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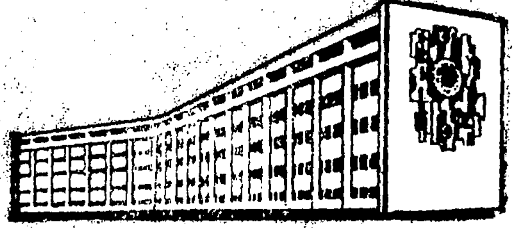
CLEBSCH-GORDAN AND RACAH COEFFICIENTS OF $SU_{pq}(2)$



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Коефіцієнти Клебша-Гордана і Рака для $SU_{pq}(2)$

Виведені явні вирази для коефіцієнтів Клебша-Гордана і коефіцієнтів Рака двох-параметричної квантової алгебри $SU_{pq}(2)$. Вони дані як скінченні суми та як скінченні базисні гіпергеометричні функції ${}_3\varphi_2$ і ${}_4\varphi_3$. Вказано як можуть бути отримані інші вирази для цих коефіцієнтів з допомогою базисних гіпергеометричних функцій.

I. I. Kachurik

Clebsch-Gordan and Racah coefficients of $SU_{pq}(2)$

Explicit expressions for the Clebsch-Gordan coefficients and for the Racah coefficients of the two-parametric quantum algebra $SU_{pq}(2)$ are derived. They are given as finite sums and as terminating basic hypergeometric functions ${}_3\varphi_2$ and ${}_4\varphi_3$. It is indicated how other expressions for these coefficients can be derived with the help of basic hypergeometric function.

Іван Іванович Качурик

Коефіцієнти Клебша-Гордана і Рака для $SU_{pq}(2)$

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1. INTRODUCTION

Quantum groups and algebras are of great significance for different branches of physics. Most important quantum group is $SU_q(2)$. The theory of representations of $SU_q(2)$ is well developed. The Wigner-Racah calculus of this quantum group is constructed. In recent years special interest has been paid to two-parametric quantum groups [1-3]. Most important one is $SU_{pq}(2)$. To apply representations of this quantum group we must develop the corresponding Wigner-Racah calculus. First of all, we must have Clebsch-Gordan coefficients and Racah coefficients of $SU_{pq}(2)$. The explicit analytical expression for Clebsch-Gordan coefficients of $SU_{pq}(2)$ was obtained in [4] by the method of projection operators. This expression could be applied for obtaining expressions for Racah coefficients of $SU_{pq}(2)$ by the methods used for one-parameter quantum group $SU_q(2)$ (see, for example, [5-7]). The simplest procedure of derivation of expressions for Racah coefficients of $SU_q(2)$ was indicated in the paper [8].It is based on an active using of basic hypergeometric functions. To apply this procedure to Racah coefficients of $SU_{pq}(2)$, we must have, in an explicit form, the expression for Clebsch-Gordan coefficients of $SU_{pq}(2)$ in another form than one given in [4]. Namely, we need the analogue of the Racah first form of Clebsch-Gordan coefficients. We cannot obtain this expression for Clebsch-Gordan coefficients from the expression in the paper [4]. Because of a mistake in formula (4.6) of this paper, the resulting expression for Clebsch-Gordan coefficients is not correct.

In this paper we derive a formula for Clebsch-Gordan coefficients (needed for derivation of expressions for Racah coefficients) by the method of highest weights. By making use of the method of the paper [8] we derive explicit expressions for Racah coefficients of $SU_{pq}(2)$. Clebsch-Gordan coefficients and Racah coefficients of $SU_{pq}(2)$ are expressed in terms of basic hypergeometric functions. They can also be expressed in terms of q-Hahn and q-Racah polynomials respectively.

2. COMMUTATION RELATIONS AND COMULTIPLICATION IN $SU_{pq}(2)$

The two-parametric quantum algebra $SU_{pq}(2)$ is generated by the elements E_+, E_-, H satisfying the relations

$$[H, E_\pm] = \pm E_\pm, \quad [E_+, E_-]_p = p^{-1}E_+E_- - pE_-E_+ = p^{-2H}\frac{q^H - q^{-H}}{q^{1/2} - q^{-1/2}}, \tag{1}$$

$$H^* = H, \quad E_\pm^* = E_\mp, \tag{2}$$

where p and q are independent real numbers. The case $p = 1$ reduces $SU_{pq}(2)$ to $SU_q(2)$.

Finite dimensional representations of $SU_{pq}(2)$ are given by integral or half-integral number l [4]. The generators E_+, E_-, H act upon the basis vectors $\mathbf{e}^l_m$ by the formulas

$$E_\pm \mathbf{e}^l_m = p^{l-m-1/2\mp 1/2}(\{l\mp m\}\{l\pm m+1\})^{1/2}\mathbf{e}^l_{m\pm 1}, \quad H\mathbf{e}^l_m = m\mathbf{e}^l_m, \tag{3}$$

where $\{x\}$ denotes the (p,q)-analogue of the number x:

$$\{x\} \equiv \{x\}_{pq} \equiv p^{1-x}[x], \quad [x] \equiv [x]_q \equiv (q^{x/2} - q^{-x/2})/(q^{1/2} - q^{-1/2}). \tag{4}$$

By replacement $q \to pq$, $p \to (q^{-1}p)^{1/2}$ the number $\{x\}$ transforms into the expression $[[x]]_{pq} = (q^x - p^{-x})/(q - p)$ used in [4]. It is easy to verify that

$$(E_\pm)^n e^l_m = p^{(nl - nm \mp n(n\pm 1))/2} \left(\frac{\{l \mp m\}!\{l \pm m + n\}!}{\{l \pm m\}!\{l \mp m - n\}!} \right)^{1/2} e^l_{m\pm n}. \tag{5}$$

Comultiplication in $SU_{pq}(2)$ is given by the formulas

$$\Delta(E_\pm) = E_\pm \otimes (q^{1/2}p^{-1})^H + (q^{1/2}p)^{-H} \otimes E_\pm, \quad \Delta(H) = H \otimes 1 + 1 \otimes H. \tag{6}$$

3. CLEBSCH-GORDAN COEFFICIENTS OF $SU_{pq}(2)$

Clebsch-Gordan coefficients (CGC's) of the tensor product of the irreducible representations T^{l_1} and T^{l_2} of $SU_{pq}(2)$ are defined by the formula

$$e^l_m = \sum_{j,k} C^{l_1 l_2 l}_{jkm}\, e^{l_1}_j \otimes e^{l_2}_k, \tag{7}$$

where $\{e^{l_1}_j\}$, $\{e^{l_2}_k\}$ and $\{e^l_m\}$ are bases of the carrier spaces for the representations T^{l_1}, T^{l_2} and T^l in which the operators E_+, E_-, H act by formulas (3). The orthogonality relations for CGC's are

$$\sum_{j,k} C^{l_1 l_2 l}_{jkm} (C^{l_1 l_2 l'}_{jkm'})^* = \delta_{ll'}\delta_{mm'}, \tag{8}$$

$$\sum_{l,m} C^{l_1 l_2 l}_{jkm} (C^{l_1 l_2 l}_{j'k'm})^* = \delta_{jj'}\delta_{kk'}, \tag{9}$$

where $j + k = j' + k' = m$.

To evaluate CGC's of $SU_{pq}(2)$ we use the method of highest weights. Acting by the operator $T^l(E_+) \equiv E^l_+$ onto both sides of the formula

$$e^l_l = \sum_{j+k=\mathrm{const}} C^{l_1 l_2 l}_{jkl}\, e^{l_1}_j \otimes e^{l_2}_k \tag{10}$$

and taking into account that $E^l_+ e^l_l = 0$, we obtain the recurrence relation

$$q^{(l+1)/2} p^{l_1 - l_2} (\{l_1 + j\}\{l_1 - j + 1\})^{1/2} C^{l_1 l_2 l}_{j-1, l-j+1, l} + (\{l_2 + l - j + 1\}\{l_2 - l + j\})^{1/2} C^{l_1 l_2 l}_{j, l-j, l} = 0. \tag{11}$$

The iteration of (11) yields

$$C^{l_1 l_2 l}_{j, l-j, l} = (-1)^{l_1 - j} q^{j(l+1)/2} p^{j(l_1 - l_2)} \left(\frac{\{l_1 + j\}!\{l_2 + l - j\}!}{\{l_1 - j\}!\{l_2 - l + j\}!} \right)^{1/2} A, \tag{12}$$

where A does not depend on j. The relation (8) gives

$$A^{-2} = \sum_j q^{j(l+1)} p^{2j(l_1-l_2)} \frac{\{l_1+j\}!\{l_2+l-j\}!}{\{l_1-j\}!\{l_2-l+j\}!}.$$

Using the q-Vandermonde sum of the basic hypergeometric series ${}_2\varphi_1$ [10] we have for A the expression

$$A = q^{-l_1(l_1+1)/4+l_2(l_2+1)/4-l(l+1)/4} p^{l(l+1)/2-l_1(l_1+1)/2-l_2(l_2+1)/2-l_1(l-l_2)}$$

$$\times(\{2l+1\}!\{l_1+l_2-l\}!/(\{l_1-l_2+l\}!\{l_2-l_1+l\}!\{l_1+l_2+l+1\}!))^{1/2},$$

which is determined up to phase factor α, $|\alpha| = 1$. Thus,

$$C^{l_1 l_2 l}_{jkl} = (-1)^{l_1-j} q^{l_2(l_2+1)/4-l_1(l_1+1)/4-l(l+1)/4+j(l+1)/2}$$

$$\times p^{l(l+1)/2-l_1(l_1+1)/2-l_2(l_2+1)/2-l_1(l-l_2)+j(l_1-l_2)}$$

$$\times \frac{\Delta(l_1 l_2 l)}{\{l_1-l_2+l\}!\{l_2-l_1+l\}!}\left(\frac{\{l_1+j\}!\{l_2+k\}!\{2l+1\}!}{\{l_1-j\}!\{l_2-k\}!}\right)^{1/2}, \qquad (13)$$

where

$$\Delta(l_1 l_2 l) = (\{l_1-l_2+l\}!\{l_2-l_1+l\}!\{l_1+l_2-l\}!/\{l_1+l_2+l+1\}!)^{1/2}.$$

Since

$$\mathbf{e}^l_m = p^{-(l-m)(l-m-1)/2}(\{l+m\}!/\{2l\}!\{l-m\}!)^{1/2}(E^l_-)^{l-m}\mathbf{e}^l_l$$

then

$$C^{l_1 l_2 l}_{jkm} \equiv ((\mathbf{e}^{l_1}_j \otimes \mathbf{e}^{l_2}_k), \mathbf{e}^l_m)$$

$$= p^{-(l-m)(l-m-1)/2}(\{l+m\}!/\{2l\}!\{l-m\}!)^{1/2}((\mathbf{e}^{l_1}_j \otimes \mathbf{e}^{l_2}_k), (E^l_-)^{l-m}\mathbf{e}^l_l)$$

$$= p^{-(l-m)(l-m-1)/2}(\{l+m\}!/\{2l\}!\{l-m\}!)^{1/2}(\mathbf{e}^l_l, (E^{\otimes}_+)^{l-m}(\mathbf{e}^{l_1}_j \otimes \mathbf{e}^{l_2}_k)), \qquad (14)$$

where according to (6)

$$E^{\otimes}_+ = E^{l_1}_+ \otimes (q^{1/2}p^{-1})^{H^{l_2}} + (q^{1/2}p)^{-H^{l_1}} \otimes E^{l_2}_+.$$

By mathematical induction the formula

$$(E^{\otimes}_+)^n = \sum_{r=0}^{n} \frac{\{n\}! q^{-r(n-r)/2} p^{r(n-r)}}{\{r\}!\{n-r\}!} (E^{l_1}_+ \otimes (q^{1/2}p^{-1})^{H^{l_2}})^r ((q^{1/2}p)^{-H^{l_1}} \otimes E^{l_2}_+)^{n-r} \qquad (15)$$

can be proved. With the help of (5) and (13) we finally obtain from (14) explicit analytic expressions for the $SU_{pq}(2)$ CGC's in the Racah first form:

$$C^{l_1 l_2 l}_{jkm}$$

$$= \delta_{m,j+k} \frac{(-1)^{l_1-j}\omega_{pq}\Delta(l_1 l_2 l)}{\{l_1-l_2+l\}!\{l_2-l_1+l\}!} \left(\frac{\{l_1-j\}!\{l_2-k\}!\{l-m\}!\{l+m\}!\{2l+1\}}{\{l_1+j\}!\{l_2+k\}!} \right)^{1/2}$$

$$\times \sum_{r=0}^{l-m} \frac{(-1)^r q^{r(l+m+1)/2} p^{-r(2l_2-2l_1-l+m+r)} \{l_1+j+r\}!\{l_2+l-j-r\}!}{\{r\}!\{l_1-j-r\}!\{l-m-r\}!\{l_2-l+j+r\}!}, \qquad (16)$$

where

$$\omega_{pq} = q^{l_2(l_2+1)/4 - l_1(l_1+1)/4 - l(l+1)/4 + j(m+1)/2}$$
$$\times p^{l(l+1)/2 - l_1(l_1+1)/2 - l_2(l_2+1)/2 + (l_2-l)(l_1+l-m) + j(l_1-l_2)}.$$

We see that the $SU_{pq}(2)$ CGC's depend on q in the same manner as in the case of the $SU_q(2)$ CGC's up to the substitution of (p,q)-factorials $\{\cdots\}!$ by q-factorials $[\cdots]!$ (see (41a) in [9]). They are expressed in terms of the same basic hypergeometric functions ${}_3\varphi_2$ (parameters and argument q of ${}_3\varphi_2$ are the same too; see (41b) in [9]):

$$C^{l_1 l_2 l}_{jkm} = \delta_{j+k,m}(-1)^{l_1-j}\omega_{pq} \frac{\Delta(l_1 l_2 l)\{l_2+l-j\}!}{\{l_1-l_2+l\}!\{l_2-l_1+l\}!\{l_2-l+j\}!}$$

$$\times \left(\frac{\{l_1+j\}!\{l_2-k\}!\{l+m\}!\{2l+1\}}{\{l_1-j\}!\{l_2+k\}!\{l-m\}!} \right)^{1/2} {}_3\varphi_2 \left(\begin{matrix} q^{m-l}, q^{j-l_1}, q^{l_1+j+1} \\ q^{l_2-l+j+1}, q^{j-l-l_2} \end{matrix} ;\ q, q \right). \qquad (17)$$

Taking into account this fact we can apply transformation formulas for ${}_3\varphi_2$ and find other expressions for CGC's (see also [11]). In such a way we obtained all (p,q)-analogues of main formulas from the theory of angular mumentum. These formulas will be given in other paper.

Explicit expressions for CGC's allow to investigate symmetry properties of CGC's. It was shown that CGC's of $SU_{pq}(2)$ satisfy the relations

$$C^{l_1 l_2 l}_{jkm} = C^{l_2 l_1 l}_{-k,-j,-m}$$

$$= (-1)^{l-l_1-k} q^{-k/2} p^{l-l_1} (\{2l+1\}/\{2l_1+1\})^{1/2} C^{l l_2 l_1}_{m,-k,j}$$

$$= C^{(l_1+l_2+j+k)/2,(l_1+l_2-j-k)/2,l}_{(l_1-l_2+j-k)/2,(l_1-l_2-j+k)/2,l_1-l_2}, \qquad (18)$$

$$C^{l_1 l_2 l}_{jkm}(p,q) = (-1)^{l_1+l_2-l} C^{l_1 l_2 l}_{-j,-k,-m}(p, q^{-1}).$$

These four relations (as in the case of Lie algebra su(2)) generate the symmetry group containing 72 elements. But now the last relation change q by q^{-1}.

The formulas representing (p,q)-CGC's admit extensions to the cases of negative integers of half-integers l_1, l_2, l, and the following "reflection symmetries" associated with parameter transformations of the type $l \to \bar{l} \equiv -l-1$ are fulfilled:

$$C^{l_1 l_2 l}_{jkm} = (-1)^{l_2-l-j} C^{\bar{l}_1 l_2 l}_{jkm} = (-1)^{l_1-j} C^{l_1 \bar{l}_2 \bar{l}}_{jkm}$$

$$(-1)^{l_2+k} C^{\bar{l}_1 l_2 \bar{l}}_{jkm} (-1)^{l_1+l_2-l} C^{\bar{l}_1 \bar{l}_2 \bar{l}}_{jkm}.$$

If $l = l_1 + s$, then connection between $C^{l_1 l_2 l}_{jkn}$ and $C^{\bar{l}_1 l_2 \bar{l}}_{jkn}$ gives

$$C^{l_1 l_2 l_1+s}_{jkn} = C^{\bar{l}_1 l_2 \bar{l}_1 - s}_{jkn},$$

it means that when l_2 is given, then it is sufficient to know CGC's for $s < 0$. CGC's for $s > 0$ are received by substitution $l_1 \to \bar{l}_1 = -l_1 - 1$.

Derivation of properties (19) uses expressions for CGC's and the relation

$$\frac{\{-n\}!}{\{-m\}!} = \frac{(-1)^{m-1}}{(-1)^{n-1}} \frac{\{m-1\}!}{\{n-1\}!} p^{(m-n)(m+n-1)},$$

which follows directly from the formulas (see [9])

$$[-n]!/[-m]! = (-1)^{m-1}[m-1]!/(-1)^{n-1}[n-1]!, \quad \{n\}! = p^{-n(n-1)}[n]!$$

Combining (18) and (19) we can find other symmetry relations for the CGC's with negative values of the parameters l_1, l_2, l. But now we must constantly replace $\bar{l}_1, \bar{l}_2, \bar{l}$ by $-l_1, -l_2, -l$ respectively in the phase factors of (18).

4. RACAH COEFFICIENTS OF $SU_{pq}(2)$

The Racah coefficients (RC's) $R(l_1 l_2 l_3, l_{12} l_{23}, l)$ of $SU_{pq}(2)$ are defined exactly in the same way as in the case of $SU_q(2)$. As for the quantum algebra $SU_q(2)$, we have

$$R(l_1 l_2 l_3, l_{12} l_{23}, l) = \sum_{i=-l_1}^{l_1} \sum_{j=-l_2}^{l_2} \sum_{k=-l_3}^{l_3} C^{l_1 l_2 l_{12}}_{ijm} C^{l_{12} l_3 l}_{mks} C^{l_2 l_3 l_{23}}_{jkn} C^{l_1 l_{23} l}_{ins}, \tag{20}$$

where $m = i + j$, $s = m + k = i + n$, $n = j + k$. As elements of a unitary matrix, $R(l_1 l_2 l_3, l_{12} l_{23}, l)$ satisfy the orthogonality relations

$$\sum_{l_{23}} R(l_1 l_2 l_3, l_{12} l_{23}, l) R(l_1 l_2 l_3, l'_{12} l_{23}, l) = \delta_{l_{12} l'_{12}}, \tag{21}$$

$$\sum_{l_{12}} R(l_1 l_2 l_3, l_{12} l_{23}, l) R(l_1 l_2 l_3, l_{12} l'_{23}, l) = \delta_{l_{23} l'_{23}}. \tag{22}$$

Since $C^{l_1 l_2 l_3}_{ijk} = 0$ if the triple $(l_1 l_2 l_3)$ does not satisfy the triangular condition $|l_1 - l_2| \leq l \leq l_1 + l_2$, then $R(l_1 l_2 l_3, l_{12} l_{23}, l) = 0$ if one of the triples $(l_1 l_2 l_{12})$, $(l_{12} l_3 l)$, $(l_2 l_3 l_{23})$ and $(l_1 l_{23} l)$ does not satisfy this condition.

$6j$-symbol for $SU_{pq}(2)$ is defined by the formula

$$\begin{Bmatrix} l_1 & l_2 & l_{12} \\ l_3 & l & l_{23} \end{Bmatrix}_{pq} = (-1)^{l_1+l_2+l_3+l}(\{2l_{12}+1\}\{2l_{23}+1\})^{-1/2} R(l_1 l_2 l_3, l_{12} l_{23}, l). \tag{23}$$

By some manipulations with formula (20), applying orthogonality relations for CGC'a, we obtain

$$R(l_1 l_2 l_3, l_{12} l_{23}, l) C^{l_{12} l_3 l}_{i+j,k,i+j+k}$$

$$= \sum_{i=-l_1}^{l_1} \sum_{j=-l_2}^{l_2} C^{l_1 l_2 l_{12}}_{i,j,i+j} C^{l_2 l_3 l_{23}}_{j,k,j+k} C^{l_1 l_{23} l}_{i,j+k,i+j+k}, \qquad (24)$$

where the summation is over values of i and j for which $i+j=$const. We now can obtain the expression for RC's in explicit form. Putting $i+j=i_{12}$, $k=l-l_{12}$ into (24) we have

$$R(l_1 l_2 l_3, l_{12} l_{23}, l)$$

$$= (C^{l_{12} l_3 l}_{i_{12}, l-i_{12}, l})^{-1} \sum_i C^{l_1 l_2 l_{12}}_{i, i_{12}-i, i_{12}} C^{l_2 l_3 l_{23}}_{i_{12}-i, l-i_{12}, l-i} C^{l_1 l_{23} l}_{i, l-i, l} \qquad (25)$$

Next we pass to $6j$-symbol

$$\left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\}_{pq} \equiv \left\{ \begin{matrix} l_1 & l_2 & l_{12} \\ l_3 & l & l_{23} \end{matrix} \right\}_{pq}$$

and use formulas (13) and (16). As a result we obtain

$$\left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\}_{pq} = q^{-a(a+1)/2} p^{c(c+1)-a(a+1)-b(b+1)-a(c-b+e-f)+c(b-c+e-d)+(d-f)(b+f-e)}$$

$$\times \frac{\Delta(abc)\Delta(cde)\Delta(aef)\Delta(bdf)\{c+d+e+1\}!}{\{a-b+c\}!\{b-a+c\}!\{a-f+e\}!\{f-a+e\}!\{b-d+f\}!\{d-b+f\}!\{d-c+e\}!}$$

$$\times \sum_{i,r} (-1)^{a+c+d+e-i+r} q^{r(f+e-i+1)+i(i+1)/2} p^{2i(a-b+d-f)-r(2d-2b-f+e-i+r)}$$

$$\times \frac{\{a+i\}!\{b+c-i+r\}!\{f+e-i\}!\{d+f-c+i-r\}!}{\{r\}!\{a-i\}!\{b-c+i-r\}!\{f-e+i-r\}!\{d-f+c-i+r\}!}. \qquad (26)$$

Let us change here the summations over i and r by the summations over $k=i-r$ and r. The summation over r with the help of q-Vandermonde sum [10] gives

$$(pq^{1/2})^{a(a+1)-k(k+1)} \frac{\{a+k\}!\{e-a+f\}!\{a+e+f+1\}!}{\{a-k\}!\{e+f+k+1\}!}.$$

By using this result, after some simple transformations in (26) we receive the following formula for $(p,q)-6j$-coefficients:

$$\left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\}_{pq} = (-1)^{a+b+d+e} p^{(a-c+d-f)(e-b+f-c)}$$

$$\times \frac{\Delta(abc)\Delta(cde)\Delta(aef)\Delta(bdf)\{a+e+f+1\}!\{c+d+e+1\}!}{\{a-b+c\}!\{b-a+c\}!\{a-f+e\}!\{b-d+f\}!\{d-b+f\}!\{d-c+e\}!}$$

$$\times \sum_r \frac{(-1)^r p^{2r(a-c+d-f)-r(r+1)} \{2b-r\}!\{a-b+c+r\}!\{d-b+f+r\}!}{\{r\}!\{a+b-c-r\}!\{b+d-f-r\}!\{c-b+f-e+r\}!\{c-b+f+e+r+1\}!} \tag{27}$$

We see that these coefficients has no q-factors before the sum (as in the case of the quantum group $SU_q(2)$).

The sum in (27) can be expressed in terms of the basic hypergeometric function ${}_4\varphi_3$ of argument q. We have

$$\left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\}_{pq} = (-1)^{a+b+d+e} p^{(a-c+d-f)(c-b+f-e)}$$

$$\times \frac{\Delta(abc)\Delta(cde)\Delta(aef)\Delta(bdf)\{2b\}!\{a+e+f+1\}!\{c+d+e+1\}!\{c-b+e+f+1\}!^{-1}}{\{a+b-c\}!\{b-a+c\}!\{a-f+e\}!\{b-d+f\}!\{b+d-f\}!\{d-c+e\}!\{c-b+f-e\}!}$$

$$\times {}_4\varphi_3 \left(\begin{matrix} q^{c-a-b}, q^{a-b+c+1}, q^{d-b+f+1}, q^{f-b-d} \\ q^{-2b}, q^{c-b+e+f+2}, q^{c-b-e+f+1} \end{matrix} ; q, q \right). \tag{28}$$

We can apply the transformation formulas for ${}_4\varphi_3$ [10] and find other expressions for RC's. In this way we have obtained (p,q)-analogues of all well known expressions for RC's of the Lie algebra su(2). These expressions will be given in another paper.

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