Exact Solutions to Operator Differential Equations

CARL M. BENDER

ABSTRACT. In this talk we consider the Heisenberg equations of motion $\dot{q} = -i[q, \mathcal{H}]$, $\dot{p} = -i[p, \mathcal{H}]$, for the quantum-mechanical Hamiltonian $\mathcal{H}(p,q)$ having one degree of freedom. It is a commonly held belief that such operator differential equations are intractable. However, a technique is presented here that allows one to obtain exact, closed-form solutions for huge classes of Hamiltonians. This technique, which is a generalization of the classical action-angle variable methods, allows us to solve, *albeit* formally and implicitly, the operator-differential equations of two anharmonic oscillators whose Hamiltonians are $\mathcal{H} = p^2/2 + q^4/4$ and $\mathcal{H} = p^4/4 + q^4/4$.

1. Introduction

The time-evolution equations

(1)
$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p}, \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q},$$

describe a classical dynamical system that evolves according to the Hamiltonian $\mathcal{H}(p,q)$. We assume that the solution to (1) satisfies the initial conditions $p(0) = p_0$, $q(0) = q_0$. Although (1) is equivalent to a single second-order differential equation, it is often possible to find a closed-form solution because the Hamiltonian $\mathcal{H}$ is a constant of the motion. Thus, we can in principle use the algebraic equation $\mathcal{H}(p,q) = E$ to solve for and eliminate one of the variables $p(t)$ or $q(t)$ in (1) and then to solve the resulting *first-order* differential equation for $q(t)$ or $p(t)$. For example, let us consider the Hamiltonian $\mathcal{H} = p^2/2 + V(q)$, for which Eq. (1) takes the form

(2)
$$\dot{q} = p, \qquad \dot{p} = -V'(q) \quad .$$

Subject Classification. Primary 81C05. Secondary 81D05, 81C10, 22E60.

Supported by US DOE Grant DE-FG02-91-ER40628.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Now, solving $\mathcal{H}(p,q) = E$ for $p(t)$ and using the first equation of (2) gives $\dot{q} = \sqrt{2[E - V(q)]}$. This is a first-order separable equation whose implicit solution satisfying $q(0) = q_0$ is

$$F[q(t)] = t + F(q_0) \quad , \tag{3}$$

where

$$F(q) = \int^q \frac{dx}{\sqrt{2[E - V(x)]}} \quad . \tag{4}$$

The quantum equations of motion for the Hamiltonian $\mathcal{H}(p,q)$ are

$$\dot{q} = -i[q, \mathcal{H}], \qquad \dot{p} = -i[p, \mathcal{H}], \tag{5}$$

where the operators $q(t)$ and $p(t)$ satisfy the equal-time commutation relation $[q(t), p(t)] = i$. It is the noncommutivity of $q(t)$ and $p(t)$ that makes (5) difficult to solve if it is a nonlinear system. Only the special case of the harmonic oscillator $\mathcal{H} = p^2/2 + q^2/2$ gives rise to the easily solvable linear equations $\dot{q} = p$, $\dot{p} = -q$, whose exact solution is

$$q(t) = q_0 \cos t + p_0 \sin t \quad , \tag{6}$$

$$p(t) = p_0 \cos t - q_0 \sin t \quad , \tag{7}$$

We propose to solve operator equations that are nonlinear by constructing a quantum analogy to Eq. (3). Specifically, we will attempt to obtain a function $F(p,q)$ of the operators $p(t)$ and $q(t)$ that satisfies

$$-i[F(p,q), \mathcal{H}(p,q)] = \frac{d}{dt} F(p,q) = 1 \quad . \tag{8}$$

Note that F is not unique; $\bar{F} = F + \phi(\mathcal{H})$, where ϕ is an arbitrary function of the Hamiltonian, also satisfies (8). However, later on we will show how to find the simplest solution for F, which we will call the *minimal* solution. If such a function F can be found then of course the solution to (8),

$$F(p(t), q(t)) = t + F(p_0, q_0) \quad , \tag{9}$$

together with

$$\mathcal{H}(p(t), q(t)) = \mathcal{H}(p_0, q_0) \quad , \tag{10}$$

constitutes an exact *implicit* solution to the operator equations of motion (5). If we can then solve (9) and (10) simultaneously for $p(t)$ and $q(t)$ as functions of p_0, q_0, and t, we have an *explicit* solution to the equations of motion. Here are some simple examples:

Example 1: $\mathcal{H} = (pe^q + e^q p)/2$. The operator equation of motion for q, $\dot{q} = e^q$, immediately suggests that $F(p,q)$ in (8) is actually a function of q only: $F(p,q) = -e^{-q(t)}$. The *explicit* solution to the operator equations of motion is

$$q(t) = -\log(e^{-q_0} - t), \qquad p(t) = p_0 - t(p_0\, e^{q_0} + e^{q_0}\, p_0)/2 .$$

Example 2: $\mathcal{H} = pe^{aq}\, p$, where a is a constant. For this Hamiltonian, a function F satisfying (5) is

$$F = -\frac{1}{a}\frac{1}{\sqrt{\mathcal{H}}}\, p(t) \frac{1}{\sqrt{\mathcal{H}}} \quad .$$

The *explicit* solution to the operator equations of motion is

$$p(t) = -a\mathcal{H}(p_0, q_0)t + p_0 , \quad q(t) = -\frac{1}{a}\ln[e^{-aq_0} - 2p_0 t + a^2 \mathcal{H}(p_0, q_0) t^2] .$$

Example 3: *Euler Hamiltonians.* We define an Euler Hamiltonian to be one in which the operator $p(t)$ is always accompanied by a multiple of $q(t)$; that is, $\mathcal{H} = \mathcal{H}(pq)$. In general, the operator equations of motion for Euler Hamiltonians can always be solved explicitly and in closed form.[1] This is true because the operator equations of motion for any Euler Hamiltonian can be written in the form

$$\dot{q} = q \sum_n \alpha_n (pq)^n , \qquad \dot{p} = -p \sum_n \alpha_n (qp)^n \quad .$$

The solution to these operator equations is

$$q = q_0 e^{\Sigma_n \alpha_n (p_0 q_0)^n t} \quad ,$$
$$p = p_0 e^{\Sigma_n \alpha_n (q_0 p_0)^n t} \quad .$$

For example, the operator equations for the Hamiltonian $\mathcal{H} = apq^2p + bqp^2q + c(pqpq + qpqp) + d(q^2p^2 + p^2q^2)$ have the form

$$\dot{q} = \alpha qpq , \qquad \dot{p} = -\alpha pqp \quad ,$$

where $\alpha = 2a + 2b + 4c + 4d$. The solution to these operator differential equations is

$$q(t) = q_0\, e^{\alpha p_0 q_0 t} , \qquad p(t) = p_0\, e^{-\alpha q_0 p_0 t} \quad .$$

A particularly interesting example of an Euler Hamiltonian is

$$\mathcal{H} = e^{c(pq+qp)}, \qquad (|c| < \pi/2) \quad .$$

The solution to the operator differential equations for this Hamiltonian is

$$q(t) = q_0\, e^{2t(\sin c)\, e^{2p_0 q_0 c}} \quad ,$$
$$p(t) = p_0\, e^{-2t(\sin c)\, e^{2q_0 p_0 c}} \quad .$$

Example 4 : $\mathcal{H} = p^\alpha q^{2\beta} p_\alpha$. For this Hamiltonian it is easy to see that a function F satisfying (5) is

$$F(p,q) = \frac{1}{4\alpha - 4\beta}\frac{1}{\sqrt{\mathcal{H}}}(pq+qp)\frac{1}{\sqrt{\mathcal{H}}} \quad . \tag{11}$$

Note that (11) ceases to exist when $\alpha = \beta$. However, the special case $\alpha = \beta$ gives an *Euler* Hamiltonian whose solution is discussed in Example 3 above. Solving $F(p,q) = t + F(p_0,q_0)$ simultaneously with $\mathcal{H}(p,q) = \mathcal{H}(p_0,q_0)$ can be complicated. However, a relatively simple case arises when $\alpha = 1$ and $\beta = N/2$. Now the explicit solution is

$$q(t) = \{[p_0 q_0 + (2-N)p_0 q_0^N p_0 t]^{-1} p_0 q_0^N p_0 [q_0 p_0 + (2-N) p_0 q_0^N p_0 t]^{-1}\}^{1/(N-2)} ,$$
$$p(t) = \{[q(t)]^{-N/2} p_0 q_0^N p_0 [q(t)]^{-N/2} - \frac{1}{4}N(N-2)[q(t)]^{-2}\}^{1/2} .$$

Example 5 : $\mathcal{H} = \alpha p^\gamma + \beta q^{-\gamma}$. Here, the form of $F(p,q)$ is similar to that in Example 4:

$$F(p,q) = \frac{1}{2\gamma}\frac{1}{\sqrt{\mathcal{H}}}(pq+qp)\frac{1}{\sqrt{\mathcal{H}}} \quad .$$

2. Operator Basis Elements and Their Algebra

Our objective is now to describe a general procedure for obtaining an operator $F(p,q)$ that satisfies (8). To do so we introduce an operator basis. The basis elements $T_{m,n}$ are defined as the sum of all possible terms containing m factors of p and n factors of q multiplied by $m!n!/(m+n)!$. $T_{m,n}$ is thus a totally symmetric Hermitian object containing $(m+n)!/(m!n!)$ individual terms. For example,

$$T_{0,0} = 1 \quad ,$$
$$T_{0,3} = q^3 \quad ,$$
$$T_{1,1} = (pq+qp)/2 \quad ,$$
$$T_{2,1} = (p^2q + pqp + qp^2)/3 \quad ,$$
$$T_{2,2} = (p^2q^2 + q^2p^2 + pqpq + qpqp + pq^2p + qp^2q)/6 \quad .$$

For two reasons, this seems to be a natural basis with which to express operators. First, $T_{m,n}$ contains positive powers of p and q, so it should be useful for constructing Taylor-like expansions of operators. Note that we can expand operators, regardless of whether they are symmetric. For example,

$$p^2q^3 = T_{2,3} - 3iT_{1,2} - 3T_{0,1}/2 \quad .$$

Second, $T_{m,n}$ satisfies an extremely useful set of commutation and anticommutation relations. Commuting with p or q has the effect of a lowering operator:

$$\begin{aligned} [T_{m,n}, p] &= in\, T_{m,n-1} \quad , \\ [T_{m,n}, q] &= -im\, T_{m-1,n} \quad . \end{aligned} \tag{12}$$

Anticommuting with p or q is analogous to applying a raising operator:

$$\begin{aligned} [T_{m,n}, p]_+ &= 2T_{m+1,n} \quad , \\ [T_{m,n}, q]_+ &= 2T_{m,n+1} \quad . \end{aligned} \tag{13}$$

The operator basis elements $T_{m,n}$ form a closed algebra under commutation and anticommutation; that is, the commutator or anticommutator of any two basis elements can be expressed as a sum of operator basis elements.

The basis elements have the crucial property that the totally symmetric operator $T_{m,n}$ can be completely reorganized using the commutation relation $[q,p] = i$ and recast in Weyl-ordered form

$$T_{m,n} = \frac{1}{2^m}\sum_{j=0}^{\infty}\binom{m}{j} p^j q^n p^{m-j} = \frac{1}{2^n}\sum_{k=0}^{\infty}\binom{n}{k} q^k p^m q^{n-k} \quad . \tag{14}$$

The operators $T_{m,n}$ have many more remarkable properties that have allowed them to play a central role in previous studies involving finite-element lattice approximations,[2] operator ordering,[3] and Hahn polynomials.[4] To summarize briefly, the operator $T_{m,m}$ can be expressed as a polynomial P of degree m in terms of the operator $T_{1,1}$:

$$T_{m,m} = P_m(T_{1,1}) \quad , \tag{15}$$

where

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= x^2 - 1 \quad , \\ P_3(x) &= x^3 - 5x \quad , \\ P_4(x) &= x^4 - 14x^2 + 9 \quad , \\ P_5(x) &= x^5 - 30x^3 + 89x \\ P_6(x) &= x^6 - 55x^4 + 439x^2 - 225 \quad . \end{aligned}$$

Equation (15) is reminiscent of the defining equation for Chebyshev polynomials $C_n(x)$: $\cos(n\theta) = C_n(\cos\theta)$.

The Hahn polynomials exhibit the following properties: First, they satisfy a three-term recursion relation:

$$P_n(x) = xP_{n-1}(x) - (n-1)^2 P_{n-2}(x) \quad .$$

Second, $P_n(x)$ satisfies a functional difference equation (rather than a differential equation):

$$(1-ix)P_n(x+2i) + (1+ix)P_n(x-2i) = (4n+2)P_n(x) \quad .$$

Third, the Hahn polynomials have a simple generating function $G(x)$:

$$\frac{e^{x\tan^{-1}t}}{\sqrt{1+t^2}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} P_n(x) \quad .$$

Fourth, the Hahn polynomials are orthogonal on the line $-\infty < x < \infty$ with respect to the weight function $w(x) = sech(\pi x/2)/2$:

$$\int_{-\infty}^{\infty} dx\, w(x) P_m(x) P_n(x) = \delta_{m,n}(n!)^2 \quad .$$

It is easy to expand functions in terms of Hahn polynomials. For example,

$$e^{cx} = \sum_{n=0}^{\infty} \frac{(\tan c)^n}{n!\cos c} P_n(x) \qquad (|c| < \pi/2) \quad .$$

Finally, the Hahn polynomials can be expressed in terms of a generalized hypergeometric function:

$$P_n(x) = \sqrt{n!}\, i^n {}_3F_2(-n, n+1, (1-ix)/4; 1/2, 1; 1) \quad .$$

3. Minimal Solution of the Harmonic Oscillator

Now we return to the problem of obtaining a solution to (8). We represent $F(p,q)$ as an arbitrary sum of operator basis elements

$$F(p,q) = \sum_{m,n} \alpha_{m,n} T_{m,n} \quad , \tag{16}$$

where $\alpha_{m,n}$ are constants to be determined from the requirement in (8) that $-i[F(p,q), \mathcal{H}(p,q)] = 1$. To illustrate, we begin by finding $F(p,q)$ for the harmonic oscillator Hamiltonian $\mathcal{H}(p,q) = p^2/2 + q^2/2$. Equations (12) and (13) make the computation very easy:

$$\frac{1}{i}[T_{m,n}, \frac{1}{2}p^2] = \frac{1}{2i}[p, [T_{m,n}, p]]_+ = \frac{n}{2}[p, T_{m,n-1}]_+ = n\, T_{m+1,n-1} \quad . \tag{17}$$

Similarly,

$$\frac{1}{i}[T_{m,n}, \frac{1}{2}q^2] = -m\; T_{m-1,n+1} \quad . \tag{18}$$

Combining (16)-(18), we see that the commutation relation in (8) takes the form

$$\sum_{m,n} \alpha_{m,n}(nT_{m+1,n-1} - mT_{m-1,n+1}) = T_{0,0} \quad . \tag{19}$$

Hence, assuming completeness, we determine that the coefficients $\alpha_{m,n}$ satisfy the linear partial difference equation

$$(n+1)\alpha_{m-1,n+1} - (m+1)\alpha_{m+1,n-1} = \delta_{m,0}\,\delta_{n,0} \quad . \tag{20}$$

This partial difference relates pairs of coefficients $\alpha_{m,n}$. If we represent the coefficients as dots on an integer planar lattice then it is clear that next-to-nearest neighboring points on a diagonal whose slope is -1 are related by (20) (see Fig. 1). As we pointed out earlier, F is *not* uniquely determined. Thus, we are free to take the simplest particular solution to (20); namely, the one having the smallest possible number of nonvanishing coefficients $\alpha_{m,n}$. We call this the minimal solution. The minimal solution to (20) is given by

$$\alpha_{-2m-1,2m+1} = (-1)^m/(2m+1)\,, \qquad m = 0, 1, 2, 3, \ldots \quad ,$$

and $\alpha_{m,n} = 0$ for other values of m, n. Thus, the formula for $F(p,q)$ in (16) becomes

$$F(p,q) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} T_{-2m-1,2m+1} \quad . \tag{21}$$

Observe that we are forced to generalize our initial assumption that, by analogy with Taylor series, our basis $T_{m,n}$ has $m, n \geq 0$. Apparently, a more accurate analogy is with Laurent series in which powers may be positive or negative. Fortunately, the formulas in (14) allow us to define $T_{m,n}$ when $m \geq 0$ and $n < 0$ and when $n \geq 0$ and $m < 0$. Moreover, the commutation and anticommutation relations in (12) and (13) continue to hold in this extended and singular basis.

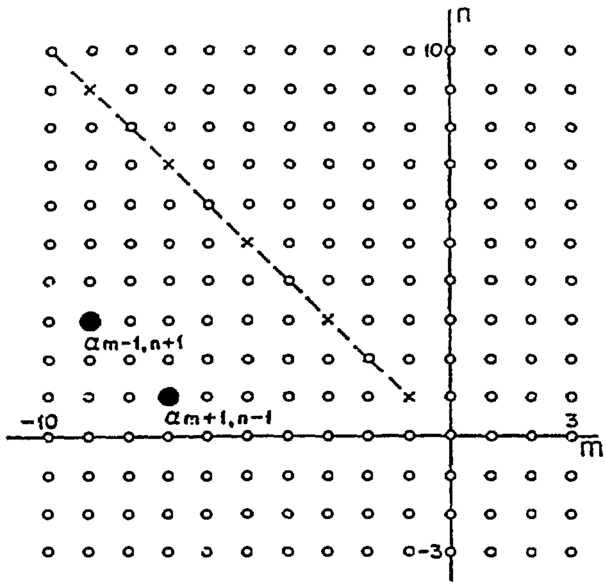

Figure 1: The partial difference equation (20) for the harmonic oscillator relates the coefficients $\delta_{m-1,n+1}$ and $\alpha_{m+1,n-1}$. the minimal solution consists of all $\alpha_{m,n}=0$ except $\alpha_{-2m-1,2m+1}$, $m=0,1,2,\ldots$, as indicated by crosses.

In order to understand the formal series in (21) we return to the exact solution to the harmonic oscillator in (6) and (7). We divide (6) by (7) and let $Z(t)=q(t)[p(t)]^{-1}$:

$$\begin{aligned} Z(t) &= (q_0\cos t+p_0\sin t)(p_0\cos t-q_0\sin t)^{-1}\\ &= (q_0\cos t+p_0\sin t)p_0^{-1}p_0(p_0\cos t-q_0\sin t)^{-1}\\ &= (Z_0\cos t+\sin t)[(p_0\cos t-q_0\sin t)p_0^{-1}]^{-1}\\ &= \frac{Z_0+\tan t}{1-Z_0\tan t}\quad . \end{aligned} \tag{22}$$

Since $Z(t)$ is a function of Z_0, $Z(t)$ must commute with Z_0. Thus, we can treat (21) as a c-number algebraic equation and solve for tan t:
$\tan t=[Z(t)-Z_0]/[1+Z(t)Z_0]$. Finally, taking the inverse tangent of this equation, we obtain

$$\arctan Z(t)=t+\arctan Z_0\quad . \tag{23}$$

This equation is an instance of (9), so we identify

$$F(p,q) = \text{arc}\,\tan\{q(t)[p(t)]^{-1}\} \quad . \tag{24}$$

Compare (21) and (24). Note that the coefficients α in (21) correspond exactly with the Taylor expansion of arc tan $x = \Sigma_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$. However, these two functions F are not the same even though they satisfy the same commutation relation in (8). As expected, they differ by a function of the Hamiltonian $\mathcal{H}$:

$$F_{\text{Eq.}(21)} - F_{\text{Eq.}(24)} = \frac{i}{2}\sum_{n=0}^{\infty} E_{2n}\mathcal{H}^{-2n-1} = \frac{i}{2\mathcal{H}}\int_0^{\infty} ds \frac{e^{-s}}{\cosh(\frac{s}{2\mathcal{H}})} \, ,$$

where E_2n are Euler numbers. (The first few Euler numbers are $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, $E_8 = 1385$, and so on.)

Even more important, the physical meaning of $F(p,q)$ is now more evident: Classically, the harmonic oscillator describes an orbit in phase space which is a circle of radius $p_0^2 + q_0^2$. The singular quantity arc tan$[q(t)/p(t)]$ is the *angle* θ of a point on this circle. Apparently, (21) is the generalization of the angle coordinate from a classical (c-number) theory to a quantum (operator) theory. This result clearly illustrates that the operator methods used here generalize the notion of classical action-angle variables to the realm of quantum mechanics.

4. Minimal Solution for Anharmonic Oscillator $\mathcal{H} = p^2/2 + q^4/4$

Next, we consider the anharmonic oscillator whose Hamiltonian is given by $\mathcal{H} = p^2/2 + q^4/4$. To find $F(p,q)$ we first evaluate the commutator

$$\begin{aligned} -i[T_{m,n}, q^4] &= -m(q^3 T_{m-1,n} + q^2 T_{m-1,n} q + q T_{m-1,n} q^2 + T_{m-1,n} q^3) \\ &= -2m(q^2 T_{m-1,n+1} + T_{m-1,n+1} q^2) \\ &= -m\,[q,[q,T_{m-1,n+1}]_+]_+ - m\,[q,[q,T_{m-1,n+1}]]_+ \\ &= -m(2[q,T_{m-1,n+2}]_+ + (m-1)i[q,T_{m-2,n+1}]) \\ &= -4mT_{m-1,n+3} + m(m-1)(m-2)T_{m-3,n+1} \, . \end{aligned} \tag{25}$$

Combining the result in (17) with that in (25), and assuming that the operators $T_{m,n}$ form a complete basis gives a recursion relation for $\alpha_{m,n}$

(26)
$$(n+1)\alpha_{m-1,n+1} - (m+1)\alpha_{m+1,n-3} + \frac{(m+3)(m+2)(m+1)}{4}\alpha_{m+3,n-1} = \delta_{m,0}\delta_{n,0} \, .$$

This partial difference equation relates triples of points on the integer lattice in Fig. 2 whose points correspond to coefficients $\alpha_{m,n}$. A careful analysis of (26) shows that a minimal solution exists consisting of certain nonzero values of $\alpha_{m,n}$. These nonzero values of $\alpha_{m,n}$ correspond to points in Fig. 2 lying in the quadrant $m < 0$, $n > 0$ and forming a triangular network as indicated in the figure.

A first step in solving (26) involves mapping the triangular network of nonvanishing values of $\alpha_{m,n}$ onto a triangular domain. On this domain the partial difference equation (26) works in much the same fashion as the difference equation that generates the binomial coefficients in Pascal's triangle. The relevant transformation of the independent variables is

$$M = -(n+2m)/6\,, \qquad N = (n-m)/6 \quad .$$

We define a new independent variable $A_{M,N}$ by

$$A_{M,N} = \alpha_{m-1,n+1} \quad ,$$

along with the constraint that $A_{M,N} = 0$ for $M < 0$, $N < 0$, and for $M > N$.

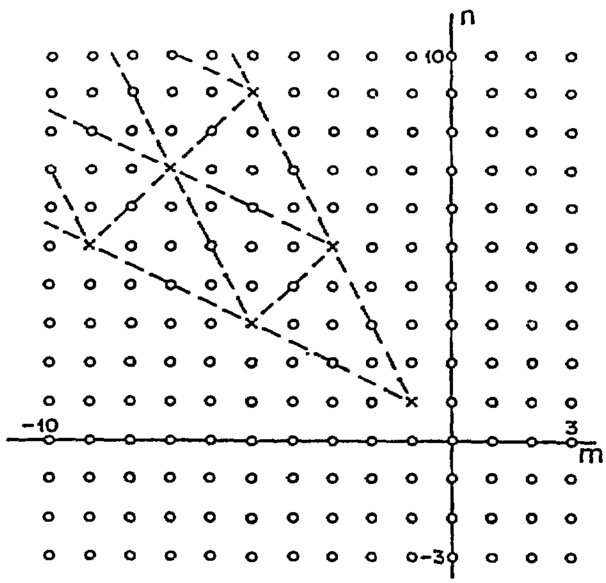

Figure 2: Triplets of points related by the partial difference equation (26) for the anharmonic oscillator with $\mathcal{H} = \frac{1}{2}p^2 + \frac{1}{4}q^4$. The minimal solution consists of nonvanishing values of $\alpha_{m,n}$ indicated by crosses; all other $\alpha_{m,n}$'s vanish.

The advantage of the M, N variables over the m, n variables is that the partial difference equation is first order:

(27)

$$(4N-2M+1)A_{M,N} + (2N+2M-1)A_{M,N-1} - \frac{1}{4}(2N+2M-1)(2N+2M-2)(2N+2M-3)A_{M-1,N-1} = \delta_{M,0}\delta_{N,0}\,.$$

A further transformation, this time of the dependent variable, reduces the partial difference equation in (27) to one whose coefficients are linear functions of M and N. We define $B_{M,N} = 2^{-N}\Gamma(1/2)A_{M,N}/\Gamma(M+N+1/2)$ and the difference equation becomes

(28) $$(4N-2M+1)B_{M,N} + B_{M,N-1} - (N+M-1) = \delta_{M,0}\delta_{N,0}\,.$$

The simplest way to express a general element $B_{M,N}$ is in terms of a generating function; to wit, we define the generating function

$$g(x,y) = \sum_{N=0}^{\infty}\sum_{M=0}^{N} B_{M,N}x^M y^N \quad .$$

From (28) we can derive a linear first order partial differential equation satisfied by $g(x,y)$:

(29) $$-x(2+xy)g_x + y(4-xy)g_y + (1+y-xy)g = 1 \quad .$$

Using the method of characteristics we cna find the unique solution to this partial differential equation that satisfies the initial condition

$$g(0,y) = \sum_{N=0}^{\infty}(-y/4)^N \frac{\Gamma(5/4)}{\Gamma(N+5/4)} = e^{-y/4}\int_0^1 du\, e^{u^4 y/4} \quad .$$

The solution is

(30)

$$g(x,y) = \int_0^1 \frac{du}{2\sqrt{(u)}\,(1-xyu)} \exp\left(\frac{2}{3x^2y} - \frac{1}{x} + \frac{(1-xy)^{3/2}(xyu-2/3)}{x^2y(1-xyu)^{3/2}}\right)\,.$$

In terms of the generating function $g(x,y)$,

(31) $$B_{M,N} = \frac{1}{M!N!}\left(\frac{\partial}{\partial x}\right)^M \left(\frac{\partial}{\partial y}\right)^N g(x,y)\Big|_{x=y=0}\,.$$

Using this expression for the coefficients $B_{M,N}$ in the formula (16) for $F(p,q)$ gives a complete and exact minimal solution to the Heisenberg operator differential equations for the anharmonic oscillator with the Hamiltonian $\mathcal{H} = p^2/2 + q^4/4$. The operator F is given by

$$F(p,q) = \sum_{N=0}^{\infty} \sum_{M=0}^{N} 2^N \frac{\Gamma(M+N+1/2)}{\Gamma(1/2)} B_{M,N} T_{-2N-2M-1,4N-2M+1} \,. \tag{32}$$

5. Minimal Solution for Anharmonic Oscillator $\mathcal{H} = p^4/4 + q^4/4$

Next, we consider the anharmonic oscillator whose Hamiltonian is given by $\mathcal{H} = p^4/4 + q^4/4$. This is the natural next step in our program of solving increasingly complex systems of quantum operator differential equations: $\mathcal{H} = p^2/2 = q^2/2$ gives rise to a two-term partial difference equation, $\mathcal{H} = p^2/2 = q^4/4$ gives rise to a three-term partial difference equation, and, as we will see, $\mathcal{H} = p^4/4 = q^4/4$ gives rise to a four-term partial difference equation. Specifically, the partial difference equation for the coefficients $\alpha_{m,n}$ in (16) has the form

$$\begin{aligned} &(n+1)\alpha_{m-3,n+1} + \frac{1}{4}(m+3)(m+2)(m+1)\alpha_{m+3,n-1} \\ &-(m+1)\alpha_{m+1,n-3} - \frac{1}{4}(n+3)(n+2)(n+1)\alpha_{m-1,n+3} = \delta_{m,0}\delta_{n,0} \,. \end{aligned} \tag{33}$$

This partial difference equation relates quartets of points, which lie at vertices of rectangles, on the integer lattice in Fig. 3 whose points correspond to coefficients $\alpha_{m,n}$. Examining (33) we see that a minimal solution exists with nonzero values of $\alpha_{m,n}$ lying in the quadrant $m < 0$, $n > 0$, and forming a wedge-shaped network as indicated in Fig. 3.

To solve (33) we use the transformation

$$N = (n-m)/8 \,, \qquad M = -(n+m)/4 \,,$$

which maps points in the m,n plane for which the coefficient $\alpha_{m,n}$ is nonzero into a triangular lattice for which N runs from 0 to infinity and M runs from 0 to $2N$. We then define a new dependent variable $C_{M,N}$ by

$$C_{M,N} = \alpha_{m-3,n+1} \quad ,$$

along with the constraint that $C_{M,N} = 0$ for $M < 0$, $N < 0$, and $M > 2N$. $C_{M,N}$ satisfies a first order partial difference equation which can be simplified further by the transformation

$$D_{M,N} = 4^{-M}\frac{\Gamma(N-M/2+3/4)\Gamma(N-M/2+5/4)}{\Gamma(N+M/2+3/4)\Gamma(N+M/2+5/4)}C_{M,N}\,.$$

The equation satisfied by $D_{M,N}$ has linear coefficients:

(34)
$$\left(N+\frac{M}{2}+\frac{1}{4}\right)D_{M,N}-\left(N-\frac{M}{2}+\frac{1}{2}\right)D_{M-1,N}$$
$$+\left(N-\frac{M}{2}-\frac{1}{4}\right)D_{M,N-1}-\left(N+\frac{M}{2}-\frac{1}{2}\right)D_{M-1,N-1}=\frac{1}{4}\delta_{M,0}\delta_{N,0}\,.$$

A general element D_M, N in this array can be expressed in terms of a generating function $h(x,y)$, where

$$h(x,y)=\sum_{N=0}^{\infty}\sum_{M=0}^{2N}D_{M,N}x^My^N\quad .$$

From (34) we can derive a first order linear partial differential equation satisfied by $h(x,y)$:

$$\frac{x}{2}\frac{1+x}{1-x}h_x+y\frac{1+y}{1-y}h_y+\frac{1+3y-4xy}{4(1-x)(1-y)}h=\frac{1}{4(1-x)(1-y)}\,. \tag{35}$$

The unique solution to this partial differential equation that satisfies the initial condition

$$h(0,y)=\sum_{N=0}^{\infty}(-y)^N\frac{\Gamma(5/4)\Gamma(N+3/4)}{\Gamma(3/4)\Gamma(N+5/4)}=\int_0^1\frac{du}{\sqrt{(1+y)^2-4yu^4}}$$

is

$$h(x,y)=\int_0^1\frac{du}{\sqrt{(1+y)^2-4yu^4(1+x-xu^2)^2}}\quad . \tag{36}$$

In terms of this generating function,

$$D_{M,N}=\frac{1}{M!N!}\left(\frac{\partial}{\partial x}\right)^M\left(\frac{\partial}{\partial y}\right)^N h(x,y)\bigg|_{x=y=0}\quad .$$