher than second order, $\tau(s)$ is a polynomial not higher than first order and λ is a constant. Equation (2.8) is equivalent to the TRR

$$(\sigma(s) + \tau(s))\ k(s+1) - (2\sigma(s) + \tau(s) - \lambda)\ k(s) + \sigma(s)\ k(s-1) = 0$$

After substitution of $\sigma(s)$, $\tau(s)$ and λ, given in Table 1 the TRR for the Krawtchouk polynomials can be written as follows

$$\mu(N-s)\ k^{(p)}_n(s+1,N) - [\mu(N-s) + s - n(1+\mu)]\ k^{(p)}_n(s,N) + s\ k^{(p)}_n(s-1,N) = 0 \quad (2.8a)$$

where $\mu = p/(1-p)$. Setting $j_1 = 1/2$, $m_1 = -1/2$, $j_2 = j$ and $j' = j - 1/2$ in (2.6) we obtain the relation for the D-functions

$$\sqrt{j-m'+1}\cos\frac{\beta}{2}\ d^j_{m,m'-1}(\beta) + \sqrt{j+m'}\sin\frac{\beta}{2}\ d^j_{m,m'}(\beta) = \sqrt{j-m}\ d^{j-\frac{1}{2}}_{m+\frac{1}{2},m'-\frac{1}{2}}(\beta) \quad (2.9)$$

which is equivalent to the difference derivative formula for the Krawtchouk polynomials [4]

$$\Delta k_n^{(p)}(s,N) = k_n^{(p)}(s+1,N) - k_n^{(p)}(s,N) = k_{n-1}^{(p)}(s,N-1), \tag{2.10}$$

where $n = j - m$, $s = j - m'$, $N = 2j$ and $p = \sin^2(\beta/2)$.

From the Rodrigues formula for the Krawtchouk polynomials

$$k_{n+1}^{(p)}(s,N) = \frac{(-1)^{n+1}\Gamma(s+1)\Gamma(N+1-s)}{(n+1)!\; N!\; p^s(1-p)^{N-n-s-1}} \nabla^{n+1}[\rho_{n+1}(s)],$$

($\rho_n(s)$ can be found in Table 1) we can obtain the additional difference derivative formula for these polynomials,

$$p(N-s+1)\; k_n^{(p)}(s,N) - s(1-p)\; k_n^{(p)}(s-1,N) = (p-1)(n+1)\; k_{n+1}^{(p)}(s,N+1) \tag{2.11}$$

which is equivalent to a relation of type (2.9), but with $j' = j + 1/2$ and $m_1 = -1/2$. The Rodrigues formula gives also the explicit form of the Krawtchouk polynomials [4]

$$\begin{aligned} k_n^{(p)}(s,N) &= \sum_{k=0}^{n}(-1)^{k+n}\frac{p^{n-k}\,(1-p)^k\;\Gamma(s+1)\Gamma(N-s+1)}{k!\,(n-k)!\;\Gamma(s+1-k)\Gamma(N+1-s-n+k)} \\ &= \frac{(-1)^n N!\; p^n}{n!\;(N-n)!}\;{}_2F_1(-n,-s;-N;1/p) \\ &= P_n^{(s-n,N-n-s)}(1-2p). \end{aligned} \tag{2.12}$$

The third line in (2.12), connecting the Krawtchouk polynomials to the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, yields us to the standard expression for the D-function in terms of the Jacobi polynomials [16]

$$\begin{aligned} d^j_{m,m'}(\beta) &= \frac{(-1)^{m-m'}}{2^m}\left[\frac{(j+m)!(j-m)!}{(j+m')!(j-m')!}\right]^{1/2} \\ &\times (1-\cos\beta)^{(m-m')/2}(1+\cos\beta)^{(m+m')/2}P_{j-m}^{(m-m',m+m')}(\cos\beta). \end{aligned} \tag{2.13}$$

The recurrent relation for the Krawtchouk polynomials, calculated by means of the method described in [4]

$$\alpha_n k_{n+1}(s) + \beta_n k_n(s) + \gamma_n k_{n-1}(s) = s k_n(s),$$

where $\alpha_n = n+1$, $\beta_n = n + p(N-2n)$, $\gamma_n = p(p-1)(N-n+1)$, is equivalent to the finite difference equation (2.8) up to the exchanges m and m'. This self-duality property yields the following symmetry property of the Krawtchouk polynomials

$$k_n^{(p)}(s,N) = (-1)^{n-s} p^{n-s} \frac{s!\,(N-s)!}{n!\,(N-n)!}\, k_s^{(p)}(n,N) \tag{2.14}$$

which is equivalent to the symmetry property of the D-functions [16]

$$d^j_{m,m'}(\beta) = (-1)^{m-m'} d^j_{m',m}(\beta). \tag{2.15}$$

Another symmetry property of the Krawtchouk polynomials, given by [4]

$$k_n^{(p)}(s,N) = (-1)^n k_n^{(1-p)}(N-s,N),$$

leads to the symmetry relation for the D-function

$$d^j_{m,m'}(\beta) = (-1)^{j+m'}\, d^j_{m,-m'}(\pi+\beta). \tag{2.16}$$

Combining relations (2.15) and (2.16) we obtain the well-known [16] symmetry property for the D-function respective to the exchange of the indices m, m' and of the sign, e.g.

$$d^j_{m,m'}(\beta) = (-1)^{m-m'} d^j_{-m,-m'}(\beta). \tag{2.17}$$

It is useful to note the explicit values of the Krawtchouk polynomials at the boundaries of the interval (i.e., $s=0$ and $s=N$) [6]

$$\begin{aligned} k_n^{(p)}(0,N) &= \frac{(-1)^n N!}{n!(N-n)!} p^n, \\ k_n^{(p)}(N,N) &= \frac{N!}{n!(N-n)!}(1-p)^n. \end{aligned} \tag{2.18}$$

Expressions (2.18) yield us to the formulas for the values of $d^j_{m,j}(\beta)$ and $d^j_{m,-j}(\beta)$, respectively, which can be calculated from (2.5) by means of the symmetry properties (2.15) and (2.17). For completeness let us write down explicitly the first three Krawtchouk polynomials, (i.e., $n = 0,1,2$)

$$\begin{aligned} k_0^{(p)}(s,N) &= 1, \\ k_1^{(p)}(s,N) &= s - pN, \\ k_2^{(p)}(s,N) &= \frac{1}{2}s^2 + (p(1-N) - \frac{1}{2})s + \frac{Np^2}{2}(N-1). \end{aligned} \tag{2.19}$$

We remark, that the coefficients given in (2.19) are, in fact, particular values of the characteristics a_n and b_n in Table 1. Indeed, the characteristics a_n and b_n belong to the expansion in s-order terms $k_n^{(p)}(s,N) = a_n s^n + b_n s^{n-1} + \ldots$ which corresponds to the expressions (2.19) for the values $n = 0, 1, 2$.

Consequently, the analysis developed in this section permits to correlate each property of the Krawtchouk polynomials to a definite property or relation for the Wigner D-functions, calculated earlier by a pure group-theoretical method. In other words, such an analysis gives a group-theoretical interpretation of the orthogonality and symmetry properties of the Krawtchouk polynomials. An additional group-theoretical interpretation was given by Koornwinder [6]. To our mind, it would be interesting to develop a similar comparative analysis, as for the q-Krawtchouk (Meixner) polynomials, on one hand, and, on the other hand, for the D-functions for the quantum groups $SU_q(2)$ ($SU_q(1,1)$). It has already been remarked [8,11], that the D-functions for $SU_q(2)$ are correlated with the q-Krawtchouk polynomials. Therefore, we investigate in more detail the properties of the q-Krawtchouk and q-Meixner polynomials of the first kind in the following sections.

3. The q-Krawtchouk polynomials and their properties

The q-Krawtchouk polynomials $k_n^{(p)}(s,N,q)$ are solutions of the finite difference equation

$$\sigma(s)\frac{\Delta}{\Delta x(s-1/2)}\left[\frac{\nabla y(s)}{\nabla x(s)}\right] + \tau(s)\frac{\Delta y(s)}{\Delta x(s)} + \lambda_n y(s) = 0 \tag{3.1}$$

on the grid $x(s) = q^{2s}$, where $\Delta x(s-\frac{1}{2}) = q^{2s}(q - q^{-1})$ and

$$\begin{aligned} \sigma(s) &= q^{2s}(1-q^{2s}), \\ \tau(s) &= \frac{1}{1-p}\left(q^s[s](1+p(q^2-1)) - pq^{N+2}[N]\right), \\ \lambda_n &= -[n]\frac{\mu q^{n+1} + q^{-n+1}}{q - q^{-1}}, \end{aligned} \tag{3.2}$$

where $\mu = p/(1-p)$ is a free parameter. They are orthogonal on the interval $s = 0, 1, \ldots, N$ with the weight function

$$\rho(s) = \frac{q^{s(s-1)}\mu^s\,[N]!}{\Gamma_q(s+1)\Gamma_q(N-s+1)} \tag{3.3}$$

and the orthogonality relation reads

$$\sum_{s=0}^{N} k_n^{(p)}(s,N,q)\ k_{n'}^{(p)}(s,N,q)\ \rho(s)\Delta x(s-\frac{1}{2}) = \delta_{nn'} d_n^2. \tag{3.4}$$

The normalization factor d_n^2 and all other characteristics and properties of the q-Krawtchouk polynomials k_n, not published in the monography [4], were calculated applying the procedures explicitly described in [4, 17]. Therefore, we do not show the detailed computations, but only present the obtained results (see Table 1) using the same notations as in [4, 17]. We note that the powers in round brackets (present in the explicit expressions of γ_n and d_n^2), i.e.,

$$(1+a)^m = (1+a)(1+q^2 a)\cdots(1+q^{2m-2}a)$$

define the so-called quasi-powers.

Moreover, we find for the q-Krawtchouk polynomials the TRR

$$\begin{aligned} &\mu q^{s+N+2}[s-N]\ k_n^{(p)}(s+1,N,q) - q^{s+2}[s]\ k_n^{(p)}(s-1,N,q) \\ &-\left(\mu q^{s+N+2}[s-N] - q^{s+2}[s] + [n](q^{2s-n+2} + \mu q^{2s+n})\right)\ k_n^{(p)}(s,N,q) = 0. \end{aligned} \tag{3.5}$$

Explicitly, the Rodrigues formula for the q-Krawtchouk polynomials can be written as

$$k_n^{(p)}(s,N,q) = \frac{(-1)^n\ (1-p)^n \Gamma_q(s+1)\ \Gamma_q(N+1-s)}{q^{s(s-1)}\ \mu^s\ [n]!\ [N]!} D^{(n)}[\rho_n(s)]. \tag{3.6}$$

The multiple derivative formula $D^{(n)}[\rho_n(s)]$ for the q-Krawtchouk polynomials reads

$$D^{(n)}[\rho_n(s)] = (-1)^n \sum_{k=0}^{n} (-1)^k \frac{[n]![N]!\mu^{s+n-k} q^{2s(n-k)+k(k-3n)+s(s-1)+\frac{3}{2}n(n+1)}}{[k]![n-k]!\Gamma_q(s+1-k)\Gamma_q(N+1-s-n+k)}. \tag{3.7}$$

The Rodrigues formula (3.6) and multiple derivative formula (3.7) permit us to write the q-Krawtchouk polynomials explicitly as

$$\begin{aligned} k_n^{(p)}(s,N,q) =&(1-p)^n \Gamma_q(s+1)\Gamma_q(N-s+1) \\ &\times \sum_{k=0}^{n} (-1)^k \frac{\mu^{n-k} q^{2s(n-k)+k(k-3n)+\frac{3}{2}n(n+1)}}{[k]![n-k]!\Gamma_q(s+1-k)\Gamma_q(N+1-s-n+k)}. \end{aligned} \tag{3.8}$$

From (3.8) we immediately calculate the values for the q-Krawtchouk polynomials

	Krawtchouk $k_n^{(p)}(x,N)$	q-Krawtchouk $k_n^{(p)}(s,N,q),\ x(s)=q^{2s}$
$\rho(x)$	$\frac{N!\ p^x\ t^{N-x}}{\Gamma(x+1)\ \Gamma(N+1-x)}$ $(p>0, t>0, p+t=1)$	$\frac{q^{s(s-1)}\mu^s[N]!}{\Gamma_q(s+1)\Gamma_q(N+1-s)}$ $(\mu=p/t=q^{2\tau})$
$\sigma(x)$	x	$q^{2s}(1-q^{2s})$
$\tau(x)$	$\frac{1}{t}(Np-x)$	$\frac{1}{t}(q^s[s](1+p(q^2-1))-pq^{N+2}[N])$
λ_n	$\frac{n}{t}$	$-[n]\frac{\mu q^{n+1}+q^{-n+1}}{q-q^{-1}}$
α_n	$n+1$	$-\frac{(q-q^{-1})[n+1]}{(1-p)q^{\tau+1}}\frac{[2n+2\tau][2n+\tau][2n+\tau+1]}{[4n+2\tau][4n+2\tau+2][n+\tau]}$
β_n	$n+p(N-2n)$	$q^{n+1}\left(\frac{[n](1+\mu q^{2N+2n})}{q^2(1+\mu q^{4n-2})}-\frac{[n+1](1+\mu q^{2N+2n+2})}{q(1+\mu q^{4n+2})}\right)$
γ_n	$pt(N-n+1)$	$-\frac{(q-q^{-1})pq^{N+3n}(1+\mu q^{4n+2})^{(N-n)}}{(1+\mu q^{4n^2-2n-2})^{(N-n+1)}}[N-n+1]$
B_n	$\frac{(-1)^n t^n}{n!}$	$\frac{(-1)^n(1-p)^n}{[n]!}$
$\rho_n(x)$	$\frac{N!\ p^{x+n}\ t^{N-n-x}}{\Gamma(x+1)\ \Gamma(N+1-n-x)}$	$\frac{(-1)^n q^{2n^2+4ns+n+s^2-s}\mu^{n+s}[N]!}{\Gamma_q(s+1)\Gamma_q(N+1-n-s)}(q-q^{-1})^n$
a_n	$\frac{1}{n!}$	$\frac{(-1)^n(1-p)^n q^{n(\tau+1)}[n+\tau-1]![4n+2\tau-2]!!}{(q-q^{-1})^n\ [2n+\tau-1]!\ [n]!\ [2n+2\tau-2]!!}$
b_n	$-\frac{Np+(n-1)(1/2-p)}{(n-1)!}$	$\frac{(-1)^{n+1}t^n q^{n(\tau+2)-1}[n+\tau-1]![4n+2\tau-2]!!(1+\mu q^{2N+2n})}{(q-q^{-1})[n-1]!\ [2n+\tau-1]!\ [2n+2\tau-2]!!\ (1+\mu q^{4n-2})}$
d_n^2	$\frac{N!(pt)^n}{n!(N-n)!}$	$\frac{(q-q^{-1})(pt)^n[N]![n+\tau-1]![4n+2\tau-2]!!(1+\mu q^{4n+2})^{(N-n)}}{q^{-n(n+\tau+2)}\ [n]!\ [2n+\tau-1]!\ [2n+2\tau-2]!!\ \Gamma_q(N-n+1)}$

at the boundaries of the interval, i.e., for $s = 0$ and $s = N$; it results

$$
\begin{aligned}
k_n^{(p)}(0,N,q) &= (1-p)^n \frac{\mu^n \Gamma_q(N+1) q^{\frac{1}{2}n(n+1)}}{[n]!\Gamma_q(N+1-n)}, \\
k_n^{(p)}(N,N,q) &= (-1)^n\,(1-p)^n \frac{\Gamma_q(N+1) q^{\frac{1}{2}n(3-n)}}{[n]!\Gamma_q(N-n+1)}.
\end{aligned}
\tag{3.9}
$$

We note, that for $q \to 1$ all results for the characteristics of the q-Krawtchouk polynomials convert into the respective relations for the classical Krawtchouk polynomials on the uniform grid [4]. In fact, there are two kinds of q-Krawtchouk polynomials. Those investigated here were introduced by Stanton [18]. They can be useful in the theory of the generalized coherent states for the $SU_q(2)$ quantum group [19]. The matter is that the simple coherent states discussed in Ref. [19] coincide with the weight function (3.3). It means that the coherent states of a more general form will be connected with the q-Krawtchouk polynomials of the first kind discussed above. The same is valid for the $SU_q(1,1)$ coherent states and the q-Meixner polynomials of the first kind considered in the next section. As for the quantum D-functions they are proportional to the q-Krawtchouk polynomials of the second kind and their consideration is beyond the frame of this paper (see for example Refs. [8,9] and the paper of Dunkl [14] for a more general discussion of orthogonal polynomials with symmetry of order three). Here and below we shall not discuss the explicit form of the polynomials expressed in terms of the basic hypergeometric series. The necessary information of this kind can be found in Refs. [5,20,21,22].

4. The q-Meixner polynomials and their properties

In this case the relations (3.2) and (3.3) are of the form

$$
\begin{aligned}
\sigma(s) &= q^{2s}(q^{2s}-1), \\
\tau(s) &= \mu q^{s+\gamma+2}[s+\gamma] - q^s[s], \\
\lambda_n &= [n]\frac{q^{-n+1} - \mu q^{2\gamma+n+1}}{q-q^{-1}}, \\
\rho(s) &= \frac{\Gamma_q(s+\gamma)\mu^s}{\Gamma_q(\gamma)\Gamma_q(s+1)}.
\end{aligned}
\tag{4.1}
$$

	Meixner $m_n^{(\gamma,\mu)}(x)$	q-Meixner $m_n^{(\gamma,\mu)}(s,q),\ x(s)=q^{2s}$
$\rho(x)$	$\frac{\mu^x\ \Gamma(\gamma+x)}{\Gamma(x+1)\ \Gamma(\gamma)}$ $(\gamma>0,\ 0<\mu<1)$	$\frac{\mu^s\Gamma_q(\gamma+s)}{\Gamma_q(\gamma)\Gamma_q(s+1)}$ $(\mu=q^{2\tau})$
$\sigma(x)$	x	$q^{2s}(q^{2s}-1)$
$\tau(x)$	$\gamma\mu-x(1-\mu)$	$\mu q^{s+\gamma+2}[s+\gamma]-q^s[s]$
λ_n	$n(1-\mu)$	$[n]\frac{q^{-(n-1)}-\mu q^{2\gamma+n+1}}{q-q^{-1}}$
α_n	$\frac{\mu}{\mu-1}$	$-\frac{\mu[\gamma+\tau+n]q^{-(\gamma+\tau+1)}}{[\gamma+\tau+2n]\ [\gamma+\tau+2n+1]}$
β_n	$\frac{n+\mu(n+\gamma)}{1-\mu}$	$q^{n+1}\left(\frac{[n](1-\mu q^{2n})}{q^2(\mu q^{2\gamma+4n-2}-1)}-\frac{[n+1](1-\mu q^{2n+2})}{q(\mu q^{2\gamma+4n+2}-1)}\right)$
γ_n	$\frac{n(n+\gamma-1)}{\mu-1}$	$-q^{2n+2}(q-q^{-1})\ [n]\ \frac{[\gamma]_n(1-\mu q^{2n})^{(n+\gamma-1)}}{[\gamma]_{n-1}(1-\mu q^{2(n+1)})^{(n+\gamma)}}$
B_n	$1/\mu^n$	$1/\mu^n$
$\rho_n(x)$	$\frac{\mu^{x+n}\Gamma(\gamma+x+n)}{\Gamma(x+1)\ \Gamma(\gamma)}$	$\frac{\mu^{s+n}(q-q^{-1})^n\Gamma_q(\gamma+s+n)}{q^{-(n^2+2n+2ns)}\Gamma_q(\gamma)\Gamma_q(s+1)}$
a_n	$\left(\frac{\mu-1}{\mu}\right)^n$	$\frac{(-1)^n[\gamma+\tau+2n-1]!}{\mu^n\ [\gamma+\tau+n-1]!}q^{n(\gamma+\tau+1)}$
b_n	$a_{n-1}\ n\left(\gamma+\frac{n-1}{2}\frac{\mu+1}{\mu}\right)$	$\frac{(-1)^n[n][\gamma+\tau+2n-1]!\ (1-\mu q^{2n})q^{n(\gamma+\tau+2)-1}}{\mu^n\ [\gamma+\tau+n-1]!\ (\mu q^{2\gamma+4n-2}-1)}$
d_n^2	$\frac{\Gamma(n+1)\ \Gamma(\gamma+n)}{\mu^n\ (1-\mu)^\gamma\ \Gamma(\gamma)}$	$q^{n(6-n)}\frac{(q-q^{-1})^{n+1}\ [\gamma]_n\ \Gamma_q(\gamma+\tau+2n)\ \Gamma_q(n+1)}{\mu^n\ (1-\mu q^{2(n+1)})^{(n+\gamma)}\ \Gamma_q(\gamma+\tau+n)}$

The corresponding interval for the discrete variable s is not bounded, i.e., $s = 0, 1, 2, \ldots$. All other properties of the q Meixner polynomials m_n are given in Table 2. These polynomials were introduced in Ref. [22] (see exercise 7.12). Let us now discuss the properties of the q-Meixner polynomials, in the same way as for the q-Krawtchouk polynomials in Section 3. Now, we have for the q-Meixner polynomials TRR

$$\begin{aligned} &\mu q^{s+\gamma+2}[s+\gamma]\, m_n^{(\gamma,\mu)}(s+1,q) + q^{s+2}[s]\, m_n^{(\gamma,\mu)}(s-1,q) \\ &- \left(\mu q^{s+\gamma+2}[s+\gamma] + q^{s+2}[s] - [n](q^{2s-n+2} - \mu q^{2\gamma+2s+n+2})\right) m_n^{(\gamma,\mu)}(s,q) = 0. \end{aligned} \tag{4.2}$$

The Rodrigues formula for the Meixner polynomials is

$$m_n^{(\gamma,\mu)}(s,q) = \frac{\Gamma_q(\gamma)\,\Gamma_q(s+1)}{\mu^{n+s}\,\Gamma_q(\gamma+s)} D^{(n)}[\rho_n(s)] \tag{4.3}$$

and the formula of multiple derivative $D^{(n)}[\rho_n(s)]$ for the q-Meixner polynomials can be expressed as

$$D^{(n)}[\rho_n(s)] = \sum_{k=0}^{n} (-1)^k \frac{[n]!\ \mu^{s+n-k}\ q^{\frac{n}{2}(n+5)-k(n-1)}\ \Gamma_q(\gamma+n+s-k)}{[k]!\ [n-k]!\ \Gamma_q(s+1-k)\ \Gamma_q(\gamma)}. \tag{4.4}$$

By means of the Rodrigues formula (4.3) and the multiple derivative formula (4.4) we can write the q-Meixner polynomials explicitly as

$$m_n^{(\gamma,\mu)}(s,q) = \frac{\Gamma_q(s+1)}{\Gamma_q(\gamma+s)} \sum_{k=0}^{n} (-1)^k \frac{[n]!\ \Gamma_q(\gamma+s+n-k)\ q^{\frac{n}{2}(n-2k+5)-k}}{\mu^k\ [k]!\ [n-k]!\ \Gamma_q(s+1-k)}. \tag{4.5}$$

From (4.5) we immediately calculate the values for the q-Meixner polynomials at the left boundary of the interval, i.e., for $s = 0$ it yields

$$m_n^{(\gamma,\mu)}(0,q) = \frac{\Gamma_q(\gamma+n)}{\Gamma_q(\gamma)} q^{\frac{1}{2}n(n+5)}. \tag{4.6}$$

Note that, for $q \to 1$, all results for the characteristics of the q-Meixner polynomials convert into the respective relations for the classical Meixner polynomials on the uniform grid [4]. The Meixner polynomials on a uniform grid emerge while discussing the Coulomb-problem in the Sturmian (Laguerre) basis [7], therefore the investigation of a q-Coulomb problem [12] in such a basis can be tackled from the q-Meixner polynomials position.

As the next step a similar analysis of the Krawtchouk (Meixner) polynomials of the second kind should be done. Then their interrelation with $SU_q(2)$ and $SU_q(1,1)$ D-functions could be investigated along the vein of our Section 2. However, for the quantum groups this problem is more difficult than in the classical case, because the matrix elements D_q^j, $j > 1/2$, expressed in terms of the functions $D_q^{1/2}$, do not commute with each other, in distinction to the commuting matrix elements $D^{1/2}$, giving the transformation of the classical Lie group $SU(2)$. This complication has to be taken into consideration before a detailed discussion of the relations between the quantum D-functions and the orthogonal polynomials will be started with. We hope to complete this work in a forthcoming paper.

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Table Captions :

Table 1 : The main characteristics of the Krawtchouk and q-Krawtchouk polynomials

Table 2 : The main characteristics of the Meixner and q-Meixner polynomials