FR9400342

July 5, 1991, Preprint YCTP-P21-91

FERMIONS AND LINK INVARIANTS

L.Kauffman
Dept.of Mathematics, Statistics and Computer Science
University of Illinois at Chicago
and
H.Saleur[1] *
Physics Department
Yale University
Connecticut

Abstract: This paper deals with various aspects of knot theory when fermionic degrees of freedom are taken into account in the braid group representations and in the state models. We discuss how the $\check{R}$ matrix for the Alexander polynomial arises from the Fox differential calculus, and how it is related to the quantum group $U_q gl(1,1)$. We investigate new families of solutions of the Yang Baxter equation obtained from "linear" representations of the braid group and exterior algebra. We study state models associated with $U_q sl(n,m)$, and in the case $n = m = 1$ a state model for the multivariable Alexander polynomial. We consider invariants of links in solid handlebodies and show how the non trivial topology lifts the boson fermion degeneracy that is present in S^3. We use "gauge like" changes of basis to obtain invariants in thickened surfaces $\Sigma \times [0,1]$.

RIMS Project "Infinite Analysis"
June 1-Aug. 31, 1991
Contributed paper 14

---

[1] On leave from SPht Ce : Saclay 91191 Gif Sur Yvette Cedex France

CEA-SPhT/91/134

* Laboratoire de la Direction des Sciences de la Matière du Commissariat à l'Energie Atomique

1 Introduction

This paper is the third in a series devoted to fermions and knot theory [1, 2].

In the current framework (see [3, 4, 5]and references therein), the Jones invariant and its generalizations are naturally connected to Lie algebras. It is a logical idea to consider also the case of super Lie algebras [6], and investigate what kind of information is carried by the fermionic degrees of freedom. As was shown in [1], the simplest super Lie algebra $gl(1,1)$ is associated with the Alexander Conway invariant. In this case a direct path was obtained between the fundamental group of the link complement and the $U_q gl(1,1)$ R matrix. It was shown also using a $det = Str$ identity how the state model of [7, 8, 9] arises from the Burau matrix, and how the $U_q gl(1,1)$ algebra describes free fermions propagating on the knot diagram. Grassmann integration allowed us to rewrite the Alexander Conway polynomial as a new determinant. In [2] this new determinant was further studied. It was shown in particular how it solves the problem of normalisation with respect to the fixed edge, and how it gives rise naturally to a two variable extension on thickened surfaces. This third paper is a sequel where we study more systematically questions raised earlier. The organization is as follows:

In the second section we discuss the crystal and its relation to Fox differential calculus. We recall [1] how the Yang Baxter solution associated with the Alexander Conway polynomial is obtained by going to the exterior algebra, that is by introducing fermions. We introduce the longitude. We present multicolor generalizations.

In section three we discuss in detail the relation between the R matrix deduced from the Burau matrix by going to the exterior algebra, and the quantum group $U_q gl(1,1)$. Various confusing issues raised in the literature are solved.

In section four we elaborate on Yang Baxter equation and exterior algebra. We discuss in particular higher dimensional representations of the crystal.

In section five we investigate the state models related to $U_q sl(n,m)$. We find that in S^3 they give the same invariants as the ones related to $U_q sl(n-m)$. Various trace identities used in [1] are proven. In the case $n = m = 1$ we discuss a state model for the multivariable Alexander polynomial.

In section six we extend these models to solid handlebodies. The non-trivial geometry lifts the above degeneracy, and a new invariant is obtained for any (n,m) pair.

In section seven we consider the effect of "gauge like" changes of basis and show that they give rise to new multivariable invariants for links in thickened surfaces.

Section eight contains some conclusions.

In independent works, graded algebras and link invariants have been addressed by the Tokyo group [10, 11] and by H.C.Lee [12]. The role of the exterior algebra, and the multivariable Alexander invariant, have also been considered by J.Murakami [13]. The subject is wide enough that there is little overlap between these contributions and ours. Solutions to the Yang-Baxter equation based on graded algebras have appeared much earlier [14, 15].

2 The link crystal and its applications

In this section we review the algebra that is related to the fundamental group of a link complement from a combinatorial point of view.

2.1 The link crystal

We first construct the **crystal** $\mathcal{C}(K)$ of a link K [17, 19] (In [16] the authors define a rack. The rack is nothing but the crystal written in exponential notation: $a\overline{b}\rceil = a^b$). The crystal is an algebraic system that classifies links up to mirror images. By adding an element corresponding to the longitude of a knot, the crystal plus longitude classify knots completely.

In order to articulate the crystal, we use an operator notation, $\overline{a}\rceil$ and $\lceil\overline{a}$. The angle bracket is referred to as the **mark**(left or right as the case may be). This notation acts as a combination parenthesis and operator symbol. The algebra will have a (non commutative) associative binary operation, denoted XY for X and Y elements of the algebra. We shall write products of the form $a * b = a\overline{b}\rceil$ and $a \bar{*} b = a\lceil\overline{b}$. The advantage of the operator notation is that it allows us to express the possibly non associative operation $*$ in an associative but non commutative context. Thus $a * (b * c) = a\overline{b\overline{c}\rceil}\rceil$ while $(a * b) * c = a\overline{b}\rceil\overline{c}\rceil$.

In the crystal associated to an oriented link diagram K there is one generator for each arc of the diagram, and one relation of the form $c = a\overline{b}\rceil$ or $c = a\lceil\overline{b}$ for each crossing. The relations depend upon orientation as indicated below

$$(1)$$

Thus we think of the overcrossing line as acting on the undercrossing line to produce the "color" of the extension of the undercrossing line on the other side. Every element of the crystal is of the form

$$\omega_0\overline{\omega_1}\overline{\omega_2}\dots\overline{\omega_n} \tag{2}$$

where the bar stands for $\rceil$ or $\lceil$, ω_0 is either a generator (arc label) or an empty word, and each ω_i is either a generator or an expression of the form (2) where ω_0 is not an empty word.

The subset of the crystal $\mathcal{C}(K)$ consisting of words of the form $\overline{\omega_1}\dots\overline{\omega_n}$ is denoted $\Pi(K)$. It is the **operator subgroup**. Crystal operations are subject to the following axioms:

$$\overline{a}\rceil\lceil\overline{a} = 1 = \lceil\overline{a}\,\overline{a}\rceil \tag{3}$$

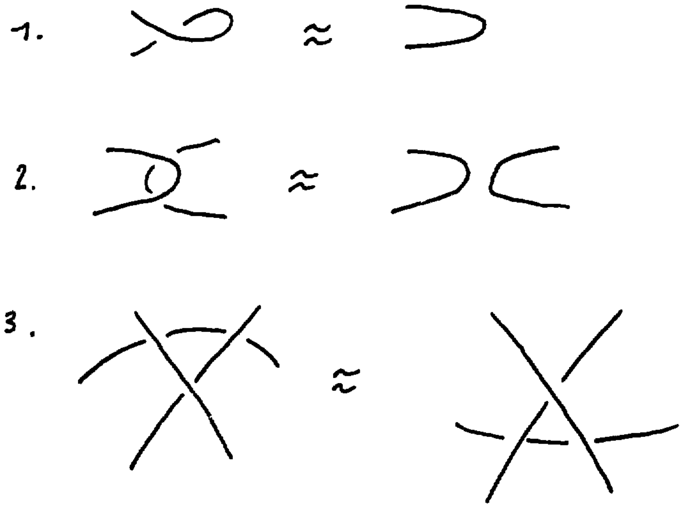
where 1 denotes the empty word, and

$$\overline{a\overline{b}\rceil}\rceil = \lceil\overline{b}\,\overline{a}\rceil\,\overline{b}\rceil$$

$$\overline{a\lceil b}\rceil = \overline{b}\rceil\,\overline{a}\rceil\,\overline{\lceil b}$$
$$\lceil\overline{ab}\rceil = \lceil b\,\overline{\lceil ab}\rceil$$
$$\overline{\lceil a\lceil b} = \overline{b}\rceil\,\overline{\lceil a}\,\overline{\lceil b} \qquad (4)$$

These axioms ensure that the crystal is invariant under the Reidemeister moves of type 2 and 3 and they demonstrate that the operator subgroup $\Pi(K)$ is indeed a group that is isomorphic with the fundamental group of the link complement in the Wirtinger representation (see [17]).

We recall briefly the Reidemeister moves.



(5)

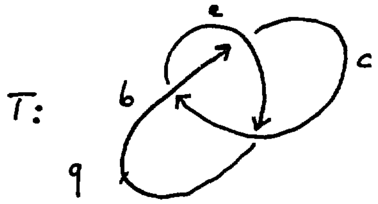
The equivalence relation generated by the Reidemeister moves 2 and 3 is called **regular isotopy**. The crystal $C(K)$ is an invariant of regular isotopy, while the operator subgroup $\Pi(K)$ is an invariant of ambient isotopy (the equivalence relation generated by all three Reidemeister moves). For example,

(6)

In the crystal $a\overline{a}\rceil \neq a$ but as an operator, $\overline{a\overline{a}\rceil}\rceil = \lceil a\overline{a}\rceil\overline{a}\rceil = \overline{a}\rceil$. This shows the invariance of $\Pi(K)$ under the first Reidemeister move.

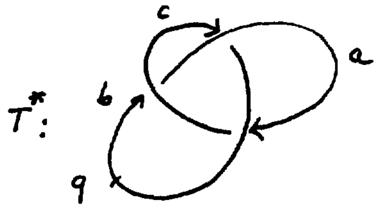
The crystal is therefore to be regarded as an invariant of framed links since regular isotopy and framing are essentially interchangeable points of view (see [17] for a discussion of this point).

However, even as a regular isotopy invariant, the crystal does not necessarily distinguish mirror images. For example the trefoil T and its mirror image T^* have isomorphic crystals, as shown below



$$c = b\overline{a}\rceil$$
$$b = a\overline{c}\rceil$$
$$a = c\overline{b}\rceil$$

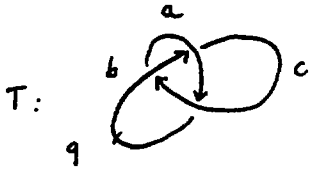
(7)



$$c\lceil\overline{a} = b$$
$$b\lceil\overline{c} = a$$
$$a\lceil\overline{b} = c$$

(8)

There is however a way to add extra information to $\mathcal{C}(K)$. We describe this for knot diagrams (one component). Choose a base point at an interior point of one of the arcs of K. Let $\lambda(K,q)$ denote the product of operators encountered at undercrossing sites as the diagram is traversed from the basepoint. For example,



$$\lambda(T,q) = \overline{a}\rceil\overline{b}\rceil\overline{c}\rceil$$

(9)

Call $\lambda(T,q)$ the longitude of K with respect to the base point q. Let $\Lambda(T) = \{g\lambda(T,q)g^{-1}; g \in \Pi(K)\}$ be the conjugacy class of $\lambda(T,q)$. It is easy to see that $\Lambda(T)$ is an invariant of regular isotopy, and that this conjugacy class is independent of the choice of base point.

Theorem 2.1: For a knot K the pair $(\mathcal{C}(K), \Lambda(K))$ is a complete invariant of regular isotopy.

Proof. It follows from [18] that the quotient of $(\mathcal{C}(K), \Lambda(K))$ by the relations $a\overline{a}\rceil = a$ (this is the quandle of K equipped with a longitude) classifies the ambient isotopy class of K. However, by using the longitude class $\Lambda(K)$ and taking $(\mathcal{C}(K), \Lambda(K))$, we can deduce the writhe of $K, \omega(K)$ as the sum of ± 1 for each left or right mark in a longitude for K. It is easy to see that knowing the writhe $\omega(K)$ discriminates the regular isotopy class within the ambient isotopy class//

Note that while we have indicated an isomorphism of $\mathcal{C}(T)$ with $\mathcal{C}(T^*)$ in the example above, we have $\lambda(T,q) = \overline{a}\rceil\overline{b}\rceil\overline{c}\rceil$ while $\lambda(T^*,q) = \lceil\overline{c}\lceil\overline{b}\lceil\overline{a}$. Thus this isomorphism does not preserve the longitudes. A more sustained argument is needed to prove that T and T^* are not ambient isotopic, but this example illustrates how the inclusion of the longitude adds needed information to the crystal. We would like to be able to deduce that T and T^* are not ambient isotopic directly from the crystal. This seems to require a modicum of group theory, and to differ very little from the classical proofs (compare [18]). Nevertheless, the longitude is instructive in showing how the isomorphism that we have given of $\mathcal{C}(T)$ and $\mathcal{C}(T^*)$ cannot be carried by an ambient isotopy since this isomorphism carries λ into λ^{-1} rather than λ.

Rather than relying on deep results in three dimensional topology, it would be very satisfying to have a combinatorial proof of theorem 2.1. One step in this direction is the direct reconstruction of a prime knot diagram from its crystal presentation and longitude. While this can be done, it is not obvious how to perform reconstruction from a crystal whose presentation has been algebraically transformed by substitutions and replacements not specified geometrically or diagramatically.

2.2 Linear representations

One of the more fruitful methods for obtaining information about the crystal is to consider representations of the form $a\overline{b}\rceil = ra + sb$, $a\lceil\overline{b} = r'a + s'b$ where r, s, r', s' belong to a commutative (see section 4 for a study of the noncommutative case) ring of operators acting on the color space of the crystal (by color space we mean all elements of the crystal of the form $\omega_0\overline{\omega_1}\ldots$ where ω_0 is a non empty generator, and each bar stands for either a right or left mark).

Lemma 2.2: In order for the formulas

$$a\overline{b}\rceil = ra + sb,\ a\lceil\overline{b} = r'a + s'b \tag{10}$$

to be a (general) representation of the crystal axioms it is necessary that r and r' be invertible, with $r' = r^{-1}$ and $s' = -r^{-1}s$ with s satisfying the identity $s^2 = (1-r)s$. Notice that if s is invertible this implies

$$a\overline{b}\rceil = ra + (1-r)b, a\lceil\overline{b} = r^{-1}a + (1-r^{-1})b \tag{11}$$

We have called this last representation the Alexander crystal [19].

Proof: First observe that $a\overline{b}\rceil\lceil\overline{b} = r'(ra+sb) + s'b = rr'a + (r's+s')b$. Since $a\overline{b}\rceil\lceil\overline{b} = a$, we conclude that $rr' = 1$ and $r's + s' = 0$. Hence $r' = r^{-1}$ and $s' = -r^{-1}s$. Second, observe

$$\begin{aligned} a\overline{b\overline{c}\rceil}\rceil &= ra + s(rb + sc) \\ &= ra + rsb + s^2c \\ a\lceil\overline{c}\,\overline{b}\rceil\overline{c}\rceil &= r[r(r'a + s'c) + sb] + sc \\ &= ra + rsb + (1-r)sc \end{aligned}$$

Thus $\overline{bc\rceil}\rceil = \lceil c\overline{b\rceil d\rceil} \rightarrow s^2 = (1-r)s$. This completes the proof//

2.3 Yang Baxter equation

It is of some interest to consider the formalism of the braid group representation that is associated with the crystal representation

$$a\overline{b}\rceil = ra + sb, a\lceil\overline{b} = r^{-1}a - r^{-1}sb \tag{12}$$

with $s^2 = (1-r)s$. In this case, we associate a linear transformation $X : V \to V$ on an elementary module V with basis $\{b, a\}$ so that $X(b) = a\overline{b}\rceil = ra + sb$, $X(a) = b$. Thus as matrices we have

$$X = \begin{pmatrix} s & r \\ 1 & 0 \end{pmatrix}, X^{-1} = \begin{pmatrix} 0 & 1 \\ r^{-1} & -r^{-1}s \end{pmatrix} \tag{13}$$

These are matrices over the ring $Z[r, r^{-1}, s]/\mathcal{I}$ where $\mathcal{I}$ is the ideal generated by $s^2 - (1-r)s$. The matrices X and X^{-1} are the elementary building blocks for a representation of the braid group that generalizes the classical Burau representation. In representing a generator σ_i (see [20]) of B_n, the Artin braid group on n strands, we take $\rho(\sigma_i)$ to be an $n \times n$ matrix with 1 on the diagonal except in the i^{th} and $i+1^{th}$ places where the 2×2 block X is inserted (X^{-1} is used for σ_i^{-1}). In [1] we discussed a solution to the Yang Baxter equation that ensued from looking at the action of the Burau representation on the exterior powers of the representation space. Here we can generalize this action as follows. Let $\hat{X}$ denote the extension of X to $\Lambda^*(V)$. Then we have

$$\hat{X}(1) = 1, \hat{X}(b) = ra + sb, \hat{X}(a) = b, \hat{X}(a \wedge b) = -r(a \wedge b) \tag{14}$$

so that as a matrix $\hat{X}$ reads

$$\hat{X} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s & r & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -r \end{pmatrix} \tag{15}$$

and once again it is easy to see that $\hat{X}$ is a solution to the Yang Baxter equation.

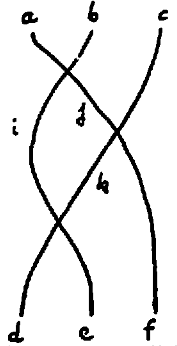
It is useful at this point to indicate a general form of solution to the Yang Baxter equation. Let $\check{R}$ denote a matrix of the form

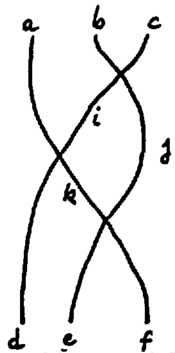
$$\check{R} = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & a & x & 0 \\ 0 & y & b & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{16}$$

Then $\check{R}$ satisfies the Yang Baxter equation

$$\sum_{ijk} \check{R}^{ab}_{ij} \check{R}^{jc}_{kf} \check{R}^{ik}_{de} = \sum_{ijk} \check{R}^{bc}_{ij} \check{R}^{ai}_{dk} \check{R}^{kj}_{ef} \tag{17}$$

which we illustrate graphically as follows





(18)

if and only if the following seven conditions on the entries of $\tilde{R}$ are satisfied

$$\begin{aligned} abx &= 0 \\ aby &= 0 \\ ab(b-a) &= 0 \\ p^2b &= pb^2 + bxy \\ n^2b &= nb^2 + bxy \\ p^2a &= pa^2 + axy \\ n^2a &= na^2 + axy \end{aligned} \tag{19}$$

(see the appendix of [9] for a proof of this statement).

For $\hat{X}$ we have $n = 1, a = s, b = 0, x = r, y = 1, p = -r$. Therefore the equations (19) become

$$r^2s = -rs^2 + sr, s = s^2 + sr \tag{20}$$

which, since we have assumed that r is invertible, are both equivalent to

$$s^2 = s(1-r) \tag{21}$$

This is the only condition needed to make $\hat{X}$ a solution to the Yang Baxter equation, and this condition is identical to the one we have assumed to create the crystal representation. When s is also invertible, we obtain the solution to the Yang Baxter equation that is related to the Alexander polynomial as discussed in [1].

It is convenient to replace $\hat{X}$ by a matrix obtained by a gauge-like change of basis. Such changes of basis will be discussed further in section 7. For 4 × 4 matrices they consist simply in multiplying the $(3,2)$ coefficient by some constant and the $(2,3)$ coefficient by the inverse constant. We leave it to the reader to check that the matrix $\tilde{R}$ below can be obtained from $\hat{X}$ by a basis change coupled with replacing s by Ω and r by q^2, with $\Omega^2 = (q - q^{-1})\Omega$

$$\tilde{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \Omega & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix} \tag{22}$$

$\tilde{R}$ is a solution to the Yang Baxter equation, and it is a form parallel to what we have used in [1] with Ω invertible.

Along with $\tilde{R}$ we have also the simpler solution when $\Omega = 0$. This corresponds to $s = 0$ and derives from the simplest crystal representation

$$a\overline{b}\rceil = ra, a\lceil\overline{b} = r^{-1}a \tag{23}$$

Of course this representation detects only the writhe of the diagram.

A state model arising from (22) would have the form $[K] = [K]_0 + [K]_1 \Omega$, where the two factors $[K]_0$ and $[K]_1$ do not involve Ω. $[K]_0$ corresponds to the state model arising from the solution $\Omega = 0$. We notice that in fact there is a more general solution of the Yang Baxter equation given by

$$\tilde{R} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \tag{24}$$

and the corresponding state summation, an invariant of regular isotopy, has the form

$$[K] = \sum_{S} [K|S] \delta^{||S||} \tag{25}$$

where δ is a third independent variable. We therefore have a three variable invariant of regular isotopy which requires further investigation.

It is interesting to see that more than one Yang Baxter solution arises from the linear representations of the crystal. This formalism also sheds some light on the possible relationships between the Jones polynomial and the Alexander polynomial. For, as we have remarked in [7, 9], the Jones ($U_q sl(2)$) and Alexander ($U_q gl(1,1)$) solutions both derive from the conditions (19) via the restrictions $b = 0, x = y = 1$, so that (19) becomes the equations

$$p^2 a = pa^2 + a, n^2 a = na^2 + a \tag{26}$$

Assuming that n, p are invertible, they become

$$a^2 = (p - p^{-1})a, a^2 = (n - n^{-1})a \tag{27}$$

Then a invertible and $p = n$ gives the $U_q sl(2)$ solution while $p = -n^{-1}$ gives the $U_q gl(1,1)$ solution. These quadratic conditions suggest that there should be a generalisation or reformulation of the crystal such that they arise from it. There would then be an algebraic line from the crystal to the Jones polynomial. At this stage we have only these hints.

2.4 Colored Links

There is one further situation in classical link theory that gives rise to nontrivial solutions of the Yang Baxter equation. This is the case of colored links, where each link component C is assigned a color, and a corresponding variable t_a. This gives rise to the classical multivariable Alexander polynomial [20, 21], and this in turn is related to a crystal representation where each crystal element a has an associated variable t_a. The representation takes the form

$$\begin{aligned} a\overline{b\rceil} &= t_b a + (1 - t_a) b \\ a\overline{\lceil b} &= t_b^{-1} a + (t_b^{-1} t_a - t_b^{-1}) b \end{aligned} \tag{28}$$

The corresponding representation of the braid group for labelled (coloured) strands is

$$X = \begin{pmatrix} 1 - t_a & t_b \\ 1 & 0 \end{pmatrix}, X^{-1} = \begin{pmatrix} 0 & 1 \\ t_b^{-1} & t_b^{-1} t_a - t_b^{-1} \end{pmatrix} \tag{29}$$

We obtain a solution $\hat{X}$ to the coloured Yang Baxter equation (see section 5) by letting X act on the exterior algebra just as before

$$\hat{X} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - t_a & t_b & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -t_b \end{pmatrix} \tag{30}$$

In section 5 we will use this solution to obtain a state model for the multivariable Alexander polynomial and compare our construction with that of Murakami [13].

2.5 Crystal and Fox differential calculus

We shall close this section with a remark about the genesis of the crystal representation mentioned above. It is most easily seen by an application of the Fox differential calculus [22]. The free differential calculus is an algebraic device that assigns a "derivative" to a non commutative product via the rule $D(xy) = D(x) + xD(y)$. The motivation for this rule comes from group theory and covering space theory. However, we can apply this derivative to the crystal

$$a\overline{b\rceil} = bab^{-1}, a\overline{\lceil b} = b^{-1}ab \tag{31}$$

where a, b are elements of a group. We shall use $D(1) = 0$, so that $0 = D(1) = D(xx^{-1}) = D(x) + xD(x^{-1})$. Hence $D(x^{-1}) = -x^{-1}D(x)$. Then we have

$$\begin{aligned} D(a\overline{b\rceil}) &= bD(a) + (1 - bab^{-1})D(b) \\ D(a\overline{\lceil b}) &= b^{-1}D(a) + (b^{-1}a - b^{-1})D(b) \,. \end{aligned} \tag{32}$$

We see that if b is replaced by t_b and a by t_a in the coefficients of $D(a)$ and $D(b)$ we obtain the pattern of the coloured crystal representation. This sketch can be expanded to give a rigorous derivation of the colored crystal representation based on the free differential calculus. The upshot is that the free calculus implicates the nontrivial two variable Yang Baxter solution $\hat{X}$. Perhaps a generalization of the free calculus can generate the Yang Baxter solution corresponding to the Jones polynomial.

3 Quantum $gl(1,1)$ and its R matrix

Having obtained a solution to the Yang Baxter equation from a linear representation of the crystal and the exterior product we can now make contact with quantum groups. The relation between the $\check{R}$ matrix

$$\check{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix} \tag{33}$$

and $U_q gl(1,1)$ seems to have been first observed in [23, 24, 25]. It was later rediscovered [26, 27, 28], sometimes with some confusion. It may be worthwhile to stress the key point of this relation here.

A first way of deriving a quantum group from the $\check{R}$ matrix is as follows (see [29] and references therein). Introduce

$$\check{R}^+ = P\check{R}P = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix} \tag{34}$$

and $L^\pm$ upper (lower) triangular matrices whose entries are operators

$$L^+ = \begin{pmatrix} L^+_{11} & L^+_{12} \\ 0 & L^+_{22} \end{pmatrix}, L^- = \begin{pmatrix} L^-_{11} & 0 \\ L^-_{21} & L^-_{22} \end{pmatrix} \tag{35}$$

such that

$$\begin{aligned} (L^\pm \otimes L^\pm)\,\check{R}^+ &= \check{R}^+\,(L^\pm \otimes L^\pm) \\ (L^- \otimes L^+)\,\check{R}^+ &= \check{R}^+\,(L^+ \otimes L^-) \end{aligned} \tag{36}$$

If in calculating the products (36) the correct Z_2 graded structure of the tensor product is used, the basis in (36) being $|00>, |01>, |10>, |11>$, where $|0>$ (resp. $|1>$) is a boson (resp. fermion), one finds from the first equation in (36)

$$\begin{aligned} L^+_{12}L^+_{11} = qL^+_{11}L^+_{12} \quad &, \quad L^-_{21}L^-_{11} = qL^-_{11}L^-_{21} \\ L^+_{12}L^+_{22} = qL^+_{22}L^+_{12} \quad &, \quad L^-_{21}L^-_{22} = qL^-_{22}L^-_{21} \\ (L^+_{12})^2 = 0 \quad &, \quad (L^-_{21})^2 = 0 \\ [L^+_{11}, L^+_{22}] = 0 \quad &, \quad [L^-_{11}, L^-_{22}] = 0 \end{aligned} \tag{37}$$

and from the second equation in (36)

$$\begin{aligned} L^-_{11}L^+_{12} &= qL^+_{12}L^-_{11} \\ L^+_{11}L^-_{21} &= qL^-_{21}L^+_{11} \\ L^-_{22}L^+_{12} &= qL^+_{12}L^-_{22} \\ L^+_{22}L^-_{21} &= qL^-_{21}L^+_{22} \end{aligned}$$

$$\begin{aligned}
[L^-_{11}, L^+_{11}] &= [L^-_{22}, L^+_{22}] = 0 \\
[L^-_{11}, L^+_{22}] &= [L^-_{22}, L^+_{11}] = 0 \\
[L^+_{12}, L^-_{21}] &= (q - q^{-1})\left(L^+_{22}L^-_{11} - L^-_{22}L^+_{11}\right)
\end{aligned} \tag{38}$$

It is always possible to choose

$$\begin{aligned}
L^-_{11}L^+_{11} &= L^+_{11}L^-_{11} = 1 \\
L^-_{22}L^+_{22} &= L^+_{22}L^-_{22} = 1
\end{aligned} \tag{39}$$

since from the above equations these products commute with all the L's. Then we set

$$L^+ = \begin{pmatrix} k^{-1} & (q-q^{-1})\eta^+ \\ 0 & \xi \end{pmatrix}, L^- = \begin{pmatrix} k & 0 \\ (q-q^{-1})\eta & \xi^{-1} \end{pmatrix} \tag{40}$$

to get

$$\begin{aligned}
\eta^2 = 0 \quad &, \quad (\eta^+)^2 = 0 \\
[k, \xi] &= 0 \\
k\eta^+ = q\eta^+ k \quad &, \quad \xi\eta^+ = q^{-1}\eta^+\xi \\
k^{-1}\eta = q\eta k^{-1} \quad &, \quad \xi^{-1}\eta = q^{-1}\eta\xi^{-1} \\
[\eta, \eta^+]_+ &= \frac{\xi k - \xi^{-1}k^{-1}}{q - q^{-1}}
\end{aligned} \tag{41}$$

ξ and k^{-1} are identical up to a scale factor. We choose

$$\xi = q^E k^{-1} \tag{42}$$

to get

$$\begin{aligned}
\eta^2 = 0 \quad &, \quad (\eta^+)^2 = 0 \\
[\eta, \eta^+]_+ &= (E)_q \\
[E, \eta] = 0 \quad &, \quad [E, \eta^+] = 0 \\
k\eta^+ = q\eta^+ k \quad &, \quad k^{-1}\eta = q\eta k^{-1}
\end{aligned} \tag{43}$$

E is a $U(1)$ generator that is scalar in any indecomposable representation. A coproduct reads

$$\begin{gathered}
\Delta\left(\eta^{(+)}\right) = q^{E/2} \otimes \eta^{(+)} + \eta^{(+)} \otimes q^{-E/2} \\
\Delta(E) = E \otimes 1 + 1 \otimes E, \Delta(k^{\pm}) = k^{\pm} \otimes k^{\pm}
\end{gathered} \tag{44}$$

and the other one

$$\Delta' = \sigma o \Delta \tag{45}$$

A universal $\mathcal{R}$ matrix satisfying [30]

$$\mathcal{R}\Delta = \Delta'\mathcal{R} \tag{46}$$

is given by

$$\mathcal{R} \propto \left[1 + (q - q^{-1}) q^{(E\otimes 1 - 1\otimes E)/2} \eta^{+} \otimes \eta\right] \left(q^{E} \otimes k^{-1}\right) \left(k^{-1} \otimes q^{E}\right) \tag{47}$$

while (46) has also the solution

$$\mathcal{R}' = \mathcal{R}^{-1}(q^{-1}) \tag{48}$$

One can set

$$k = q^{N} \tag{49}$$

with

$$[N, \eta^{+}] = \eta^{+}, [N, \eta] = -\eta \tag{50}$$

The above relations define $U_q gl(1,1)$. More generally we shall deal with the quantum group $U_q sl(n,m)$ for $n \neq m$ but $U_q gl(n,n)$ for $n = m$.

The same analysis without treating the grading properly gives, setting

$$L^{+} = \begin{pmatrix} k^{-1} & (q - q^{-1})X \\ 0 & \xi \end{pmatrix}, L^{-} = \begin{pmatrix} k & 0 \\ (q - q^{-1})Y & \xi^{-1} \end{pmatrix} \tag{51}$$

the relations

$$\begin{aligned} X^2 = 0 \quad &, \quad Y^2 = 0 \\ [k, \xi] \quad &= \quad 0 \\ kX = qXk \quad &, \quad \xi X = -q^{-1} X \xi \\ k^{-1} Y = q Y k^{-1} \quad &, \quad \xi^{-1} Y = -q^{-1} Y \xi^{-1} \\ [X, Y] \quad &= \quad \frac{k\xi - k^{-1}\xi^{-1}}{q - q^{-1}} \end{aligned} \tag{52}$$

Now set

$$k = q^{N} \tag{53}$$

where

$$[N, X] = X, [N, Y] = -Y \tag{54}$$

Also set

$$\begin{aligned} X &= (-1)^{N} \eta^{+} \\ Y &= \eta \\ \xi &= (-1)^{N} q^{E} k^{-1} \end{aligned} \tag{55}$$

It is then easy to check that the commutator in the last line of (52) becomes an anticommutator

$$[\eta, \eta^{+}] = (-1)^{N} \frac{k\xi - k^{-1}\xi^{-1}}{q - q^{-1}} = (E)_q \tag{56}$$

such that (52) coincides with (43).

Therefore the algebras one finds by taking or not the grading of the tensor product into account are isomorphic. The relation to $U_q gl(1,1)$ is simply more transparent in the first case.

Another approach is to consider instead the dual [29]. Solving

$$\check{R}\, T \otimes T = T \otimes T \check{R} \tag{57}$$

where

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{58}$$

one finds, if the Z_2 graded structure of the tensor product is used

$$\begin{array}{rcl} ab = qba & , & ac = qca \\ bd = q^{-1}db & , & cd = q^{-1}dc \\ b^2 = 0 & , & c^2 = 0 \\ bc & = & -cb \\ ad - da & = & (q - q^{-1})cb \end{array} \tag{59}$$

dual to $U_q gl(1,1)$ with the fundamental representation

$$\eta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \eta^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, N = \begin{pmatrix} n-1 & 0 \\ 0 & n \end{pmatrix} \tag{60}$$

(where n is a free parameter), the product

$$fg(x) = (f \otimes g)\Delta(x) \tag{61}$$

and the coproduct (44). The quantum determinant

$$D = ad^{-1} - bd^{-1}cd^{-1} \tag{62}$$

is central.

The same calculation without taking into account the Z_2 grading [9] gives, if

$$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{63}$$

the following relations

$$\begin{array}{rcl} \alpha\beta = q\beta\alpha & , & \alpha\gamma = q\gamma\alpha \\ \beta\delta = -q^{-1}\delta\beta & , & \gamma\delta = -q^{-1}\delta\gamma \\ \beta^2 = 0 & , & \gamma^2 = 0 \\ \beta\gamma & = & \gamma\beta \\ \alpha\delta - \delta\alpha & = & (q - q^{-1})\gamma\beta \end{array} \tag{64}$$

One can easily find the fundamental representation of $U_q gl(1,1)$ in disguised form

$$X = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, k = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}, \xi = \begin{pmatrix} q & 0 \\ 0 & -1 \end{pmatrix} \tag{65}$$

and check that it is dual to (52).

$gl_q(1,1)$ can be derived from (64) by introducing a pair of additional numbers ψ, ψ^+ that mutually anticommute and

$$a = \alpha, d = \delta, b = \psi\beta, c = \psi^+\gamma, d = \psi\psi^+\delta \tag{66}$$

4 Exterior product and the Yang Baxter equation

The purpose of this section is to elaborate further on solutions of the Yang Baxter equation obtained by going to the exterior algebra.

Def.4.1: Consider X, a $2p \times 2p$ matrix. Introduce X_{12}, X_{23} , the $3p \times 3p$ matrices with block diagonal form

$$X_{12} = \begin{pmatrix} X & 0 \\ 0 & 1_p \end{pmatrix}, X_{23} = \begin{pmatrix} 1_p & 0 \\ 0 & X \end{pmatrix}$$

If

$$X_{12}X_{23}X_{12} = X_{23}X_{12}X_{23} \tag{67}$$

then X is said to provide a linear representation of the braid group.

For n strands, setting

$$X_i = \begin{pmatrix} 1_{(i-1)p} & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & 1_{(n-1-i)p} \end{pmatrix} \tag{68}$$

for $i = 1, \ldots, n$ one gets a representation of B_n with matrices of size $pn \times pn$ increasing linearly with n.

Prop.4.2: Write X in $p \times p$ block form

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \tag{69}$$

Then the equation (67) holds if and only if the following seven matrix equations are satisfied

$$\begin{aligned} x_{11}^2 + x_{12}x_{11}x_{21} &= x_{11} \\ x_{22}^2 + x_{21}x_{22}x_{12} &= x_{22} \\ x_{11}x_{12} + x_{12}x_{11}x_{22} &= x_{12}x_{11} \\ x_{22}x_{21} + x_{21}x_{22}x_{11} &= x_{21}x_{22} \\ x_{21}x_{11} + x_{22}x_{11}x_{21} &= x_{11}x_{21} \\ x_{12}x_{22} + x_{11}x_{22}x_{12} &= x_{22}x_{12} \\ x_{12}x_{21} + x_{11}x_{22}x_{11} &= x_{21}x_{12} + x_{22}x_{11}x_{22} \end{aligned} \tag{70}$$

If X satisfies (67) so does the matrix obtained by the same change of basis on all the x's: $x_{ij} \rightarrow \Omega x_{ij} \Omega'$.

Proof: A direct calculation which we omit//

Before turning to some solutions of (70) we now make contact with the standard solutions of the Yang Baxter equation. These involve operators acting in some tensor product $W^{\otimes 3}$ with $Y_{12} = Y \otimes 1$, $Y_{23} = 1 \otimes Y$, Y acting in $W^{\otimes 2}$, such that $Y_{12}Y_{23}Y_{12} = Y_{23}Y_{12}Y_{23}$. A representation of the braid group on n strands is then obtained by matrices $Y_i = \ldots \otimes 1 \otimes Y \otimes 1 \ldots$ of dimension increasing exponentially with n.

We first introduce, if V is the vector space of dimension $3p$ with basis $v_1 \dots v_{3p}$ where X_{12}, X_{23} act, the exterior algebra

$$\Lambda^* = V^0 \oplus V^1 \oplus \dots \oplus V^{3p} \tag{71}$$

where V^k is the subspace generated by vectors $v_{i_1} \wedge \dots \wedge v_{i_k}$ and $V^1 = V$. One has $dim\Lambda^* = (2^p)^3$. We then define the extension of X_{12} to the exterior product by

$$\hat{X}_{12}\left(v_{i_1} \wedge \dots \wedge v_{i_k}\right) = X_{12}\left(v_{i_1}\right) \wedge \dots \wedge X_{12}\left(v_{i_k}\right) \tag{72}$$

and similarly for X_{23}. Then one has

$$\hat{X}_{12}\hat{X}_{23}\hat{X}_{12} = \hat{X}_{23}\hat{X}_{12}\hat{X}_{23} \tag{73}$$

since (67) holds.

To consider $\hat{X}$ as a standard solution to the Yang Baxter equation we still need to define the space W and a basis such that $\hat{X}_{12}, \hat{X}_{23}$ act as $Y \otimes 1$ and $1 \otimes Y$ for some Y. This is done by building the Λ^* basis as

$$\begin{aligned}(1, v_1, \dots v_p, \dots, v_1 \wedge \dots \wedge v_p) \quad &\wedge \quad (1, v_{p+1} \dots, v_{2p}, \dots, v_{p+1} \wedge \dots \wedge v_{2p}) \\ &\wedge \quad (1, v_{2p+1}, \dots, v_{3p}, \dots, v_{2p+1} \wedge \dots \wedge v_{3p})\end{aligned} \tag{74}$$

where each exterior product has increasing v indices, elements are varied in the first set, then the second, then the third. If $p = 1$, for instance, this gives

$$1, v_1, v_2, v_1 \wedge v_2, v_3, v_1 \wedge v_3, v_2 \wedge v_3, v_1 \wedge v_2 \wedge v_3 \tag{75}$$

We now identify the set of vectors in (74) as the basis of $W^{\otimes 3}$ written

$$(\omega_1 \dots \omega_{2^p}) \otimes (\omega_1 \dots \omega_{2^p}) \otimes (\omega_1 \dots \omega_{2^p}) \tag{76}$$

elements being varied in the first set, then the second, then the third. For $p = 1$ this gives

$$\omega_1 \otimes \omega_1 \otimes \omega_1, \omega_2 \otimes \omega_1 \otimes \omega_1, \omega_1 \otimes \omega_2 \otimes \omega_1, \omega_2 \otimes \omega_2 \otimes \omega_1, \omega_1 \otimes \omega_1 \otimes \omega_2, \omega_2 \otimes \omega_1 \otimes \omega_2, \omega_1 \otimes \omega_2 \otimes \omega_2, \omega_2 \otimes \omega_2 \otimes \omega_2 \tag{77}$$

If Y is the matrix representation of $\hat{X}$ acting in

$$(1. v_1, \dots, v_p, \dots, v_1 \wedge \dots \wedge v_p) \wedge (1, v_{p+1}, \dots, v_{2p}, \dots, v_{p+1} \wedge \dots \wedge v_{2p}) \tag{78}$$

then in the basis (74) $\hat{X}_{12}$ is represented by the matrix Y repeated 2^p times, while $\hat{X}_{23}$ has each element of Y repeated 2^p times. They can thus be represented as matrices for $Y \otimes 1_{2^p}$, $1_{2^p} \otimes Y$. Hence

Prop.4.3: Lifting the action of X_{12}, X_{23} to Λ^* by the above procedure provides, in the basis (74), matrices $\hat{X}_{12}, \hat{X}_{23}$ that can be considered as satisfying the standard Yang Baxter equation.

Example: In the case $p = 1$, it is easy to see that all solutions to (70) are equivalent to

$$X = \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \tag{79}$$

where t is a free parameter. The above matrix $Y = \hat{X}$ is then

$$Y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -t \end{pmatrix} \tag{80}$$

We thus recover the results of section 2. This $p = 1$ solution generalizes in the following way.

Prop.4.4: If x_{22}=0, x_{11} is invertible, x_{11}, x_{12}, x_{21} are diagonalizable, then all solutions to (70) are equivalent to

$$X = \left(\begin{array}{ccc|ccc} 1-t_1 & & & t_1 & & \\ & \ddots & & & \ddots & \\ & & 1-t_p & & & t_p \\ \hline 1 & & & 0 & & \\ & \ddots & & & \ddots & \\ & & 1 & & & 0 \end{array}\right) \tag{81}$$

where $t_1 \ldots t_p$ are free parameters.

Proof: Indeed if $x_{22} = 0$, the system (70) is equivalent to

$$[x_{12}, x_{21}] = 0, [x_{11}, x_{12}] = 0, [x_{11}, x_{21}] = 0, x_{11}^2 + x_{11}x_{12}x_{21} = x_{11}$$

and since x_{11} is invertible the last condition can be rewritten

$$x_{11} = 1 - x_{12}x_{21}$$

We can now simultaneously diagonalize x_{11}, x_{12}, x_{21} (without affecting the Yang Baxter equation, as noticed above) such that, if the eigenvalues of x_{11} are written $1 - t_i$, those of x_{12} (resp. x_{21}) are written $\alpha t_i^{1/2}$ (resp. $\alpha^{-1}t_i^{1/2}$), X reads

$$X = \left(\begin{array}{ccc|ccc} 1-t_1 & & & \alpha t_1^{1/2} & & \\ & \ddots & & & \ddots & \\ & & 1-t_p & & & \alpha t_p^{1/2} \\ \hline \alpha^{-1}t_1^{1/2} & & & 0 & & \\ & \ddots & & & \ddots & \\ & & \alpha^{-1}t_p^{1/2} & & & 0 \end{array}\right)$$

A gauge-like change of basis $v_i \to \frac{\alpha}{t_i^{1/2}} v_i$ gives the desired form//

We therefore obtain multiparameter solutions of the Yang Baxter equation, however rether trivial. As the $p = 1$ case was related to the one-dimensional representation of the link crystal, so is (81) related to p-dimensional ones. In this case a strand carries a color that lives in a p dimensional vector space. From prop.4.3 we see that the Yang Baxter equation, or alternatively the invariance under type 3 moves forces these representations to factorize onto the $p = 1$ ones.

Following [1] it is easy to see that the link invariant obtained through (81) is the product of Alexander-Conway invariants

$$\nabla_L\left(t_1 - t_1^{-1}\right)\ldots\nabla_L\left(t_p - t_p^{-1}\right) \tag{82}$$

We explained in section 3 that the $p = 1$ Y matrix was related, after a gauge like change of basis, to the $\check{R}$ matrix of the quantum algebra $U_q gl(1,1)$

$$\check{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix} \tag{83}$$

where $t = q^{-2}$. In a similar fashion, for $p > 1$, the Y matrix can be considered after the appropriate change of basis as the $\check{R}$ matrix for the product $U_{q_1}gl(1,1) \times \ldots U_{q_p}gl(1,1)$ with relations

$$\begin{aligned} \left[\eta_i, \eta_j^+\right] &= \delta_{ij}\left(E_i\right)_{q_i},\ i,j = 1\ldots p \\ \left[\eta_i, \eta_j\right] &= 0 \\ \left[\eta_i^+, \eta_j^+\right]_+ &= 0 \end{aligned} \tag{84}$$

and the coproduct

$$\Delta\left(\eta_i^{(+)}\right) = q^{E_i/2} \otimes \eta_i^{(+)} + \eta_i^{(+)} \otimes q^{-E_i/2} \tag{85}$$

$\check{R}$ has eigenvalues

$$1, -t_i, t_i t_j, \ldots, \prod_{i_1 < \ldots < i_k} t_{i_1} \ldots t_{i_k}, \ldots, (-1)^p t_1 \ldots t_p$$

each 2^p times degenerate.

Another multiparameter family of solutions is provived by

Prop. 4.5: $X = \begin{pmatrix} 0 & x_{12} \\ x_{21} & 0 \end{pmatrix}$, where $[x_{12}, x_{21}] = 0$, satisfies (67).

In the case $p = 1$ the only non-trivial solution of this form is $X = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$, which is equivalent to $X = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ with $a = (bc)^{1/2}$. Then

$$\hat{X} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & -a^2 \end{pmatrix} \tag{86}$$

For arbitrary p, assuming x_{12}, x_{21} are diagonalizable, solutions are equivalent to

$$X = \left(\begin{array}{ccc|ccc} 0 & & & a_1 & & \\ & \ddots & & & \ddots & \\ & & 0 & & & a_p \\ \hline a_1 & & & 0 & & \\ & \ddots & & & \ddots & \\ & & a_p & & & 0 \end{array}\right) \tag{87}$$

The last set of solutions we shall exhibit is given by

Prop. 4.6: $X = \begin{pmatrix} x & y \\ y & x \end{pmatrix}$ with $x^2 = x, xy = 0, x \neq 0$ satisfies (67)

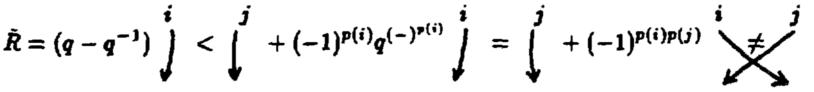
For $p = 2$ this is equivalent to

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & a & 0 & 0 \end{pmatrix} \tag{88}$$

5 State model for $U_q sl(n,m)$ R matrix

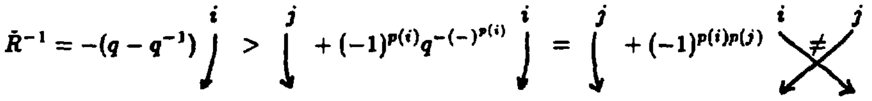
5.1 Unbalanced case

The $U_q sl(n,m)$ $\check{R}$ $(n \neq m)$ matrix [31, 1] (in the product of two fundamental representations) reads in the graphical representation devised in [9]



$$\check{R} = (q-q^{-1}) \overset{i}{\downarrow} < \overset{j}{\downarrow} + (-1)^{p(i)} q^{(-)^{p(i)}} \overset{i}{\downarrow} = \overset{j}{\downarrow} + (-1)^{p(i)p(j)} \overset{i \quad j}{\underset{\neq}{\times}} \tag{89}$$

where the labels take values $i = 1, \ldots, n+m$, the first $i = 1, \ldots, n$ being bosonic coordinates $(p(i) = 0)$, the last $i = n+1, \ldots, n+m$ being fermions $(p(i) = 1)$. For the inverse operator one has



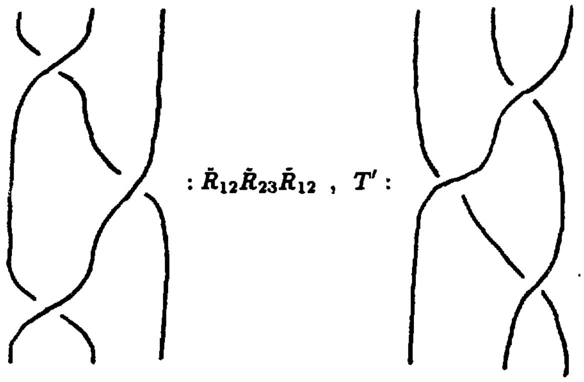
$$\check{R}^{-1} = -(q-q^{-1}) \overset{i}{\downarrow} > \overset{j}{\downarrow} + (-1)^{p(i)} q^{-(-)^{p(i)}} \overset{i}{\downarrow} = \overset{j}{\downarrow} + (-1)^{p(i)p(j)} \overset{i \quad j}{\underset{\neq}{\times}} \tag{90}$$

such that

$$\check{R} - \check{R}^{-1} = (q-q^{-1})1 \tag{91}$$

The matrices (89,90) differ from the $U_q sl(n+m)$ ones in that m of their diagonal terms take the values $-q^{-1}$ instead of q. The $(-1)^{p(i)p(j)}$ introduced in (89) for algebraic consistence (i.e. $\check{R} \rightarrow$ graded permutation operator as $q \rightarrow 1$) are not meaningful and can be suppressed if necessary. We shall use in the following the notation $z = q - q^{-1}$.

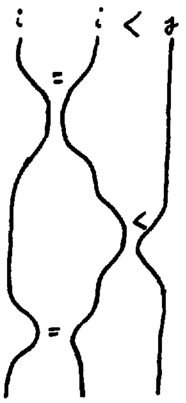
That $\check{R}$ satisfies the Yang Baxter equation is easily checked by expanding graphically the two triangles



T : $\check{R}_{12}\check{R}_{23}\check{R}_{12}$, T' : $\check{R}_{23}\check{R}_{12}\check{R}_{23}$ (92)

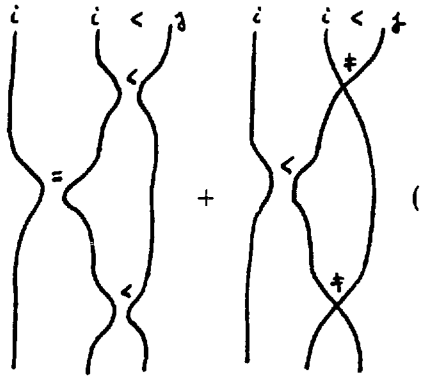
and comparing factors. Here is an example with no permutation of the coordinates

$T:$



$zq^{2(-)^{p(i)}} = q^3 - q$ for i a boson, $q^{-1} - q^{-3}$ for i a fermion

$T':$



$$
\begin{aligned}
(-1)^{p(i)} q^{(-)^{p(i)}} z^2 &+ q - q^{-1} \\
&= q^3 - q \text{ for i a boson} \\
&= q^{-1} - q^{-3} \text{ for i a fermion}
\end{aligned}
$$

Other cases are easily checked in a similar way.

Consider now as in [9] K an oriented link diagram and U the associated universe. A **state** S of U is a labelling of the edges of U with elements from the set $i = 1, \ldots, n+m$ so that the labels are strongly preserved at each vertex. The **splitting** $L(S)$ of the state S is the labelled universe obtained frrom S by splicing each vertex where the spins do not cross over. $L(S)$ is a collection of simple closed curves C in the plane labelled by a single spin index i_C from the index set. For a given state S denote by $[K|S]$ the product of matrix elements arising from each vertex, $\check{R}$, $\check{R}^{-1}$ being associated with a positive or negative crossing, with conventions

(93)

We also use the following conventions for rotation numbers

rot = 1 :  rot = -1 : 

(94)

We then have, generalising [9] to the graded case, and supposing $n > m$ for definiteness

Theorem 5.1: Introduce the label weight

$$\begin{aligned} \Gamma(i) &= -n + m - 1 + 2i \text{ ,i a boson ; } i = 1, \ldots, n \\ \Gamma(i) &= 3n + m + 1 - 2i \text{ ,i a fermion ; } i = n+1, \ldots, n+m \end{aligned} \tag{95}$$

and the state summation

$$[K] = \sum_S [K|S] \prod_C q^{\Gamma(ic)rot(C)} (-1)^{p(ic)} \tag{96}$$

Then

$$P_K = \left(q^{n-m}\right)^{-w(K)} \frac{[K]}{[0]},\ [0] = (n-m)_q \tag{97}$$

is an ambient isotopy invariant of the oriented link K satisfying the skein relation

$q^{n-m}P$  $-q^{-(n-m)}P$  $= (q - q^{-1})P$  (98)

It coincides for any $n, m \in N$ such that $n - m = N \geq 2$ with the invariant obtained from $U_q sl(N)$ in the fundamental representation.

Before proving this let us write down the weights Γ

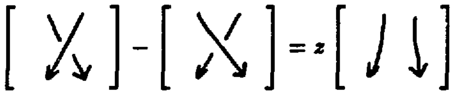
$$\begin{array}{ll}
\text{bosons} & : \left\{ \begin{array}{ll} i = 1 & \Gamma = -(n-m-1) \\ i = 2 & \Gamma = -(n-m-1) + 2 \\ & \\ i = n-m & \Gamma = n-m-1 \end{array} \right. \\
\text{bosons} & : \left\{ \begin{array}{ll} i = n-m+1 & \Gamma = n-m+1 \\ i = n-m+2 & \Gamma = n-m+3 \\ & \\ i = n & \Gamma = n+m-1 \end{array} \right. \\
\text{fermions} & : \left\{ \begin{array}{ll} i = n+1 & \Gamma = n+m-1 \\ & \\ i = n+m-1 & \Gamma = n-m+3 \\ i = n+m & \Gamma = n-m+1 \end{array} \right.
\end{array} \tag{99}$$

Example:

For $sl(5,2)$ we have $i = 1,2,3,4,5,6,6,7$ and $\Gamma(i) = -2,0,2,4,6,4$. As it will turn out, the last m bosonic and the m fermionic labels with identical weights Γ cancel each other, leaving effectively only $n - m = N$ labels.

Proof:

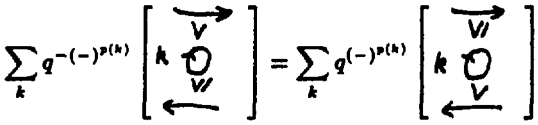
(i) By the same arguments as in the non-graded case [9] one finds that the state summation $[K]$ is invariant under the Reidemeister moves $3A, 3B, 2A$ and satisfies



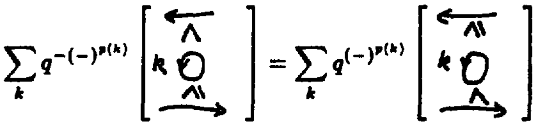
(100)

As in [9] a link diagram in bracket $[K]$ stands for its state summation, a split state in brackets $[L(S)]$ for the product of symmetry factors.

(ii) Invariance under $2B$ is satisfied provided

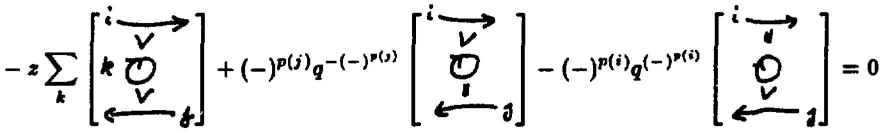


(101)



(102)

Let us consider the first orientation. (102) is equivalent to



(103)

or

$$-(q-q^{-1})\sum_{k=j+1}^{i-1}(-)^{p(k)}q^{-\Gamma(k)} + q^{-\Gamma(j)-(-)^{p(j)}} - q^{-\Gamma(i)+(-)^{p(i)}} = 0 \tag{104}$$

This is easily proven case by case. Let us consider first $i = j+1$. Then if both i and $j+1$ are bosons (resp. fermions) $\Gamma(i) = \Gamma(j) + 2$ (resp. -2), $(-)^{p(i)} = (-)^{p(j)}$ so (103) holds. If j is a boson, i a fermion, then $j = n, i = n+1$. Then $\Gamma(j) = \Gamma(i)$ while $(-)^{p(j)} = -(-)^{p(i)}$. Again the left member is zero. If $i \geq j+2$, we have to consider three cases: $(j+1,\ldots,i-1)$ are bosons, or fermions, or $j+1$ is a boson while $i-1$ is a fermion. Let us discuss the first case. Then

$$\sum_{k=j+1}^{i-1}(-)^{p(k)}q^{-\Gamma(k)} = q^{n-m+1}\sum_{k=j+1}^{i-1}q^{-2k}$$

After multiplication by $q - q^{-1}$ and expansion of the product, only the first and last terms remain to give

$$(q - q^{-1}) \sum_{k=j+1}^{i-1} (-)^{p(k)} q^{-\Gamma(k)} = q^{n-m+1}\left(q^{-2j-1} - q^{-2i+1}\right) = q^{-\Gamma(j)-1} - q^{-\Gamma(i)+1}$$

such that the left member of (103) is indeed zero. The second case is just symmetric. In the third case

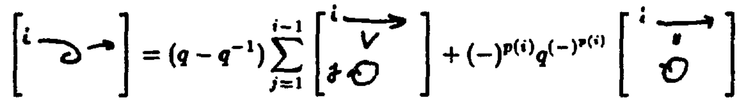
$$\sum_{k=j+1}^{i-1} (-)^{p(k)} q^{-\Gamma(k)} = q^{n-m+1} \sum_{k=j+1}^{n} q^{-2k} - q^{-3n-m-1} \sum_{k=n+1}^{i-1} q^{2k}$$

After multiplication by $(q - q^{-1})$ and expansion of the product

$$\begin{aligned}(q - q^{-1}) \sum_{k=j+1}^{i-1} (-)^{p(k)} q^{-\Gamma(k)} &= q^{n-m+1}\left(q^{-2j-1} - q^{-2n-1}\right) - q^{-3n-m-1}\left(q^{2n+1} - q^{2i-1}\right) \\ &= q^{-\Gamma(j)-1} - q^{-\Gamma(i)-1}\end{aligned}$$

such that again the left member of (103) vanishes. The other orientation case is left to the reader.

(iii)We have shown so far that $[K]$ is an invariant of regular isotopy. We must now consider the effect of type 1 moves. Consider



(105)

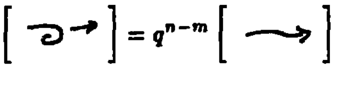
Suppose $i = 1, \ldots, n$ is bosonic. We get on the right hand side of (105) $\left[\, i \longrightarrow \right]$ times the numerical factor

$$q^{n-m+1}\left[(q - q^{-1}) \sum_{j=1}^{i-1} q^{-2j} + q^{-2i+1}\right] = q^{n-m}$$

independent of i. If $i = n+1, \ldots, n+m$ is fermionic, we get $\left[\, i \longrightarrow \right]$ times

$$q^{n-m+1}(q - q^{-1}) \sum_{j=1}^{n} q^{-2j} - q^{-3n-m-1}\left[(q - q^{-1}) \sum_{j=n+1}^{i-1} q^{2j} - q^{2i-1}\right] = q^{n-m}$$

independent of i again. Hence



(106)

One would find in a similar fashion

$\Big[$  $\Big] = q^{-(n-m)} \Big[$  $\Big]$ (107)

Hence if we denote by $w(k)$ the writhe, $q^{-(n-m)w(K)}[K]$ is invariant of ambient isotopy. For the unknot

$\Big[$  $\Big] = \sum_i q^{\Gamma(i)}(-)^{p(i)} = (n-m)_q$

$\Big[$  $\Big] = \sum_i q^{-\Gamma(i)}(-)^{p(i)} = (n-m)_q$ (108)

Defining $P_K = (q^{n-m})^{-w(K)} \frac{[K]}{[0]}$ we get a normalized invariant satisfying all desired relations//

The case $n < m$ can be worked out in a similar way. It can also be obtained from $n > m$ by exchanging bosons and fermions, and q and $-q^{-1}$.

It is interesting to compare the above state model with the one for $U_q sl(n+m)$. In the latter case, the same labels $i = 1, \ldots, n+m$ have weights $\gamma(i) = -(n+m+1-2i)$. At the higher symmetry point $q = i$, the $U_q sl(n+m)$ $\check{R}$ matrix and the $U_q sl(n,m)$ $\check{R}$ matrix coincide (up to irrelevant signs). One has moreover

$$\prod_C i^{\gamma(i_C) rot(C)} = (-)^m \sum{}^{rot(C)} \prod_C i^{\Gamma(i_C) rot(C)} (-)^{p(i_C)}$$

since for $i = 1, \ldots, n$, $\gamma(i) = \Gamma(i) - 2m$, for $i = n+1, \ldots, n+m-1$, $\gamma(i) = \Gamma(i) - 4n - 2m - 2 + 4i$, $\sum_C rot(C) = rot(K)$ independent of the state S. Hence, taking into account the writhe dependent prefactor and the normalization gives

$$P_K^{sl(n+m)}(q = i) = (-)^{m[rot(K)-1-w(K))} P_K^{sl(n,m)}(q = i) \tag{109}$$

or, using the relation $rot(K) - 1 - w(K) =$ number of components $- 1$ mod 2

$$P_K^{sl(n+m)}(q = i) = (-)^{m[\text{number of components}-1]} P_K^{sl(n,m)}(q = i) \tag{110}$$

5.2 Balanced case

We now consider the balanced case $n = m$ where $[K]$ is identically zero. As in [1] we define the invariant, which turns out to be the Alexander-Conway invariant, by turning to tangles.

Theorem 5.2: In the balanced case $n = m$ the state summation

$$[K] = \sum_S [K'|S] \prod_C q^{\Gamma(i_C) rot(C)} (-)^{p(i_C)}$$

where K' is a tangle with exterior strands carrying a fixed label that can be any of the $i = 1, \ldots, 2n$ equals the Alexander Conway invariant ∇_K

Proof: Invariance under moves of type 2, 3 follows at once from the above considerations. Invariance under type 1 holds because in the balanced case $n = m$ in (106,107). Finally $[K]$ satisfies the desired Skein relation and $\left[\bigcirc \right] = 1$//

In the simplest case $n = m = 1$ we find $\Gamma(1) = \Gamma(2)$ hence

$$\nabla_K = q^{rot(K)} \sum_S [K'|S] \prod_C (-)^{p(i_C)} \tag{112}$$

Introduce new labels $\sigma = \pm 1$ defined by $\sigma = 1$ for $i = 1$, $\sigma = -1$ for $i = 2$. Then $p(i) = \frac{1-\sigma}{2}$ and the sum (112) reads as well

$$\nabla_K = (iq^{-1})^{-rot(K)} \sum_S [K'|S] \prod_C i^{\sigma_C rot(C)} \tag{113}$$

in agreement with [9].

Example: Let us consider the trefoil

and $sl(2,2)$. Labels are $i = 1, 2, 3, 4$ with $\Gamma = 1, 3, 3, 1$. Suppose we assign label 1 to the outer strands in the state model. Then 2 states contribute to the summation

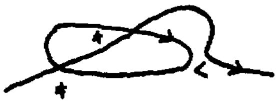
$$z(q^{-3} - q^{-3} - q^{-1}) = -1 + q^{-2}$$

$$q^3 q^{-1} = q^2$$

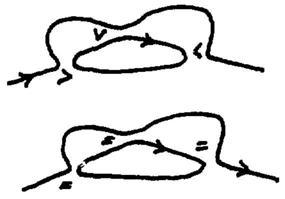
and indeed $\nabla = 1 + (q - q^{-1})^2$. If we assign the label 2 to the outer strands, 5 states contribute

$$zq^{-1}$$

$$z(-q^{-3} - q^{-1})$$



$$zq^{-1}$$



z^3q^{-1}

q^3q^{-3}

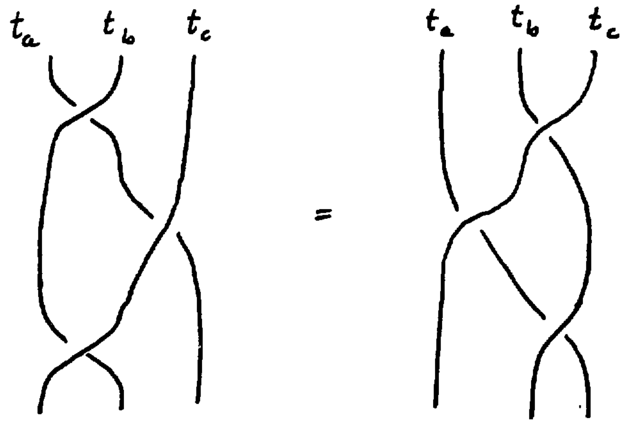
and $\nabla = 1 + z(q^{-1} - q^{-3}) + z^3q^{-1} = 1 + z^2$.

5.3 Multivariable Alexander Polynomial

In the case $n = m = 1$ the $U_qgl(1,1)$ algebra has all its irreducible representations of dimension 1 and 2. In particular the tensor product decomposes as $2 \otimes 2 = 2 \oplus 2$. It is therefore as easy to work with an $\check{R}$ matrix acting in the product of two different representations, that is carrying different colours. As explained in section 2 from the point of view of Fox differential calculus one has matrices

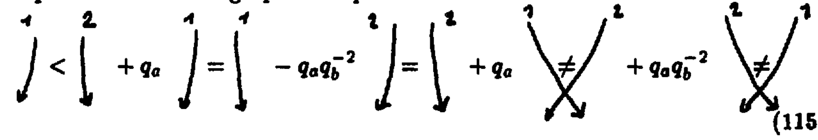
$$\check{R}^{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-t_a & t_b & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -t_b \end{pmatrix} \tag{114}$$

that satisfy the coloured Yang Baxter equation, which we represent as follows



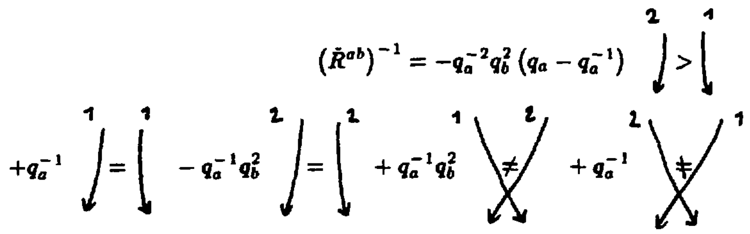
Setting as before $t = q^{-2}$ we find in the graphical representation

$$\check{R}^{ab} = (q_a - q_a^{-1})$$



(115)

while the inverse matrix reads



(116)

we have then

Theorem 5.3: Let

$$\Delta = \frac{1}{q_a - q_a^{-1}} \prod_C q_{a_C}^{rot(C)} \sum_S [K'|S] \prod_C (-)^{p(i_C)} \tag{117}$$

where C denotes the various components that can carry different colours a_C and the strand opened to define the tangle K' carries the color a. Then Δ coincides with the multivariable Alexander polynomial as in [21, 13].

Many state models can be devised to get this invariant. In fact it it easy to check that the matrices

$$\check{R}^{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - t_a & t_b^x & 0 \\ 0 & t_a^y & 0 & 0 \\ 0 & 0 & 0 & -t_b^x t_a^y \end{pmatrix} \tag{118}$$

satisfy the coloured Yang Baxter equation for any numbers x, y such that $x + y = 1$.

6 $U_q sl(n,m)$ Link invariants in solid handlebodies

As was observed in the last section, the $U_q sl(n,m)$ $\check{R}$ matrix does not provide new invariants for links in S^3. The additional degrees of freedom carried by the state model loops are not detected by the invariant , which is the same as the $U_q sl(N)$ one for any n, m such that $n - m = N \geq 2$.

The $U_q sl(n,m)$ models however split when generalized to links in solid handlebodies of genus $g \geq 1$ and give rise then to new invariants for any $n = N + m, m$. To start with we consider the solid torus, $g = 1$. We have

Theorem 6.1: Introduce formal variables a_i, a_i^{-1}, $i = 1, \ldots, n+m$ and the state summation

$$[K] = \sum_S [K|S] \prod_C q^{\Gamma(i_C) rot(C)} (-)^{p(i_C)} \prod_{C'} a_{i_{C'}}^{rot(C')} \tag{119}$$

where the circuits C' are non-contractible. Then

$$P_K = \left(q^{n-m}\right)^{-w(K)} \frac{[K]}{(n-m)_q} \in Z\left[q, q^{-1}, a_i, a_i^{-1}\right] \tag{120}$$

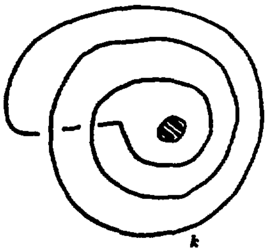
is:

(i)invariant of ambient isotopy
(ii)factorizes for products of links
(iii)satisfies the skein relation

$q^{n-m}P$  $- q^{-(n-m)}P$  $= (q - q^{-1})P$  (121)

(iiii)P  $= P$  $= 1$

(iiiii)for the loop winding k times counterclockwise (here is an example for $k = 3$)



one has

$$P = \frac{q^{-(n-m)(k-1)}}{(n-m)_q} \sum_* \left(q - q^{-1}\right)^{r-1} \prod_{j=1}^{k} \left[a_{i_j} q^{\Gamma(i_j)} q^{(-)^{p(i_j}}\right]^{t_j} (-)^{p(i_j)} q^{-(-)^{p(i_j)}} \tag{122}$$

where the star sum means the following conditions

$$\begin{gathered} 1 \leq r, t_1, \ldots, t_r \\ t_1 + \ldots + t_r = k \\ 1 \leq i_1 \ldots < i_k \leq m + n \end{gathered} \tag{123}$$

while for the loop winding clockwise the same results with $q \to q^{-1}$, $a_i \to a_i^{-1}$ holds. These are conversely a consistent axiomatic definition of P.

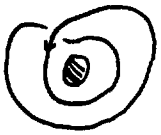
Examples:

For the single loop encircling the hole

one has

$$P = \frac{\sum_{i=1}^{n} a_i q^{\Gamma(i)} - \sum_{i=n+1}^{n+m} a_i q^{\Gamma(i)}}{(n-m)_q} \tag{124}$$

and for the loop winding twice

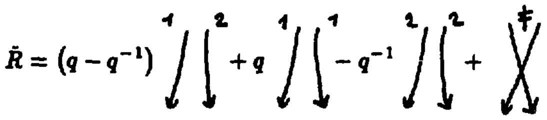


$$P = \frac{q^{-(n-m)}}{(n-m)_q} \left[(q-q^{-1}) \sum_{i<j} a_i q^{\Gamma(i)} (-)^{p(i)} a_j q^{\Gamma(j)(-)^{p(j)}} + \sum_i q^{(-)^{p(i)}} a_i^2 q^{2\Gamma(i)} \right] \tag{125}$$

The normalization is such that the S^3 invariant is obtained by letting the a's go to one. The same invariant has been obtained by Turaev [32] in the $U_q sl(N)$ case. His approach does not use the state model and is therefore a little more intricate.

Proof: The present state model is just a natural extension of the old one where some loops carry now a different weight. The proof of ambient isotopy and skein relation is thus immediate since the arguments used in proving th 5.1 depend on local loop states only, they carry on as well in this case. Factorisation follows from the very definition of the state model. Finally the k winding loop value is obtained by a direct calculation which we omit//

Of special interest is the balanced case $n = m$. The generalisation of the Alexander Conway invariant to solid handlebodies was indeed not obtained in [32] as the corresponding solution of the Yang Baxter equation was not yet known. We restrict for simplicity to $n = m = 1$ here, with the R matrix



(126)

then

Theorem 6.2: Set

$$\nabla = q^{rot(K)} \sum_S [K|S] \prod_C (-)^{p(i_C)} \prod_{C'} a_{i_{C'}}^{rot(C')} \tag{127}$$

which is a Laurent polynomial $\in Z\left[q, q^{-1}, a_1, a_1^{-1}, a_2, a_2^{-1}\right]$, then one has the following properties:

(i)∇ is invariant of ambient isotopy

(ii)factorizes for products of links

(iii)satisfies the skein relation

∇  $-\nabla$  $= \left(q - q^{-1}\right) \nabla$  (128)

(iiii)∇ vanishes for a contractible loop

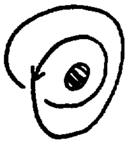
(iiii)For the loop winding k times counterclockwise

$$\nabla = q^k \left(a_1 - a_2\right) \sum_{i+j=k-1} \left(q a_1\right)^i \left(q^{-1} a_2\right)^j \tag{129}$$

while for the loop winding clockwise the same results holds with $q \rightarrow q^{-1}$,$a_i \rightarrow a_i^{-1}$. These are conversely a consistent axiomatic definition of ∇.

Examples:

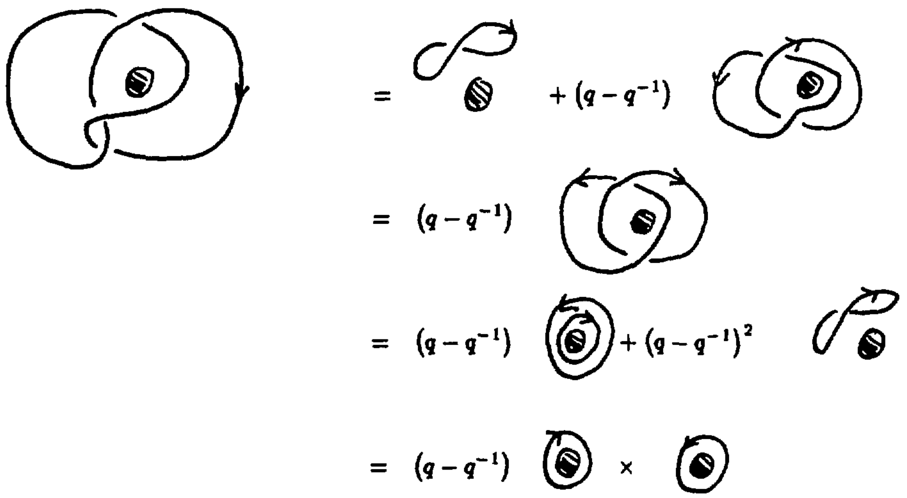
$$\nabla = q\left(a_1 - a_2\right) \tag{130}$$



$\nabla = q^2 \Big[$  $\left(q - q^{-1}\right) a_1 \left(-a_2\right) +$  $q a_1^2 - q^{-1} a_2^2 \Big]$

$= q^2 \left(a_1 - a_2\right) \left(q a_1 + q^{-1} a_2\right)$ (131)

and here is an example of a direct axiomatic calculation



$$= \left(q - q^{-1}\right)\left(2 - a_1 a_2^{-1} - a_1^{-1} a_2\right)$$

The proof follows the same lines as before, and we omit it. Notice that the definition of ∇ did not require the introduction of tangles. For a link that is contractible in the solid torus, ∇ would therefore vanish.

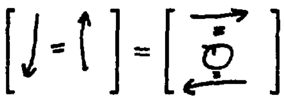
The method generalises to higher genus. It requires the introduction of variables a_i^α for $i = 1, \ldots, n+m$, $\alpha = 1, \ldots, g$, defining an invariant in $Z\left[q, q^{-1}, a_i^\alpha, (a_i^\alpha)^{-1}\right]$ by the summation

$$[K] = \sum_S [K|S] \prod_C q^{\Gamma(i_C) rot(C)} (-)^{p(i_C)} \prod_{C'} \prod_{\text{holes}(C')} \left(a_{i_{C'}}^{\alpha_{C'}}\right)^{rot(C')} \tag{132}$$

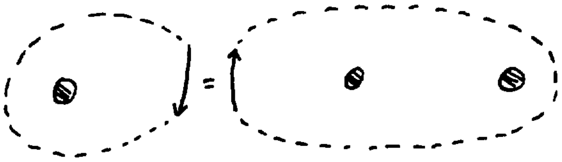
where holes (C') are the holes encircled by C' and

$$P_K = \left(q^{n-m}\right)^{-w(K)} \frac{[K]}{(n-m)_q} \tag{133}$$

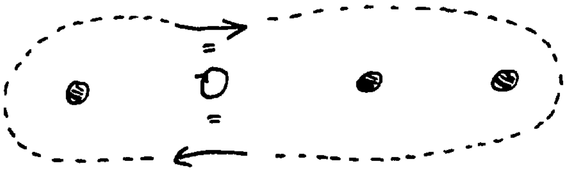
The proof of P_K invariance requires a little additional care if $g > 1$. This is because for the type $2B$ moves one requires


(134)

If both loops on the left encircle some holes as



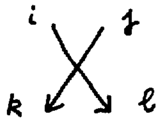
then on the right one gets



This requires the weight of non-contractible loops to be a product of independent contributions for each hole enclosed, as in the above formula.

7 Changes of basis and invariants of links in thickened surfaces

We have used several times in this paper what we called gauge-like changes of basis. Consider an $\check{R}$ matrix acting in $V^{\otimes 2}$ as giving the weights of some crossings in a link diagram (in the physics literature one would refer to Boltzmann weights of vertex models). We restrict to cases where there is charge conservation, that is, one can find labels for the basis states in V such that



$$\check{R}^{ij}_{kl} = 0 \text{ if } i + j \neq k + l \tag{135}$$

then one has

Prop 7.1: If $\check{R}$ satisfies the Yang Baxter equation so does the matrix $\check{R}'$ obtained by

$$\check{R}'^{ij}_{kl} = \alpha^{l+i-j-k} \check{R}^{ik}_{kl} \tag{136}$$

for any complex number α.

In the following we suppose moreover, as is the case for $U_q sl(N)$ or $U_q sl(n,m)$ $\check{R}$ matrices, that the labels are strongly preserved at each vertex. In that case the only vertices affected by the transformation are of the form

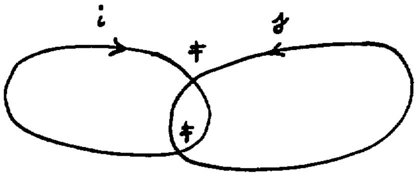
(137)

Example:

Starting with the $\check{R}$ matrix associated with the Alexander polynomial one gets the matrices

$$\check{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & \alpha^2 & 0 \\ 0 & \alpha^{-2} & 0 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix} \tag{138}$$

If there was a loop weight such that the state model deduced from the matrix $\check{R}$ as in Theorem 5.1 gives rise to a link invariant, then it is easy to check that replacing the $\check{R}$ weigths by $\check{R}'$ weights as in Prop. 7.1 gives also an invariant. Actually, for links in S^3 this invariant coincides with the old one. This is because when the labels are strongly preserved, the gauge transformation affects only vertices of the type (137) and on the plane where the link diagram is drawn loops cross an even number of times. Therefore the weights $\alpha^{2(i-j)}$ and $\alpha^{-2(i-j)}$ come always in pairs:

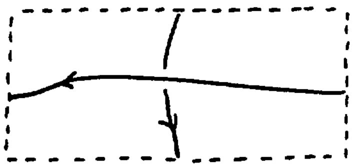


(139)

The interest of the gauge transformation becomes clear if one turns to links in thickened surfaces of the form $\Sigma \times [0,1]$ where Σ is an oriented two-dimensional surface. The link diagram is then drawn on Σ where loops in general can have an odd number of intersection points and therefore the additional weights α do not cancel out. The definition of the state model that gives rise to link invariants requires however a well defined concept of rotation number, which is possible if the the surface is parallelisable [33] eg. a torus or a closed surface of genus $g \geq 2$ with a puncture. We restrict here to some examples with the torus.

Examples:

Consider the link



(140)

and the state model for the $\check{R}$ matrix derived from the $U_q sl(2)$ one by the procedure of Prop. 7.1

$$\check{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & \alpha^2 & 0 \\ 0 & \alpha^{-2} & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \quad . \tag{141}$$

One has then, with the same definition as in Theorem 5.1

$$(2)_q P = 2q^{-1} + q^{-2}\left(\alpha^2 + \alpha^{-2}\right) \tag{142}$$

since all involved rotation numbers are zero. Similarly for the $\check{R}$ matrix deduced from the $U_q gl(1,1)$ one

$$\nabla = q - q^{-1} - \left(\alpha^2 + \alpha^{-2}\right) \tag{143}$$

The additional variable α seems to have a non trivial role and distinguishes links in $\Sigma \times [0,1]$, although its meaning is not as clear as for the a variables introduced in section 6. For further aspects of this invariant in the $U_qgl(1,1)$ case see [2]. Many interesting aspects have also been considered in [34]. The quantum group meaning of the $\check{R}$ matrices modified as in Prop. 7.1 has been addressed in [35].

8 Conclusion

It must be emphasized that the introduction of fermionic degrees of freedom is not a refinement but a natural and necessary step from the point of view of classical knot theory: the Alexander invariant corresponds to the balanced case $n = m = 1$. The peculiarities of $U_qgl(1,1)$ representation theory explain for instance the possibility of its multivariable extension. In the case $n = m > 1$ only the invariant built with the fundamental representation does coincide with the Alexander polynomial. Considering the case of higher representations opens the exciting possibility of including this polynomial in a hierarchy, whose topological meaning has still to be elucidated. The non-balanced case $n \neq m$ is especially interesting in solid handlebodies. Keeping $n - m$ fixed one can for instance capture more information on noncontractible links by increasing n without changing the value of the invariant for contractible links. To complete the picture one should now understand the role of supergroups in the quantum field theory approach [36] (this has been partly addressed in [30]) and study 3-manifold invariants.

References

[1] L.Kauffman, H.Saleur, "Free Fermions and the Alexander Invariant" , to appear in *Comm.Math.Phys.*

[2] F.Jaeger, L.Kauffman, H.Saleur, "The Alexander Invariant in S^3 and in Thickened Surfaces, a New Determinant Formulation", to appear

[3] M.Wadati, T.Deguchi, Y.Akutsu, *Physics Reports,* **180** (1989)

[4] Proceedings of the 13^{th} Johns Hopkins Workshop on "Knots, Topology and Quantum Field Theory", June 1989, L.Lusanna Editor, World Scientific

[5] V.Jones, *Ann.of Math.*, **126**, 335 (1987)

[6] V.Kac, *Comm.Math.Phys.*, **53**, 31 (1977)

[7] L.Kauffman, *Contemp.Math.*, **96** (1989)

[8] F.Jaeger, "Composition Products and Models for the Homfly Polynomial", preprint (1987)

[9] L.Kauffman, "Knots, Abstract Tensors and the Yang Baxter Equation", in ref. 4

[10] T.Deguchi, *J.Phys.Soc.Japan*, **58**, 3441 (1989)

[11] Y.Akutsu, T.Deguchi, *J.Phys.*, **A23**, 1861 (1990)

[12] H.C.Lee, "Invariants of Quantum Groups, Link Polynomials of $U_q(gl(N+1,C);L)$ and Holonomy in $SU(N+1-L,L)$ Chern Simons Theory", Chalk River preprint TP 90 1205 (1990)

[13] J.Murakami, "A State Model for the Multivariable Alexander Polynomial", and "On Local Relations to determine the Multivariable Alexander Polynomial of Coloured Links", preprints Osaka (1990)

[14] C.L.Schultz, *Phys.Rev.Lett.*, **46**, 629 (1981)

[15] J.Perk, C.L.Schultz, in "Non Integrable Systems - Classical and Quantum Theory", M.Jimbo, T.Miwa Editors (1981)

[16] R.Fenn, C.Rourke, Research Announcment (1991)

[17] L.Kauffman, "On Knots", *Ann.of Math.Study*, **15**, Princeton University Press (1987)

[18] D.Joyce, *J.Pure and App.Alg.*, **23**, 37 (1982)

[19] L.Kauffman, "Knots and Physics", World Scientific (1991)

[20] D.Rolfsen, "Knots and Links", Publish or Perish (1976)

[21] L.Kauffman, "Formal Knot Theory", Princeton University Press Math. Notes, **30** (1983)

[22] R.H.Crowell, R.H.Fox, "Introduction to Knot Theory", Blaisdell Pub. Co. (1963)

[23] H.Saleur, "Symmetries of the XX chain and Applications", in Trieste Conference on Recent Developments in Conformal Field Theories, October 89, S.Randjbar Daemi et al. Eds.,World Scientific

[24] W.Schmidke, S.Vokos, B.Zumino,*Z.Phys.*, **C48**, 249(1990)

[25] E.Corrigan, D.Fairlie, P.Fletcher, R.Sasaki, *J.Math.Phys*, **31**, 776 (1980)

[26] L.Liao,X.C.Song, "Quantum Lie Superalgebras and Non Standard Braid Group Representations", preprint AS-ITP-90-38

[27] N.Jing,M.L.Ge,Y.S.Wu, "New Quantum Group Associated with a Non Standard Braid Group Representation", preprint IASSNS-HEP-90/31

[28] Y.Cheng, M.L.Ge, K.Xue, "New Solutions of Yang Baxter Equation", preprint ITP-SB-90-38

[29] V.G.Drinfeld, "Quantum Groups", in Proceedings of the International Congress of Mathematicians, Berkeley (1986)

[30] L.Rozansky, H.Saleur, "Quantum Field Theory for The Multivariable Alexander Conway Polynomial", preprint YCTP P20 91

[31] P.Kulish, N.Reshetikhin, *Lett. in Math.Phys.*, **18**, 143 (1989)

[32] V.G.Turaev, "Skein Quantization of Poisson Algebras of Loops on Surfaces", LOMI preprint (1989)

[33] B.L.Reinhart, *Ann.Inst.Fourier*, **10**, 155 (1960)

[34] P.Cotta Ramusino, M.Rinaldi, "On the ALgebraic Structure of Links Diagrams on a Two Dimensional Surface", preprint HUTMP-90/B263 (1990)

[35] P.Kulish, private communication

[36] E.Witten, *Comm.Math.Phys.*, **121**, 351 (1989)