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DIFFERENTIAL EQUATION METHOD.
THE CALCULATION OF N-POINT DIAGRAMS

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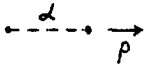
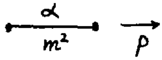
1993

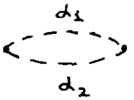
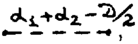
Here we present a method which allows one to obtain the results for the massive diagrams without calculating the complicated D-space (in case of the dimensional regularization) Feynman integrals using the rule of integration by parts [1, 2, 3] . Throughout the article we use the following notation. The use of dimensional regularization is assumed. All the calculations are performed in momentum space of dimension $D = 4 - 2\epsilon$. The dotted and solid lines of a diagram correspond to the massless and massive (for simplicity euclidean) propagators, respectively,

$$\frac{1}{[p^2]^\alpha} = $$  , $$\frac{1}{[p^2+m^2]^\alpha} = $$ 

α and m are called the index and mass of this line, respectively. Further (except specially described places) all the solid lines contain the same mass m . Index of a line is equal to 1 and masses are not marked.

Let us consider the rules of the Feynman diagrams calculation.

Rule 1. Massless loops are integrated due to the graphical identity

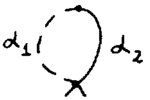
 $= A(\alpha_1, \alpha_2)$  ,

where

$$A(\alpha_1, \alpha_2) = \pi^{D/2}\frac{a(\alpha_1)a(\alpha_2)}{a(\alpha_1+\alpha_2-D/2)}, \quad a(\alpha) = \frac{\Gamma(D/2-\alpha)}{\Gamma(\alpha)},$$

Γ is the Euler Γ – function

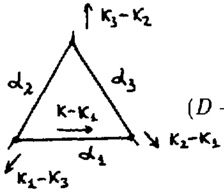
Rule 2. Massive tadpoles are integrated due to the graphical identity

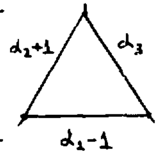
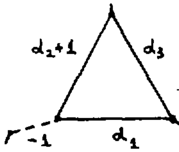
 $$= B(\alpha_1, \alpha_2)\frac{1}{(m^2)^{\alpha_1+\alpha_2-D/2}},$$

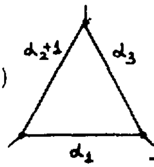
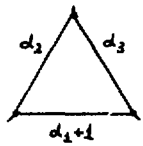
where

$$B(\alpha_1, \alpha_2) = \pi^{D/2}\frac{\Gamma(D/2-\alpha_2)\Gamma(\alpha_1+\alpha_2-D/2)}{\Gamma(\alpha)\Gamma(D/2-1)}$$

Rule 3. For a triangle the following recurrent relation is valid (here the line with index α_i also contains the mass m_i)



$(D - 2\alpha_1 - \alpha_2 - \alpha_3) = \alpha_2 \Big[$  $-$  $-$

$(m_1^2 + m_2^2)$  $\Big] + (\alpha_2 \leftrightarrow \alpha_3) - 2m_1^2\alpha_1$ 

We should stress the fact that the lower line of the triangle is distinguished and two other lines are similar. The lower line is called "distinguished line".

The rule 3, i.e. the rule of integration by parts, can be obtained multiplying the subintegral magnitude by number $D = \frac{d}{dq_\mu}(q-p)^\mu$ and using the relationship $\int d^Dq$ div ( ) $= 0$ for the regularized Feynman integrals.

The key idea of this method: using the rules 1-3 the results for different massive Feynman dyagrams are obtained without the complicated D-space integrals calculation . Notice that all the information about the D-space Feynman integrals structure, is already contained in the rules 1 and 2 for the massless loop and massive tadpole, respectively.

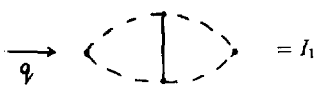
Consider a few specific examples.

A. Propagator-type diagrams. (see also [4])

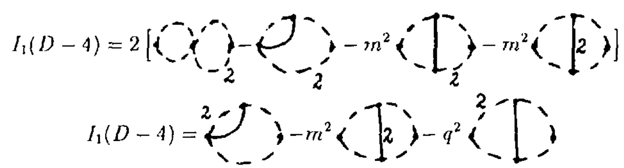
We divide all the diagrams into three types:

- the master diagrams which can not be obtained as loops and chains combination,
- the diagrams which are the combination of loops and chains,
- the diagrams which contain only loops.

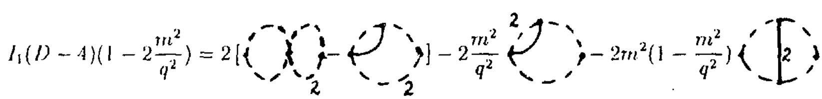
1. As the master diagram example we consider



Applying sequentially the rule 3 to the right triangle with the vertical and sidelong distinguished lines, we get



Combining these two eq.s, we obtain



The last diagram in the right hand side (r.h.s.) is the differential with respect to m^2 of the initial diagram I_1. Hence, the r.h.s. has the form

$$2f_1 + 2m^2(1 - m^2/q^2)\frac{d}{dm^2}I_1,$$

where the function f_1 does not contain master diagrams.

Thus,we get the differential equation with respect to m^2 (DE) for the initial diagram. The result for I_1 is

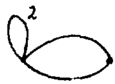
$$I_1(t) = \left[\frac{t^2}{(1-t)(1-2t)}\right]^{\epsilon}\left(C_1 + \int_0^t \frac{d\psi f_1(\psi)}{\psi^{2\epsilon}(1-\psi)^{1-\epsilon}(1-2\psi)^{1-\epsilon}}\right), \quad (t = \frac{q^2}{q^2+m^2})$$

The constant C_1 can be obtained by comparison with massless limit $t = 1$.

2. As the example of the second type of diagrams we consider the diagram

 $= I_2$

Applying rule 3 to the loop with the massless distinguished line, we have

$I_2(D-3) =$ 

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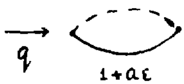
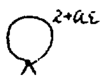
3. The examples of the third type of diagrams.

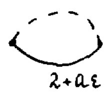
a). A simple loop with one massive line.



$= I_3$

Applying rule 3 with the massless distinguished line, we get

$I_3(1-(a+2)\epsilon) = (1+a\epsilon)\Big[$ 

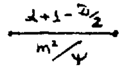
$-(q^2+m^2)$ 

$\Big]$

By analogy with subsect.1 we have

$$(q^2)^{(a+1)\epsilon} I_3(t) = (1/t)^{1-(a+2)\epsilon}\int_0^t \frac{d\psi}{\psi^{\epsilon}(1-\psi)^{(a+1)\epsilon}}(1+a\epsilon)B(2+a\epsilon,0)$$

Notice that sometimes this is a more convenient form of the diagram I_3

$= \int_0^1 \frac{d\psi}{\psi^{\alpha+1-D/2}(1-\psi)^{\epsilon}}\,\alpha B(\alpha+1,0)$ 

(1)

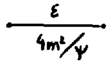
It is clear to see that rule 3 allows us to reduce L-loop diagram to the $(L-1)$-loop diagram with one propagator which contains the integration parameter in its mass.

b). A simple loop with two massive lines.

 $= I_4$

By analogy with the previous subsection we get

$$I_4 = (4^\epsilon/2)B(2,0)\int_0^1 \frac{d\psi}{\psi^\epsilon(1-\psi)^{1/2}} \quad (2)$$



c). The two-loop diagram

$= I_5,$

which appears in the r.h.s. of the DE for many complicated diagrams.

Differentiating eq.(2) with respect to the mass and applying this result along with the eq.(1) to the diagram I_5, we get

$$I_5 = 4^\epsilon \Gamma(1+\epsilon)\int_0^1 \frac{d\psi}{\psi^{1+\epsilon}(1-\psi)^{1/2}} =$$

$$4^\epsilon\Gamma(2\epsilon)\int_0^1 \frac{d\psi}{\psi^{1+\epsilon}(1-\psi)^{1/2}}\int_0^1 \frac{d\sigma\sigma^{-\epsilon}}{[q^2(1-\psi)\sigma+4m^2]^{2\epsilon}}$$

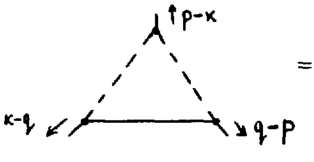
After the integration we have

$$I_5 = \frac{\Gamma^2(1+\epsilon)}{(1-\epsilon)(m^2)^{2\epsilon}}\Big[\frac{1}{2\epsilon^2} - 1 - p\ln(\frac{1+\sqrt{p}}{1-\sqrt{p}}) - \frac{1+p}{4p}(\ln(\frac{1+\sqrt{p}}{1-\sqrt{p}}))^2 + O(\epsilon)\Big],\ (p = \frac{q^2}{q^2+4m^2})$$

Thus, the usage of the integration by parts rule allows one to obtain the DE for the massive diagrams with r.h.s. containing more simple diagrams.

B. Vertex-type diagrams. (*see also* [5])

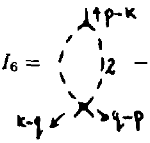
4. Consider one-loop vertex-type diagram with one massive line.

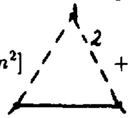


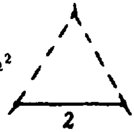
$= I_6$

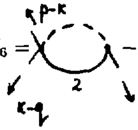
Applying rule 3 with massive and one massless distinguished lines, we have

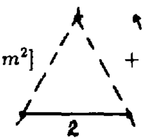
$$(D-4)I_6 = \quad -[(p-q)^2+m^2] \quad + (p \leftrightarrow k) - 2m^2 \quad (3)$$

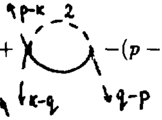






$(D-4)I_6 =$ 

$-[(p-q)^2+m^2]$ 

$+$ 

$-(p-k)^2$ 

(4)

Combining three eq.s (3), (4) and (4) with substitution ($p \leftrightarrow k$) and putting $q=0$ without the lack of generality, we get

$$(D-4)I_6\psi_6 = \Phi_6 - 2\phi_6 \frac{d}{dm^2} I_6$$

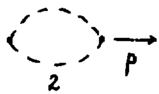
Here

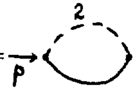
$$\psi_6 = 1-(k^2+p^2+2m^2)/(k-p)^2,\ \phi_6 = m^2 + (k^2+m^2)(p^2+m^2)/(k-p)^2$$

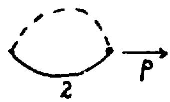
and

$$\Phi_6 = f_2(p-k) - [(k^2+m^2)f_1(k) + (p^2+m^2)f_1(p)]/(p-k)^2,$$

where

$f_2(p) = 2$ 

$, f_1(p) =$ 

$+$ 

Defining the variable $x = m^2/(p-k)^2$ for the initial diagram in the symmetrical point $k^2 = p^2 = (k-p)^2 = \mu^2$, we get

$$\phi_3(x) = (1+x+x^2)\mu^2$$

$$I_6(x) - I_6(0) = [J_1(x,0) - 2J_1(x,1)],$$

where

$$J_1(x,a) = \int_0^x \frac{d\psi}{1+\psi+\psi^2} \ln(\psi+a)$$

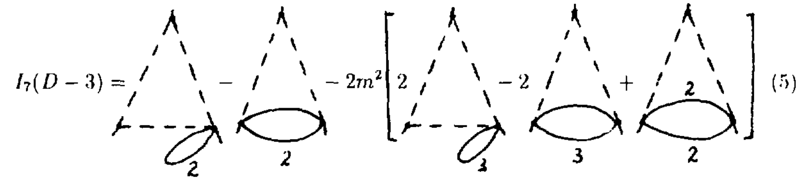
The value of diagram $I_6(0)$ is found in papers [6, 5].

5. Here we consider the diagram

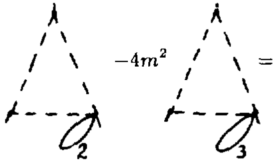

$= I_7,$

which is contained in the ghost-gluon- ghost vertex in QCD.

Applying rule 3 to diagram I_7 with massive distinguished lines we get



The first and third diagrams in r.h.s. of eq.(5) can be expressed by rule 2 in the form



$$= \frac{\Gamma(1+\epsilon)}{(m^2)^\epsilon}(1/\epsilon + 2)I_6(0)$$

These diagrams contain ultraviolet infinity of the initial diagram. All the other diagrams in r.h.s. of eq.(5) are ultraviolet-finite. Using eq.(2) we get that three other diagrams from r.h.s. of eq.(5) give the following contribution:

$$-\int_0^1 \frac{d\tau}{\tau}\sqrt{1-\tau}\, I_6(4x/\tau)$$

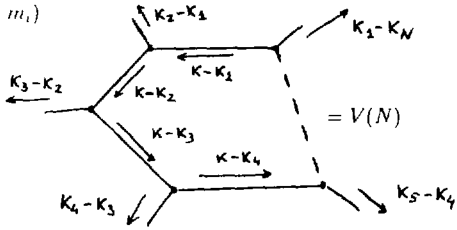
The full contribution of the initial two-loop diagram can be presented in the form

$$I_7(x) = \frac{\Gamma(1+\epsilon)}{(m^2)^\epsilon}\left[(1/\epsilon + 2)I_6(0) - \int_0^1 \frac{d\tau}{\tau}\sqrt{1-\tau}\, I_6(4x/\tau)\right]$$

Thus, the usage of the rule of integration by parts, allows one to obtain the DE for the massive vertex-type diagrams with r.h.s. containing the propagator-type diagrams only.

C. N-point diagrams (see also [7])

Consider of N- point diagram (here the line with momentum $k - k_i$ contains also the mass m_i)



Applying rule 3 with the distinguished line with momentum $k - k_1$, we get

$$V(N)(D-N-1) = V_2^{(1)}(N-1) - \left[(k_1-k_2)^2 + m_1^2 + m_2^2\right]V_2(N) +$$
$$(2 \leftrightarrow 3) + \ldots + (N-1 \leftrightarrow N) - 2m_1^2 V_1(N)$$

Hereafter the upper symbol (in brackets) marks the initial diagram line which is canceled by the application of rule 3. The lower one marks the line which has index 2. Applying also rule 3 with the other distinguished lines, we get the matrix equation

$$\hat{A}\vec{V}(N) = \vec{C}, \left(\vec{V}(N) = [\begin{matrix} V_1 \\ \vdots \\ V_N \end{matrix}]\right) \tag{6}$$

where $C_i = \sum_{j \neq i} V_j^{(i)}(N-1) - (D-N-1)V(N)$

$$a_{ii} = 2m_i^2; \quad a_{ij}(i \neq j) = (k_i - k_j)^2 + m_i^2 + m_j^2$$

The solution of eq.(6) is a standard one

$$V_i(N) = \frac{\det \hat{A}_i}{\det \hat{A}},$$

where the matrix $\hat{A}_i$ equals to one $\hat{A}$ with respect elements a_{ij} which are replaced by elements C_j. We have

$$-\det \hat{A} \ \frac{d}{da}\vec{V}(N) = \sum_1^N \det \hat{A}_i, \ C_i = -\frac{d}{da}V^{(i)}(N-1) - (D-N-1)V(N) \tag{7}$$

It is clear to see that the eq.s (7) allow us to reduce the N-point diagram to $(N-1)$-point diagrams.

Resume. The usage of the rule of the integration by parts allows one to obtain the DE for the massive diagrams with r.h.s.containing more simple diagrams. Using this procedure several times one can get the loops with some number of lines. These loops are integrated either by means of eq.s (1) and (2) decreasing step by step the number of loops, or using the method of Feynman parameters for a more complicated case.

The main difference from the massless case is the necessity to integrate the final result with respect to the mass several times. Thus, this method (at least, in principle) is as powerful in the calculation of massive Feynman diagrams as the rule of integration by parts in the massless case.

The essential property of the massive case is the continuous application of the rule of integration by parts to the both: complicated and simple diagrams. In the massless case rule 3 is used only with master diagrams.

The essential difference from the ordinary methods (for example, Feynman parameters method) of calculation of the massive Feynman diagrams is the appearance of the complicated functions in the final result only. Hence, these functions do not interfere the process of calculation.

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Котиков А.В. E2-93-414
Метод дифференциальных уравнений.
Расчет N-точечных диаграмм

Дан новый метод вычисления фейнмановских диаграмм, который является достаточно простой процедурой голучения результата без непосредственного вычисления D-мерных (в рамках размерной регуляризации) интегралов. В качестве иллюстрации проведен расчет некоторых диаграмм.

Работа выполнена в Лаборатории сверхвысоких энергий ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 1993

Kotikov A.V. E2-93-414
Differential Equation Method.
The Calculation of N-Point Diagrams

A new method of massive Feynman diagrams calculation is presented. It provides a fairly simple procedure to obtain the result without the D-space integral calculation (for the dimensional regularization). Some diagrams are calculated as an illustration of this method capacities.

The investigation has been performed at the Laboratory of Particle Physics, JINR.

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