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ABSTRACT

Understanding the movement of windblown soil particles and the resulting formation of complex surface features are among the most intriguing problems in dryland research. This understanding can only be achieved through physical and mathematical modelling and must also involve observational data and laboratory experiments. Some current mathematical models that have contributed to the basic understanding of the transportation and deposition of soil particles by wind are presented and solved in these notes.

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A Introduction

Drylands cover nearly one third of the Earth's land surface and about sixty countries have arid or semi-arid regions. In spite of this fact, however, the study of the physical and ecological processes associated with arid regions were never taken seriously by the international community before the fifties. In many arid zones these processes often have detrimental effects on productivity, human settlements and communications. The first worldwide effort to study the problems of arid regions was started in 1951 when UNESCO launched its major scientific program on arid lands. The program continued for nearly a decade during which several research reports as well as a newsletter were published. A few years later a tragic drought hit the southern edge of the Sahara desert. The severity of the drought led to a famine during the years 1969-1973, as a result of which about a quarter of a million people lost their lives. The region affected, which is now known as the Sundano-Sahelian region, lies below the Sahara drylands desert and runs across the entire African continent.

The Sahelian tragedy focused a great international attention on the drylands problems culminating in the famous UN Conference on Desertification, which took place in Nairobi, Kenya, in September 1977. The conference was attended by representatives of nearly 100 countries – thus providing an international forum for discussion of arid-lands problems. It was estimated at the conference that during the last 50 years about 650,000 square kilometres of once fertile land south of the Sahara was lost to the desert. The conference recognized the global nature of the desertification problem and suggested a plan of action to combat desertification. Millions of dollars have subsequently been spent on conferences, workshops and projects aimed at combating desertification and adopting the plan of action suggested by the UN conference. Most of this effort, however, has been directed towards ecological, socio-economic and management problems of dry environments – leaving the fundamental physical problems of climatic changes, soil erosion by wind, sand transport and dunes migration largely untouched. There is an unfortunate general belief among the majority of those concerned with the desertification problem that deserts are manmade. In Africa, in particular, the poor Nomads and their animals were largely blamed for desertification. It is not surprising, therefore, that after nearly two decades of intensive work by ecologists, social scientists, socio-economists, planners and politicians no effective solution to the desertification problem has yet materialized. Indeed until now the southward and eastward expansion of the Sahara Desert is posing one of the most serious and challenging problems facing the entire international community.

In December 1980 the International Centre for Theoretical Physics, in Italy, initiated a new fundamental approach to the problems of drylands. A Workshop entitled 'The Physics of Desertification' was held and was attended by approximately 80 scientists of various academic backgrounds including climatologists, geomorphologists, ecologists, geographers, soil scientists, engineers, physicists and mathematicians – thus providing the first opportunity ever for a multidisciplinary international group of theoretical and applied scientists to exchange and coordinate their views, transcending all traditional professional boundaries. The Workshop recognized that a fundamental understanding of the problems of drylands can only be achieved through a broadly based interdisciplinary approach in which the desertification mechanism should be regarded as a combination of *both* physical and ecological processes. The interdisciplinary research projects must be guided by physical and mathematical modelling. In particular, the Workshop identified three areas of research in which our basic knowledge is insufficient. These are:

- Theoretical and experimental studies to determine accurately the appropriate meteorological and soil factors responsible for soil erosion and dust production;
- comprehensive theoretical, experimental and observational studies to analyse the movements of sand grains near the surface in order to derive an accurate expression for the rate of sand transport;
- detailed theoretical, experimental and field studies to understand the formation and movement of various type of sand features.

To my knowledge, the first proper scientific study of sand movement in the Sahara was carried out by Bagnold during the thirties. Bagnold spent several years in the Sahara desert during which he analysed the modes of sand movement and the formation and growth of sand dunes. Most of his work is contained in his famous book 'The Physics of Blown Sand and Desert Dunes', which he published in 1941. Referring to the Sahara desert, he wrote in the introduction of this classic book 'In places vast accumulations of sand weighing million of tons move in regular formation over the surface of the country, growing, retaining their shape, even breeding, in a manner which, by its grotesque imitation of life, is vaguely disturbing to an imaginative mind'.

Two basic sand features are immediately recognized in the Sahara desert: sand ripples with length varying from a few inches to several feet and with length-height ratio about 10; and sand dunes which are much bigger than ripples – the smallest dunes are much longer than the longest ripples. Dunes occur in the Sahara in a considerable variety of forms. These forms are usually determined by the strength and direction of wind, the amount and characteristics of sand available and the physical obstacles in the area. In the case of a unidirectional steady flow of wind and a limited supply of sand crescent-shaped dunes (Barchan dunes) are often found. If the undirectional wind is strong enough isolated dome–shaped dunes are formed. In areas where the sand is abundant and the dominant wind shifts slightly in direction, the so–called linear dunes (also known as seif dunes) are usually found. The crests of these dunes are sharp and their lengths vary from a few hundred metres to a few hundred kilometres. Linear dunes are the most abundant in the Sahara and in certain areas they cluster to form large dune fields. If the wind blows from several directions complex patterns of crescent–shaped dunes are often found in the Sahara. Such forms are called star dunes; they have a high peak at the centre out of which three or four long arms extend radically.

Other large desert regions in the world have windblown features similar to the Sahara. In fact these features are also observed in other parts of our solar system. The largest single dune field in the solar system is the one surrounding the north pole of Mars.

In many places in and around the Sahara the movement of sand creates a serious threat to agricultural projects, human settlements and communications. In Sudan, for example, the country's most valuable gum–arabic belt is overwhelmed by sand dunes. Several villages, roads and railways in North Africa are invaded by shifting sand. Near the banks of the Nile in Egypt and northern Sudan farmers have been fighting a losing battle against windblown sand (El–Baz, 1977).

Various methods have been tried to slow down drifting sand. Perhaps the most successful, but very expensive method is the one used in Iran and Saudi Arabia, where crude oil is sprayed over mobile sand. The most popular method in and around the Sahara, however, is to plant shelter belts (e.g. Eucalyptus or Acacia trees) in the path of advancing sand dunes. Shelter belts act as physical barriers as well as wind breakers, thus causing the wind to deposit the sand before the shelter belt. It is essential, however, before embarking on expensive sand stabilization projects in a particular area, to understand fully the physical mechanism of sand dunes formation and movement in that area.

B Soil Transport Rates

Understanding the mechanism of soil transport in dryland areas is one of the most intriguing problems facing mathematicians, physicists and geologists alike. Perhaps the first and simplest step is to try to understand the free interaction between wind and sand in the absence of any obstacles.

In his article 'A Further Journey in the Libyan Desert', R.A. Bagnold wrote (Bagnold, 1933):

> In the western desert of Egypt the free interplay of sand and wind has been allowed to continue for a vast period of time, and here, if anywhere, it should be possible in the future to discover the laws of sand movement and the growth of dunes.

Most of our current understanding of this free interplay between sand and wind is based largely on Bagnold's work, which is summarized in his classical book 'The Physics of Blown Sand and Desert Dunes' (Bagnold, 1941). The essential goal of sand physics today is to follow Bagnold's work and reach a clear understanding of the mechanism responsible for the formation, development and movement of various types of sand features in deserts and arid regions. The first step towards this goal was taken by Bagnold – he studied in detail, both in the laboratory and in the field, the motion of individual sand grains under the action of a uniform wind of various strengths. He identified three modes of movement of sand particles: saltation, surface creep and suspension. A saltating grain (between 0.1 mm and 0.5 mm in diameter) follows a trajectory which rises steeply from the sand surface and is carried by wind to a height of a few feet. The grain then starts to lose its height due to gravity until it collides with the surface and sets other particles in motion. Recent measurements show that on a sandy surface over 90 percent of saltating grains move below 30 cm. Light particles (less than 0.1 mm in diameter) are carried by the wind to very high levels and form dust. Heavy particles (over 0.5 mm in diameter) roll along the surface when acted upon by a strong wind and/or when bombarded by saltating particles. Near the surface the process of suspension is insignificant and as far as the deformation of the surface is concerned, one can safely regard saltation and surface creep as the dominant physical processes in action.

Bagnold conducted a simple analysis using saltation mechanics and derived a simple formula for the rate of sand transport per unit width per unit time

$$T = C\sqrt{\frac{d}{D}}\,\frac{\rho}{g}\,(u^*)^3, \qquad u^* = \sqrt{\frac{\tau_0}{\rho}}, \tag{1}$$

where C is an empirical coefficient depending on the characteristics of sand particles, D is a standard sand grain diameter (e.g. 0.25 mm); d is the diameter of sand in question; τ_0 is the shear stress at the sand surface due to wind action, and ρ is the air density.

Although Bagnold's description of the three modes of particles' motion is now generally accepted, his estimate of the rate of sand transport (Eq.(1)) is far from being satisfactory. Several authors have attempted improvements on Bagnold's work and have obtained various formulae relating the horizontal mass flux of erodible soil to the observed and threshold wind speed (for a comprehensive review, see Greeley and Iversen, 1985). A generalized Bagnold formula which incorporates most of these expressions as special cases may be written as

$$q = q_0(u - u_T)^n \, u^m, \qquad n + m = 3 \tag{2}$$

in which u and u_T are the observed and threshold wind speeds and n, m and q_0 are constants whose values depend on soil and flow properties. Until now, however, there is no relation that accurately predicts the rate of sand transport. Thus the whole field of saltation mechanics and sand transport laws are wide open for further intensive research.

The empirical formula (2) is valid for a single mean value of the wind speed. Field observations and wind data, however, reveal that soil transport is commonly caused by a highly variable wind flow. One method of characterizing the variability of the wind is to assume a statistical distribution for the wind speed. In particular, Johnson (1978) has shown that the two-parameter Weibull distribution fits wind data reasonably well.

The probability density function of the Weibull distribution is given by

$$f(u) = \frac{k}{c}\left(\frac{u}{c}\right)^{k-1} \quad \exp\left\{-\left(\frac{u}{c}\right)^k\right\} \tag{3}$$

where k and c are the shape and scale parameters. The average wind speed $\bar{u}$ can easily be obtained by integrating (3) and the result is

$$\bar{u} = \int_0^\infty f(u)du = c\,\Gamma\left(1 + \frac{1}{k}\right) \tag{4}$$

where Γ is the gamma-function.

For a statistically distributed wind flow governed by the density function (3) we can compute the average rate of soil transport over a certain period (Skidmore, 1986; Gillette and Passi, 1988; Babiker et al., 1987)

$$E = \langle\frac{q}{q_0}\rangle = \frac{k}{c}\int_{u_T}^\infty \left(\frac{u}{c}\right)^{k-1} (u - u_T)^n u^m \quad \exp\left[-\left(\frac{u}{c}\right)^k\right] du\,. \tag{5}$$

In terms of the dimensionless parameter $R = u_T/\bar{u}$ we can write (5) in the form

$$E = \langle\frac{q}{q_0}\rangle = k\left(\frac{\bar{u}}{c}\right)^k (\bar{u})^{m-n} \int_R^\infty (x - R)^n \, x^{m+k-1} \, \exp\left[-\left(\frac{\bar{u}}{c}\right)^k x^k\right] dx\,. \tag{6}$$

Exact analytic values for the integral in (6) can be derived for the following two special cases

i) $k = 1, n = 1, m = 2$

In this case (6) can easily be evaluated in closed form to give (see Gradshteyn and Ryzhik, 1965)

$$E = (\bar{u})^3 \quad (R^2 + 4R + 6)\,e^{-R} \tag{7}$$

ii) $k = 2, n = 1, m = 2$

This case was discussed in some detail by Gillette and Passi (1988); who obtained a value for the integral in terms of the incomplete gamma function. The integral may also be evaluated in closed form (see Babiker et al., 1987) to give

$$E = \frac{1}{2}\left(\frac{3}{\sqrt{\pi}}\,\bar{u}\right)^3 \left\{\frac{3\sqrt{\pi}}{2}\,[1 - \phi(\lambda)] + \lambda\exp(-\lambda^2\right\} \tag{8}$$

where $\lambda^2 = \frac{\pi}{4} R^2$.

For other values of k, n and m (6) has been evaluated numerically (see Babiker et al., 1987) and the results are summarized in Figs.1 and 2. These results can be used in the following two ways:

a) If at a given site the mean wind speed is determined from measurements, the Weibull parameters k and c may be estimated using regression analysis (Johnson, 1978; Skidmore, 1986). If at the same time the threshold speed is estimated by using wind erosion groups or dry sieving (Gillette and Passi, 1988), the curves in Figs.1 and 2 may then be used to estimate the average soil loss for different models of particle transport rates.

b) If the average soil loss is measured at a particular location and the relevant parameters are estimated, the curves in Figs.1 and 2 may be used to test the accuracy of different expressions for the horizontal particle transport rate.

In general, the values obtained show that for a fixed Weibull parameter k and a fixed threshold speed u_T, the average soil transported increases as m increases and n decreases (for values of n and m satisfying $n + m = 3$). On the other hand, for fixed values of n and m the average soil transported decreases as the Weibull parameter k increases for any value of the threshold speed u_T.

C Formation and Propagation of Sand Dunes

As a result of the processes of saltation and surface creep sand beds start to deform, first into ripples and then into dunes. Various ripples and dunes are also observed to move slowly and regularly across the surface of the desert. Theoretical studies on the deformation of sand beds by wind started with the work of Exner in 1925 using a simple hydraulic model. Various related models were subsequently developed by several authors (Kennedy, 1964; Reynolds, 1965; Engelund, 1970; Engelund and Fredsoe, 1971; Reynolds, 1976 and Richards, 1979), using the equations of hydrodynamics and stability analysis. However, further development in this line of approach has been hampered by the lack of a precise expression for the rate of sand transport. Among all theoretical models studied, Kennedy's potential flow model is the simplest and most widely studied. Its predictions are also in broad agreement with some experimental and observational data. Below we shall describe this model and its predictions.

The model

Kennedy considered an incompressible inviscid flow with uniform horizontal mean velocity $\tilde U$ over a sand surface consisting of non–cohesive particles (see Fig.3 below).

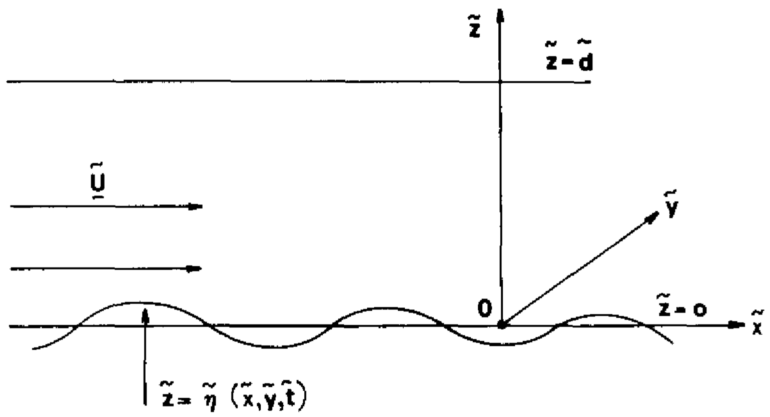

Fig.3

We shall use a Cartesian system of coordinates $O(\tilde x, \tilde y, \tilde z)$ such that $O\tilde x$ is parallel to $\tilde U$ and $O\tilde z$ is in the vertical direction. The topography of the surface is given by

$$\tilde z = \tilde\eta(\tilde x, \tilde y, \tilde t)\ . \tag{9}$$

The medium is bounded above by a rigid lid at a height $\tilde d$, where $|\tilde\eta| \ll \tilde d$. In the regions $\tilde\eta \le \tilde z \le \tilde d$ the flow velocity $\tilde{\underline{u}}$ satisfies the continuity equation

$$\nabla \cdot \tilde u = 0\ . \tag{10}$$

For irrotational motion we can express $\tilde{\underline{u}}$ as

$$\tilde{\underline{u}} = \nabla\, \tilde\Phi \tag{11}$$

in which case (10) becomes

$$\nabla^2\, \tilde\Phi = 0\ . \tag{12}$$

We shall assume that the presence of the surface topography $\tilde z = \tilde\eta(\tilde x, \tilde y, \tilde t)$ causes small perturbations to the basic flow $\underline{\tilde U}$. In this case we can express $\tilde\Phi$ in the form

$$\tilde\Phi = \tilde U\, \tilde x + \tilde\phi \tag{13}$$

where $\tilde\phi$ is the perturbed part of the potential.

Substituting (13) in (12) we find that the potential flow is governed by the Laplacian equation

$$\nabla^2\, \tilde\phi = 0\ . \tag{14}$$

Eq.(14) is to be solved subject to the boundary conditions that the component of velocity normal to the surface at $\tilde z = \tilde d$ and $\tilde z = \tilde\eta$ must be zero. These two boundary conditions can be written as

$$\frac{\partial\tilde\phi}{\partial\tilde z} = 0 \qquad \text{at} \quad \tilde z = \tilde d \tag{15}$$

and

$$\frac{\partial\tilde\phi}{\partial\tilde z} = \frac{\partial\tilde\eta}{\partial\tilde t} + \tilde U\, \frac{\partial\tilde\eta}{\partial\tilde x} + \frac{\partial\tilde\phi}{\partial\tilde x}\,\frac{\partial\tilde\eta}{\partial\tilde x} + \frac{\partial\tilde\phi}{\partial\tilde y}\,\frac{\partial\tilde\eta}{\partial\tilde y} \quad \text{at} \quad \tilde z = \tilde\eta\ . \tag{16}$$

In addition the motion of sediment near the surface $\tilde z = \tilde\eta$ must satisfy the continuity equation

$$\tilde B\, \frac{\partial\tilde\eta}{\partial\tilde t} + \nabla\cdot\underline{\tilde G} = 0 \quad \text{at} \quad \tilde z = \tilde\eta \tag{17}$$

where $\tilde G$ is the rate of sediment transport per unit volume. B is the bulk specific weight of the sediment. Assuming that the sediment is predominantly transported in the direction of the mean flow $\tilde U$, we can reduce (17) to the simpler relation

$$\tilde B\, \frac{\partial\tilde\eta}{\partial\tilde t} + \frac{\partial\tilde G}{\partial\tilde x} = 0 \quad \text{at} \quad \tilde z = \tilde\eta\ . \tag{18}$$

To close the problem we must adopt one of the empirical relations for the rate of sediment transport $\tilde q$ as given in (2). Following Kennedy, 1964, we find it convenient to use the relations

$$\tilde G = \tilde m(\tilde u - \tilde U_c)^n = \tilde m\left(\tilde U + \frac{\partial\tilde\phi}{\partial x} = \tilde U_c\right)^n \tag{19}$$

where $\tilde U_c$ in the threshold speed, $\tilde m$ is a dimensional coefficient and n is a dimensionless exponent.

Next we find it convenient in the analysis below to cast equations (14), (15), (16) and (18) in dimensionless forms. Taking $\tilde U, \tilde d$ and $\tilde d/\tilde U$ to be units of velocity, distance and time, respectively, these equations assume the dimensionless forms:

$$\nabla^2\,\phi \;=\; 0 \tag{20}$$

$$\frac{\partial\phi}{\partial z} \;=\; 0 \qquad \text{at} \quad z = 1 \tag{21}$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial x} + \frac{\partial\phi}{\partial x}\,\frac{\partial\eta}{\partial x} + \frac{\partial\phi}{\partial y}\,\frac{\partial\eta}{\partial y} \quad \text{at} \quad z = \eta \tag{22}$$

$$\frac{\partial\eta}{\partial t} + H\,\frac{\partial T}{\partial x} = 0 \quad \text{at} \quad z = \eta \tag{23}$$

where $H = \dfrac{\tilde m(\tilde U - \tilde U_c)^n}{\tilde U \tilde B \tilde d}$ is a dimensionless constant and T is the dimensionless volume rate of sediment transport

$$T = \left\{1 + \frac{\partial\phi}{\partial x}\Big/(1 - U_c)\right\}^n\ . \tag{24}$$

The non-linear parts of the boundary condition (22) cause considerable complications in finding solutions to the full system of equations (20)–(24). In the following two subsections we shall first consider the linearized system and carry out a simple linear stability analysis to study the growth (or decay) rate of the amplitude of the perturbed potential. Secondly, we shall study the non-linear evolution of the system using a multiple scale-technique commonly used in studying non-linear problems in fluid dynamics

Solutions of the linear problems

The linearized form of the equations (20)–(24) is

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0 \qquad \text{in} \qquad 0\le z\le 1 \tag{25}$$

$$\frac{\partial\phi}{\partial z}=0 \qquad \text{at} \qquad z=1 \tag{26}$$

$$\frac{\partial\phi}{\partial z}=\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial x} \qquad \text{at} \qquad z=0 \tag{27}$$

$$\frac{\partial\eta}{\partial t}+H_c\frac{\partial\phi}{\partial x}=0 \qquad \text{at} \qquad z=0 \tag{28}$$

where $H_c=\dfrac{Hn}{1-U_c}$.
In obtaining (28) we used the following series expansion for (24)

$$T=\left\{1+\frac{n}{1-U_c}\frac{\partial\phi}{\partial x}+O\left(\frac{\partial\phi}{\partial x}\right)^2\right\}\ . \tag{29}$$

In his linear stability analysis Kennedy evaluated the horizontal perturbed speed of the flow, $\dfrac{\partial\phi}{\partial x}$, at $x-\delta$ where δ defines the dimensionless distance by which the local sediment transport lags behind the local velocity of the bed. We shall see later that, while δ plays a key role in the linear stability theory, it has no effect in the non-linear solutions obtained in the last part of this section.

To study the stability of the linear system (25)–(28) we shall assume that the surface and the flow potential have the forms

$$\left.\begin{aligned}\eta &= A(t)\ e^{ik(x-U_bt)}\\ \phi &= B(z,t)\ e^{ik(x-U_bt)}\end{aligned}\right\} \tag{30}$$

where U_b is the speed of the bed, k is the wave number and A is the amplitude to be determined. Eq.(25) then reduces to

$$\frac{\partial^2 B}{\partial z^2}-k^2B=0 \tag{31}$$

whose solution can be written as

$$B(z,t)=C(t)\cosh(kz)+D(t)\sinh(kz) \tag{32}$$

Applying the boundary condition (26) to (32) gives

$$B(z,t)=C(t)\{\cosh(kz)-\lambda\sinh(kz)\},\quad \lambda=\frac{\sinh(k)}{\cosh(k)}\ . \tag{33}$$

We now apply the boundary condition (27) to get

$$-\lambda k\ C(t)=\frac{dA}{dt}-ik\ U_bA+ik\ A=\frac{dA}{dt}+ik\ A(1-U_b)\ .$$

Thus

$$\phi=\frac{1}{\lambda k}\left\{\frac{dA}{dt}+ik\ A(1-U_b)\right\}\{\lambda\sinh(kz)-\cosh(kz)\}\ e^{ik(x-U_bt)}\ . \tag{34}$$

The differential equation for the amplitude $A(t)$ can now be obtained by applying the last boundary condition (28). The result is

$$\frac{dA}{dt}=(P+iQ)A \tag{35}$$

where

$$P = \frac{k\ H_c}{\lambda}\cos(k\delta)\Big/\left(1+\frac{H_c^2}{\lambda^2}-2\frac{H_c}{\lambda}\sin(k\delta)\right) \tag{36}$$

$$Q = k\left[-U_b+\frac{H_c^2}{\lambda^2}(1-U_b)+\frac{H_c}{\lambda}(2U_b-1)\sin(k\delta)\right]\Big/\left(1+\frac{H_c^2}{\lambda^2}-2\frac{H_c}{\lambda}\sin(k\delta)\right)\ . \tag{37}$$

Since the oscillatory time dependence of η has already been prescribed by Eq.(30), we can set $Q=0$ to get the following equation for U_b

$$U_b=\left[\frac{H_c^2}{\lambda^2}(1-U_b)+\frac{H_c}{\lambda}(2U_b-1)\sin(k\delta)\right]\Big/\left(1+\frac{H_c^2}{\lambda^2}-2\frac{H_c}{\lambda}\sin(k\delta)\right)\ . \tag{38}$$

The solution of (35) then takes the form

$$A(t)=A_0\ e^{Pt}\ . \tag{39}$$

As expected from the linear stability theory Eq.(39) shows that a small disturbance at an initially flat sand surface can cause the amplitude to grow ($P>0$) or decay ($P<0$). The maximum growth is reached for $\delta=\dfrac{L\pi}{2k}$ ($L=1,3,5\ldots$), giving the value A_0 for the maximum amplitude. These values of δ correspond to neutral stability. For a detailed analysis of the various bed forms predicted by the linear theory the reader is referred to the review articles by Kennedy (1964) and Eltayeb *et.al.* (1985).

Solution of the non-linear system

A solution of the non-linear set of equations (20)–(24) can be derived using the method of multiple scales commonly used in studying non-linear systems. The details of the derivation is very cumbersome and is therefore not reproduced here. The reader is referred to the paper by Eltayeb and Hassan (1981) for these details.

The small expansion parameter required for the multiple scale technique is defined here as

$$\epsilon = \tilde{h}/\tilde{d} \tag{40}$$

where $\tilde{h}$ is the dimensional amplitude of the surface undulations and $\tilde{d}$ is the height of the layer.

We now express η and ϕ in powers of ϵ as follows:

$$\eta = \sum_{j=n}^{\infty} \sum_{n=0}^{\infty} \epsilon^j \left(\eta_{nj}\, E^n + \bar{\eta}_{nj}\, E^{-n}\right) \tag{41}$$

$$\phi = \sum_{j=n}^{\infty} \sum_{n=0}^{\infty} \epsilon^j \left(\phi_{nj}\, E^n + \bar{\phi}_{nj}\, E^{-n}\right) . \tag{42}$$

The coefficients η_{nj} are functions of X, Y, τ and ϕ_{nj} are functions of X, Y, z, τ. The variables X, Y and τ are defined by

$$\left.\begin{aligned} X &= \epsilon\,(x - U_g t),\ Y = \epsilon\, y,\ \tau = \epsilon^2\, t; \\ E &= \exp ik(x - U_b t) \end{aligned}\right\} \tag{43}$$

and the group velocity U_g is defined by

$$U_g = \frac{\partial}{\partial k}\,(k\, U_b) . \tag{44}$$

Eltayeb et al. (1985) have shown that the scaling (43) is the only system that provides non-trivial and well-posed set of equations.

Substituting expressions (41) and (42) into equations (20)-(24) and equating to zero the coefficients $\epsilon^l E^r (l, r = 0, 1, 2, 3, \ldots)$ we obtain a hierarchy of systems of equations which can be solved seriatim. To close the leading order system we need to consider seven equations resulting from the systems $(l, r) = (0,1), (0,2), (1,1), (1,2), (2,2), (0,3), (1,3)$.

The seven equations can be reduced to three coupled second order partial differential equations for ϕ_{01}, η_{02} and η_{11}, which in turn reduce to two equations upon applying the physical boundary conditions

$$\eta_{02}, \frac{\partial \phi_{01}}{\partial X}, \frac{\partial \phi_{01}}{\partial Y} \longrightarrow 0 \quad \text{as} \quad X^2 + Y^2 \longrightarrow \infty . \tag{45}$$

The two equations can be written as

$$i\,\frac{\partial A}{\partial \tau} + a_1\,\frac{\partial^2 A}{\partial X^2} + a_2\, A \frac{\partial \phi_{01}}{\partial X} + a_3\,\frac{\partial^2 A}{\partial Y^2} + a_4\, A|A|^2 = 0 \tag{46}$$

$$b_1\,\frac{\partial^2 \phi_{01}}{dX^2} + \frac{\partial^2 \phi_{01}}{\partial Y^2} + b_2\,\frac{\partial}{\partial X}\,|A|^2 = 0 \tag{47}$$

where

$$A = -\frac{i\, U_b}{k\, H_c}\,\eta_{11} \tag{48}$$

and $a_1 - a_5, b_1$ and b_2 are constants which depend on k, U_b and U_g.

If we further neglect variations in the Y direction (i.e. set $\frac{\partial}{\partial Y} = 0$) the two equations (46) and (47) reduce to

$$i\,\frac{\partial A}{\partial \tau} + a_1\,\frac{\partial^2 A}{dX^2} + \beta\, A|A|^2 = 0 \tag{49}$$

where $\beta = \left(a_4 - \frac{b_1 a_2}{b_2}\right)$.

Eq.(49) is readily recognized as a non-linear Schrödinger equation. In the theory of water waves this equation has been studied extensively by Hasimoto and Ono (1972).

A simple plane wave solution can easily be derived for the case in which the amplitude A depends on τ only. This leads to the Stokes wave train solution

$$A = A_0\, e^{ip\tau}, \qquad p = \beta\, A_0^2 . \tag{50}$$

It can be shown that (see for example Hasimoto and Ono, 1972) the solution (50) is unstable only if

$$a_1 \beta > 0 . \tag{51}$$

Eq.(49) is also known to have the following solitary wave type solutions in the region $a_1\beta > 0$

$$\text{where} \quad \left.\begin{aligned} A(X,\tau) &= B(X)\, e^{ip\tau}, a_1\beta > 0 \\ B(X) &= \left(\frac{2p}{\beta}\right)^{1/2} sech\left[\left(\frac{p}{a_1}\right)^{1/2} X\right] \end{aligned}\right\} . \tag{52}$$

The solution (52) shows that, in the region of unstable wave train, ripples (or small dunes) of Stokes train type can eventually degenerate into a single dome-shaped solitary dune with amplitude $B(X)$ and width $\left(\frac{a_1}{p\,\epsilon^2}\right)^{1/2}$. Such solitary dunes are well known in certain dryland regions where the sand is formed in small quantities.

D Suspension Transport of Soil Particles

For light sand-sized particles the dominant mode of transport is that of suspension. In this section we shall study a diffusion model appropriate for the transport of particles in suspension. The model is widely used in studying problems of wind erosion and atmospheric pollution (see, e.g. Pasquill, 1962; Gillette and Goodwin, 1974; Hassan and Eltayeb, 1991, 1993; Eltayeb and Hassan, 1992). The model may also be used in studying vertical transport of blowing snow (see, e.g. Takeuchi, 1980; Schmidt, 1982).

The diffusion equation

Consider a Cartesian system of coordinates $O(x, y, z)$ in which Oz is vertically upwards and Ox is horizontal. The concentration of sand particles which simultaneously diffuse and settle under wind action and gravitational force is governed by the diffusion equation (cf. Pasquill, 1962)

$$\frac{\partial c}{\partial t} + \nabla \cdot (c\mathbf{v}) = \nabla \cdot (\mathbf{D} \cdot \nabla c) - \mathbf{w} \cdot \nabla c \tag{53}$$

where $c(x,y,z,t)$ is the concentration of sand, $\mathbf{v}(x,y,z)$ the velocity of the wind, $\mathbf{w}(x,y,z)$ is the settling velocity and $\mathbf{D}(x,y,z)$ is the spatial diffusion tensor.

We shall assume a *steady* state in which:

(i) the wind is unidirectional and varies with height only, i.e.

$$\mathbf{v} = [U(z), 0, 0] \; ; \tag{54}$$

(ii) the sedimentation velocity is a function of particle radius only and directed downwards, i.e.

$$\mathbf{w} = (0, 0, -W) \; ; \tag{55}$$

(iii) the diffusion tensor $D_{ij} = 0$ if $i \neq j$, and

$$D_{11} = D_x, \quad D_{22} = D_y, \quad D_{33} = D_z \; ; \tag{56}$$

(iv) the concentration and the diffusion tensor are homogeneous in the y direction.

The application of the conditions (i)–(iv) to Eq.(53) yields

$$U \frac{\partial c}{\partial x} = \frac{\partial}{\partial x}\left(D_x \frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial z}\left(D_z \frac{\partial c}{\partial z}\right) + W \frac{\partial c}{\partial z} \, . \tag{57}$$

In most practical cases the transport of soil in the direction of the wind (as determined by the expression $U\,\partial c/\partial x$) dominates over turbulent diffusion in the same direction [i.e. the term $\partial(D_x\,\partial c/\partial x)/\partial x$]. This allows a further simplification to (57). Thus

$$U \frac{\partial c}{\partial x} = \frac{\partial}{\partial z}\left(D_z \frac{\partial c}{\partial z}\right) + W \frac{\partial c}{\partial z} \, . \tag{58}$$

Formulation and solution of the diffusion model

We consider a source of dust of infinite length situated on the ground surface along the horizontal line, $0 \leq x \leq \infty$, $z = z_0$ (z_0 being the roughness height). Particles of uniform size and weight are released from the source under the action of a unidirectional wind flow, $U(z)\hat{\mathbf{x}}$. The particles are subsequently transported downwind and are allowed to diffuse and settle under the action of gravity.

The steady–state distribution of the dust particles is governed by the diffusion equation (58).

Assuming for tractability the customary forms for wind speed and eddy diffusivity

$$U(z) = \beta m^m, \quad D_z = \lambda z \; , \tag{59}$$

where β, m and λ are positive constants, we can write Eq.(58) in the form

$$\frac{\partial c}{\partial X} = \frac{\partial^2 c}{\partial Z^2} + \frac{(1+2\nu)}{Z} \frac{\partial c}{\partial Z} \, , \tag{60}$$

where

$$X = \frac{\lambda(m+1)^2}{4\beta}\, x, \quad Z = z^{m+1/2}, \quad \nu = \frac{W}{\lambda(m+1)} \, . \tag{61}$$

We next solve Eq.(60) subject to the boundary conditions

$$\begin{aligned} c(0, Z) &= 0, \\ c(X, \infty) &= 0, \\ c(X, Z_0) &= g(X), \end{aligned} \tag{62}$$

where $Z_0 = z_0^{m+1/2}$ and $g(X)$ is a prescribed analytic function.

An analytic solution of the diffusion Eq.(60) with the boundary conditions (62) can be derived using the method of Laplace transform.

Taking the Laplace transform of (60) in X and using the first boundary condition in (62) we find that the transformed concentration function, $\bar{c}(p,z)$, satisfies the differential equation

$$\frac{\partial^2 \bar{c}}{\partial Z^2} + \frac{(1+2\nu)}{Z} \frac{\partial \bar{c}}{\partial Z} - p\bar{c} = 0 \; , \tag{63}$$

where $\bar{c}(p,Z) = \int_0^\infty c(X,Z) e^{-pX}\, dX$. The solution of (63) which satisfies the second condition in (62) has the form

$$\bar{c}(p, Z) = A(p) Z^{-\nu} K_\nu(p^{1/2} Z) \; , \tag{64}$$

where K_ν is the modified Bessel function of the second type, and $A(p)$ is a constant which can easily be determined using the third boundary condition in (62).

At $Z = Z_0$ we have

$$\begin{aligned} \bar{c}(p, Z_0) &= A(p) Z_0^{-\nu} K_\nu(p^{1/2}, Z_0) \\ &= \bar{G}(p, Z_0) \end{aligned} \tag{65}$$

where $\bar{G}(p, Z_0)$ is the Laplace transform of $g(X, Z_0)$. Eqs.(64) and (65) then give

$$\begin{aligned} \bar{c}(p, Z) &= \left(\frac{Z_0}{Z}\right)^\nu \cdot p\bar{G}(p, Z_0) \cdot \frac{K_\nu(p^{1/2} Z)}{pK_\nu(p^{1/2} Z_0)} \\ &= \left(\frac{Z_0}{Z}\right)^\nu p\bar{G}(p, Z_0) \cdot \bar{H}(p; Z) \, . \end{aligned} \tag{66}$$

In order to invert (66) to obtain $c(X, Z)$ we first consider the case

$$g(X) = 1 \, .$$

Here

$$\bar{G}(p, Z_0) = \frac{1}{p} \tag{67}$$

and the inverse of $\bar{H}(p, Z)$ is given by

$$H(X, Z) = \left(\frac{Z_0}{Z}\right)^\nu + \frac{2}{\pi} \int_0^\infty Q(Z, y; Z_0)\, \frac{e^{-y^2 X}}{y}\, dy \, , \tag{68}$$

in which

$$Q(Z,y;Z_0) = \frac{[J_\nu(yZ)Y_\nu(yZ_0) - J_\nu(yZ_0)Y_\nu(yZ)]}{\{J_\nu(yZ_0)\}^2 + \{Y_\nu(yZ_0)\}^2} . \tag{69}$$

Here $J_\nu(x)$ and $Y_\nu(x)$ are the usual Bessel functions. It follows from (66)-(69) that

$$c(X,Z) = \left(\frac{Z_0}{Z}\right)^\nu \left\{ \left(\frac{Z_0}{Z}\right)^\nu + \frac{2}{\pi}\int_0^\infty Q(Z,y;Z_0)\,\frac{e^{-y^2X}}{y}\,dy \right\} , \tag{70}$$

which satisfies all the conditions (62).

Next we derive an expression $c(X,Z)$ for the concentration of dust particles when $g(X)$ is a differentiable general function.

Here we use the result (68) and the convolution theorem to find that

$$c(X,Z) = \left(\frac{Z_0}{Z}\right)^\nu \int_0^X g'(t)\; H(X-t,z)dt \tag{71}$$

so that

$$c(X,Z) = \left(\frac{Z_0}{Z}\right)^\nu \left\{ \left(\frac{Z_0}{Z}\right)^\nu g(X) + \frac{2}{\pi}\int_0^\infty \times \left[Q(Z,y;Z_0)\,\frac{e^{-y^2X}}{y}\int_0^X g'(t)\,e^{y^2t}dt \right] dy \right\} \tag{72}$$

where the accent denotes differentiation with respect to the argument t.

The analytic expressions for the concentration of dust $c(X,Z)$ obtained here are valid for all values of the (scaled) roughness height Z_0.

When $Z_0 \to 0$ the asymptotic forms of Bessel functions $J_\nu(x), Y_\nu(x)$ (see Abramowitz and Stegun, 1965) can be used to show that (70) and (72) reduce to

$$c(X,Z) \simeq \left(\frac{Z_0}{Z}\right)^{2\nu} \frac{\Gamma(\nu, Z^2/4X)}{\Gamma(\nu)}, \quad Z_0 \to 0 \tag{73}$$

and

$$c(X,Z) \simeq \frac{2^{-2\nu}}{\Gamma(\nu)}\, Z_0^{2\nu} \int_0^X \frac{g(t)e^{-Z^2/4(X-t)}}{(X-t)^{\nu+1}}\,dt, \quad Z_0 \to 0 . \tag{74}$$

where Γ is the Gamma function.

If the concentration is allowed to vary in the vertical direction only (Gillette and Goodwin, 1974) the results (73) can be further simplified by taking the limit $X \to \infty$. We thus find

$$c(\infty,Z) = \left(\frac{Z_0}{Z}\right)^{2\nu} \frac{\Gamma(\nu,0)}{\Gamma(\nu)} = \left(\frac{Z_0}{Z}\right)^{2\nu} . \tag{75}$$

Obviously Eq.(75) can also be deduced from (73) by setting the wind speed equal to zero (i.e. $\beta = 0$) in which case the parameter

$$\frac{Z^2}{4X} = \frac{\beta Z}{\lambda x} = 0 .$$

It should be noted that, since the factor $\frac{\Gamma(\nu,Z^2/4X)}{\Gamma(\nu)}$ is less than one, the two-dimensional result (73) gives lower values for the concentration of soil particles than those predicted by the one-dimensional result (75).

The analytic expression (72) has been evaluated numerically to illustrate the behaviour of the concentration $c(X,Z)$ as a function of X and Z for various values of the roughness height Z_0. The results are summarized in Figs.4 and 5.

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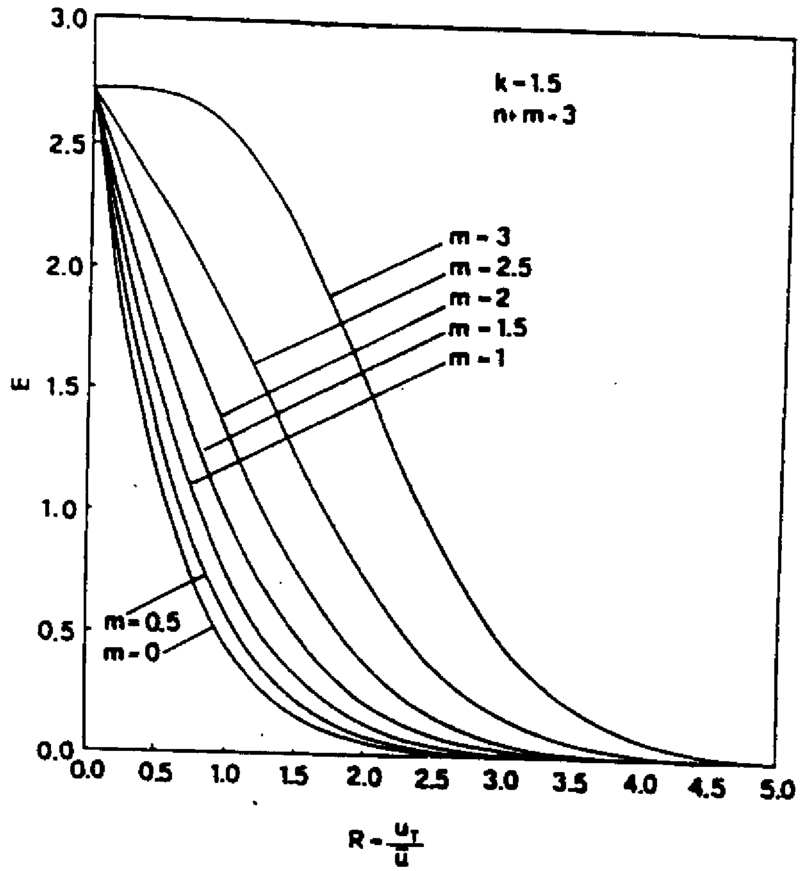

Fig.1 The average horizontal flux E plotted against R, where R is equal to the threshold speed/average wind speed for a fixed value of the Weibull parameter k and various values of m.

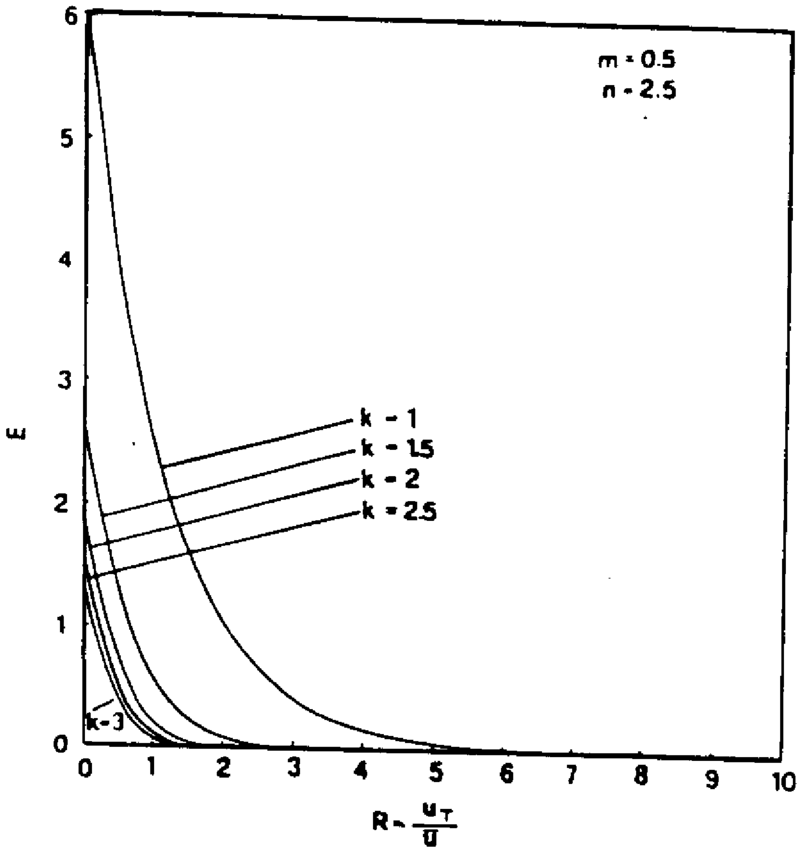

Fig.2 The average horizontal flux E plotted against R for various values of the Weibull parameter k.

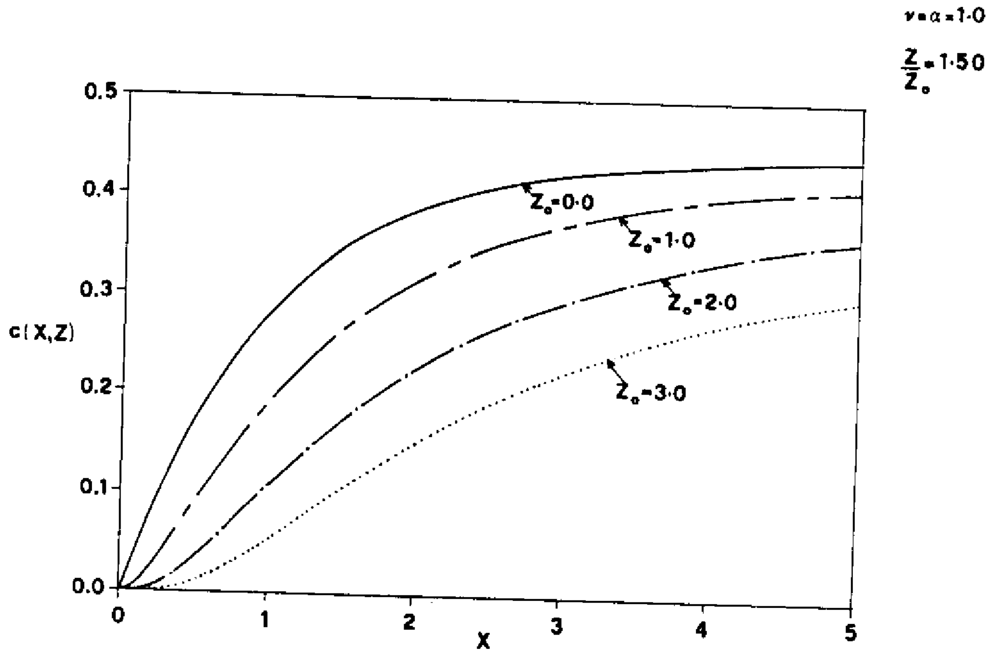

Fig.4 The concentration $c(X,Z)$ as a function of X when $Z - 0$ takes the values 0.0, 1.0, 2.0 and 3.0 and $Z/Z_0 = 1.5$. Here $\nu = \alpha = 1.0$ and the source strength is given by: $g(X) = 1 - e^{-X/\alpha}$.

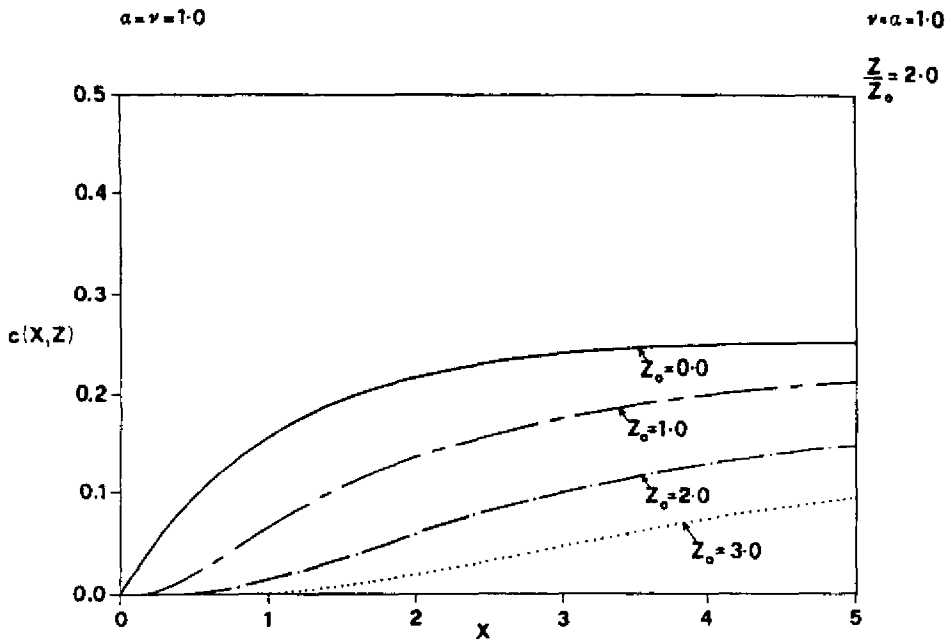

Fig.5 The concentration $c(X, Z)$ as a function of X when $Z - 0$ takes the values 0.0, 1.0, 2.0 and 3.0 and $Z/Z_0 = 2.0$.