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JACOBSON RADICALS IN MATRIX NEAR RINGS

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ABSTRACT

This paper was motivated by the question: Does, like the ring case, the process of taking Jacobson Radicals and constructing matrix near rings coincide?

We present here, for the class of weakly distributive d.g. near rings, a conditionally affirmative answer to this question. Our main result is as follows:

Let R be a weakly distributive d.g. near ring with identity. If $\overline{J(R)} \supseteq \delta_1 M_n(R)$, the derived group of $M_n(R)$, then $\overline{J(R)} = J(M_n(R))$ where $\bar{I}$ is defined as a subnear ring of $M_n(R)$ generated by $\{f_{ij}^a; a \in I, 1 \leq i, j \leq n\}$.

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1 Introduction

We call (R,+,.) a right near ring if (R,+) is a group which is not necessarily abelian, (R,.) is a semigroup and $(x+y)z = xz + yz \ \forall \ x, y, z \in R$. A normal subgroup (I,+) of (R,+) is an ideal of R if $IR \subseteq I$ and $x(a+y) - xy \in I \ \forall \ x, y \in R$ and $a \in I$. A group G is an R-module if R is homomorphic to M(G), the set of all functions from G to itself. An R-module G is monogenic if there exits g in G such that $Rg = G$. A monogenic R-module is of type 0 if G is simple; it is of type 1 if G is simple and for all g in G either $Rg = G$ or $Rg = \{0\}$; it is of type 2 if G has no non-trivial proper R-submodule. For $\nu = 0, 1, 2$, we define the ν-radical of R as $J_\nu(R) = \{Ann_R G$; G is an R-module of type $\nu\}$. If R has an identity element then $J_1(R) = J_2(R)$.

R is called distributively generated (d.g. in short) if (R,+) is generated as a group by (S,.), a semigroup of distributive elements of R. For d.g. near rings, we define the distributor series by $D^0(I) = I, D^{j+1}(I) = \text{Gp} < (R : D^j(I), D^j(I) >^R$ where $< X >^R$ is the normal subgroup of R generated by X and $(R : X, Y) =< (r : x, y) = r(x+y) - ry - rx$; $r \in R, x \in X, y \in Y >$. R is called weakly distributive (w.d. in short) near ring if $D^n(R) = \{0\}$ for some integer n. For w.d. near rings, all three radicals coincide.

All these results are available in Meldrum [3].

$M_n(R)$, the near ring of n x n matrices, is defined as a subnear ring of $M(R^n)$, generated by the set $\{f_{ij}^r : r \in R, 1 \leq i, j \leq n\}$ where $R^n = \oplus_1^n (R, +)$, the direct sum of n copies of (R,+), $f_{ij}^r = \iota_i f^r \pi_j$ with $f^r(x) = rx \ \forall \ x \in R$, ι_j and π_j are j-th co-ordinate injection and projection functions respectively. If R is d.g. by S then $M_n(R)$ is d.g. by $\{f_{ij}^s : s \in S, 1 \leq i,j \leq n\}$, and every matrix in $M_n(R)$ can be expressed as a sum of matrices of the form f_{ij}^s and their additive inverses. $\overline{I}$ is defined as a subnear ring of $M_n(R)$, generated by the set $\{f_{ij}^a : a \in I, 1 \leq i,j \leq n\}$ and I^+ is defined as an ideal generated by $\{f_{ij}^a : a \in I, 1 \leq i,j \leq n\}$. We define $I^* = (I^n : R^n) = \{X \in M_n(R) : X\alpha \in I^n \; \forall \, \alpha \in R^n\}$. I^* is an ideal of $M_n(R)$ and $I^+ \subseteq I^*$. See Meldrum and van der Walt [4], van der Walt [6] and Abbasi, Meldrum and Meyer [2] for further details.

2 Main Results

We first collect all the results of our interest which are available in van der Walt [5] and Abbasi, Meldrum and Meyer [1] and [2] respectively.

2.1 Lemma

If R is a near ring with identity, then $(J_2(R))^* = J_2(M_n(R))$.

2.2 Lemma

If R is a w.d. near ring with identity, then $J(R)^* = J(M_n(R))$.

2.3 Lemma

If R is a d.g. near ring with identity and $I^+ \supseteq \delta_1(M_n(R))$, *then* $I^+ = I^*$.

Before extending the preceding result to get our key lemma, we first present this result.

2.4 Lemma

If R is a d.g. near ring with identity and $\overline{I} \supseteq \delta_1(M_n(R))$, *then* $\overline{I}$ *is an ideal of* $M_n(R)$.

Proof: The fact that $(\overline{I}, +)$ is a normal subgroup of $M_n(R)$ follows from group theory and the rest can be checked by using the hypothesis, theorem 1.2 of Abbasi, Meldrum and Meyer [2], induction on weight of matrices and matrix computation rules given in lemma 3.1 of Meldrum and van der Walt [4].

2.5 Lemma

Let R be a d.g. near ring with identity and $\overline{I} \supseteq \delta_1(M_n(R))$ *then* $\overline{I} = I^*$.

Proof: This follows from lemmas 2.3 and 2.4 as $\overline{I} = I^+$.

We can also prove this result independently by using exactly the same technique of proof as that of lemma 2.3.

Now preceding result together with lemma 2.1 gives us

2.6 Theorem

If R is a d.g. near ring with identity and $\overline{J_2(R)} \supseteq \delta_1(M_n(R))$ *then* $\overline{J_2(R)} = J_2(M_n(R))$.

Our main result is an immediate consequence of lemmas 2.5 and 2.2.

2.7 Theorem

Let R be a w.d. near ring with identity. If $\overline{J(R)} \supseteq \delta_1(M_n(R))$ *then* $\overline{J(R)} = J(M_n(R))$.

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