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ABSTRACT

In this paper some generalized four parameter phase transformations of a Dirac spinor are considered. It is shown that a corresponding compensating transformation of the electromagnetic field which restores the invariance of the Dirac–Maxwell equation might exist, provided some consistency conditions are satisfied by the parameters of the transformations. These transformations are used further to consider the Maxwell equations under the assumption that a Bosonization takes place. Only one of the considered cases proves to have a solution (the other cases show to be trivial) which although unphysical is obtained explicitly.

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1 Introduction

In this paper we would like to point on the possible invariance of the classical Dirac-Maxwell spinor equation under the action of four-parameter Abelian local phase transformations. These transformations must generalize the usual $U(1)$ local gauge transformations of spinor electrodynamics in a sense that they should not only alter the mutual phase of all four spinor components, but also reflect on the phases of these components separately. As a result the corresponding generalized "gauge" transformation of the electromagnetic field shall contain terms that do not keep invariant the electromagnetic field strength tensor unlike the usual gauge invariance.

These generalized phase transformations of the spinor field will have the form

$$\Psi'(x) = \Phi(x)\psi(x), \tag{1}$$

where $\Phi(x) \in GL(4,C)/GL(4,R)$ (in general) is the phase matrix, while $\psi(x)$ and $\Psi'(x)$ are classical, i.e commuting Dirac spinors. This formula was used in paper [1] to introduce a polar decomposition of a Dirac spinor by setting $\psi(x)$ to be a free Majorana spinor rather than a Dirac one. The latter proved to be usefull in combination with the Bose ansatz (i.e. making the free Majorana spinor to be a constant). Thus an exact solution of the equations of classical spinor electrodynamics was obtained in paper [2].

It seems however that the solution obtained is neither due to the polar decomposition, nor to the Bosonization, but rather to the phase matrix itself. The reason is that in a sense it can be considered (just formally) as an analogue of the nonlinear operator, which was introduced in the papers of Flato, Simon, Pinczon and Taflin [3][4][5][6] in order to factorize the solution of a nonlinear equation as a product of this nonlinear operator and a field representing the initial conditions (not necessarily the free field initial conditions). All this is motivating us to consider such generalized phase transformations.

The fact that we will consider classical spinor fields should not be confusing. Indeed even in quantum theory the quantized spinor field is represented as a sum of products of the creation (annihilation) operators, which are the anticommuting quantities, and the polarization functions (the spin-state functions). The latter functions are in fact classical spinors and carry the essential information on the spin state and the Lorentz transformation properties. From that point of view our construction can be used in the quantized theory too.

For our purposes it is convenient to represent the Dirac spinor as a sum of a real and imaginary parts

$$\psi(x) = \chi(x) + i\eta(x), \tag{2}$$

where $\chi(x)$ and $\eta(x)$ are two Majorana (i.e. real) spinors. The reason is that the calculations are extremely simplified if a specific, based on the properties of the classical Majorana spinors, representation of the $gl(4,C)$ algebra is used (see [7]). The main features of this representation shall be briefly sketched in section 2.

In section 3 we shall discuss the generalized phase transformations of spinors in connection to the classical Dirac-Maxwell spinor equation. We shall show that these transformations split that equation into an algebraic system, which determines the compensating transformation of the electromagnetic field (thus restoring the invariance), and a system of differential conditions (in general nonlinear) on the parameters of the phase transformation, which keeps the information on the initial fields. The latters are the consistency conditions for the Dirac-Maxwell equation to be invariant under the generalized phase transformations.

In section 4 we shall consider very briefly these differential consistency conditions for some particular choice of the initial spinor and electromagnetic fields.

Finally in section 5 we shall consider the Maxwell equation for one particular and rather extreme choice of the initial fields. This case (actually we make use of the Bosonization ansatz) has the advantage that it allows the analysis to be performed to the very end, i.e. to obtain an exact, although unphysical and quite unusual, solution of the whole system of equations.

2 A simple representation of the $gl(4, C)$ algebra in the Majorana spinor space

In what follows the metric is of signature $g_{\mu\mu} = (+, -, -, -)$ and the Levi-Civita tensor is normalized by the condition $\varepsilon_{0123} = -\varepsilon^{0123} = -1$. The $\gamma-$matrices are in the Majorana representation (i.e. all imaginary). In that representation $\gamma^0\gamma^\mu$ and $\gamma^0\sigma^{\mu\nu}$ are symmetric while γ^0, $\gamma^0\gamma^5$ and $\gamma^0\gamma^\mu\gamma^5$ are antisymmetric.

The representation which we shall use is defined in the Majorana spinor space. For this reason we shall briefly sketch some of the properties of Majorana spinors. A complete basis in that space is given by the spinors $\chi, \eta, \chi_5 = \gamma_5\chi$, and $\eta_5 = \gamma_5\eta$, provided the first two of them satisfy the condition

$$\eta \neq (a + b\gamma_5)\chi, \tag{3}$$

where a and b are real scalar functions[1]. This condition ensures that the following two bilinear forms (scalar and pseudoscalar)

$$\rho \equiv i\overline{\chi}\eta \qquad \rho_5 \equiv i\overline{\chi}\gamma_5\eta \tag{4}$$

cannot both be zero. As we shall see this is enough to provide the completeness of the basis.

Now we shall introduce the vector bilinear forms[2] of χ and η

$$J_\mu = \overline{\chi}\gamma_\mu\chi, \quad H_\mu = \overline{\chi}\gamma_\mu\eta = \overline{\eta}\gamma_\mu\chi, \tag{5}$$

$$\widetilde{J}_\mu = \overline{\eta}\gamma_\mu\eta, \quad H_{5\mu} = \overline{\chi}\gamma_\mu\gamma_5\eta = -\overline{\eta}\gamma_\mu\gamma_5\chi \tag{6}$$

and give a brief summary of their properties. Using the Fierz decomposition [8] of the product of two classical (i.e. commuting) spinors in terms of $\gamma-$matrices we can see that these four vectors form an orthonormal light-cone tetrad, i.e.

[1] Clearly, this is no more than a slight extension of the linear independance condition for the spinor case.

[2] The second pair of Majorana spinors χ_5 and η_5 does not add new bilinear forms.

$$J^\mu \widetilde{J}_\mu = J^\mu H_\mu = J^\mu H_{5\mu} = \widetilde{J}^\mu H_\mu = \widetilde{J}^\mu H_{5\mu} = H^\mu H_{5\mu} = 0, \tag{7}$$

$$J^\mu \widetilde{J}_\mu = 2R^2, \qquad H^\mu H_\mu = H^{5\mu} H_{5\mu} = -R^2, \tag{8}$$

where

$$R^2 = \rho^2 + \rho_5^2. \tag{9}$$

Similarly we can prove that these vectors satisfy the completeness condition

$$\frac{1}{2}\{J_\mu \widetilde{J}_\nu + \widetilde{J}_\mu J_\nu\} - H_\mu H_\nu - H_{5\mu} H_{5\nu} = g_{\mu\nu} R^2. \tag{10}$$

So we see that ρ and ρ_5 should not be simultaneously zero, otherwise $R = 0$ and the two light-cone vectors would be colinear, while H_μ and $H_{5\mu}$ would be either colinear to the only existing in that case light-cone vector, or zero.

Now we can introduce the normalized vectors

$$2j_\mu = R^{-1} J_\mu, \quad \widetilde{j}_\mu = R^{-1} \widetilde{J}_\mu, \quad h_\mu = R^{-1} H_\mu, h_{5\mu} = R^{-1} H_{5\mu} \tag{11}$$

and apply again the Fierz formula to obtain the following identities

$$\not{j}\chi = 0, \qquad \not{j}\eta = -2ie^{\varphi_5\gamma_5}\chi, \tag{12}$$

$$\widetilde{\not{j}}\eta = 0, \qquad \widetilde{\not{j}}\chi = 2ie^{\varphi_5\gamma_5}\eta, \tag{13}$$

$$\not{h}\chi = ie^{\varphi_5\gamma_5}\chi, \qquad \not{h}_5\chi = -ie^{\varphi_5\gamma_5}\chi_5, \tag{14}$$

$$\not{h}\eta = -ie^{\varphi_5\gamma_5}, \qquad \not{h}_5 = -ie^{\varphi_5\gamma_5}\eta_5, \tag{15}$$

where $\not{v} = v^\mu \gamma_\mu$ as usually and the function φ_5 is defined by

$$\rho = R\cos\varphi_5 \qquad \rho_5 = R\sin\varphi_5. \tag{16}$$

Actually, we can always put $\varphi_5 = 0$ without any loss of generality. Indeed, consider the transformations

$$\chi \to \chi' = e^{-\frac{1}{2}\varphi_5\gamma_5}\chi, \quad \eta \to \eta' = e^{-\frac{1}{2}\varphi_5\gamma_5}\eta.$$

Clearly, these transformations keep the vector bilinear forms invariant and act on ρ and ρ_5 only, i.e.

$$\rho \to \rho' = R, \qquad \rho_5 \to \rho_5' = 0.$$

In what follows we shall consider $\varphi_5 = 0$ for simplicity.

Now it is clear from the completeness condition (10) and the normalization condition (11) that γ_μ can be represented in the form

$$\gamma_\mu = \frac{1}{2}\left[j_\mu \widetilde{\not{j}} + \widetilde{j}_\mu \not{j}\right] - h_\mu \not{h} - h_{5\mu} \not{h}_5. \tag{17}$$

Then having in mind the identities (12)-(15) and the fact that we have set $\varphi_5 = 0$ we obtain for the action of γ_μ on the spinors χ and η

$$\gamma_\mu \chi = i\{j_\mu \eta - h_\mu \chi + h_{5\mu} \chi_5\}, \tag{18}$$

$$\gamma_\mu \eta = -i\{\tilde{j}_\mu \chi - h_\mu \eta - h_{5\mu} \eta_5\}. \tag{19}$$

The action on χ_5 and η_5 is readily obtained from the above two formulae.

We see that the right hand sides of (18) and (19) are linear combinations of the spinors χ, η, χ_5 and η_5. And since any of the $gl(4, C)$ matrices can be expressed in terms of the γ-matrices and their products we can argue that the action of any $gl(4, C)$ matrix on these Majorana spinors would lead again to a linear combination of these spinors. This proves that χ, η, χ_5 and η_5 form endeed a complete basis of the Majorana spinor space and actually formulae (18) and (19) are defining an irreducible representation of the $gl(4, C)$ algebra in this space.

Now we shall for convenience introduce a new orthonormal vector tetrad through the expressions

$$k_\mu = \frac{1}{2}(j + \tilde{j})_\mu, \quad l_\mu = \frac{1}{2}(j - \tilde{j})_\mu, \quad m_\mu = h_\mu, \quad n_\mu = h_{5\mu}. \tag{20}$$

Obviously this is not a light-cone tetrad already[3]. Then we define

$$J_1 = k_{[\mu} l_{\nu]} \sigma^{\mu\nu}, \quad {}^*J_1 = J_1 \gamma_5, \quad J_1^2 = -{}^*J_1 = -1, \tag{21}$$

$$J_2 = k_{[\mu} m_{\nu]} \sigma^{\mu\nu}, \quad {}^*J_2 = J_2 \gamma_5, \quad J_2^2 = -{}^*J_2^2 = -1, \tag{22}$$

$$J_3 = k_{[\mu} n_{\nu]} \sigma^{\mu\nu}, \quad {}^*J_3 = J_3 \gamma_5, \quad J_3^2 = -{}^*J_3^2 = -1, \tag{23}$$

where the square brackets denote the antisymmetrization of the corresponding indices. What is important for our purposes is that the set of matrices 1, γ_5, J_k and ${}^*J_k$ (for fixed, but arbitrary $k = 1, 2, 3$) forms an abelian subalgebra of $gl(4, C)$. And this is exactly what we shall use to construct the phase matrix of the generalized phase transformations, which we shall discuss in the next section. For this reason we write down the explicit formulae for the action of these matrices on the basis spinors in the Majorana space. We have

$$J_1 \chi = -i\chi, \quad J_1 \eta = i\eta, \quad {}^*J_1 \chi = -i\chi_5, \quad {}^*J_1 \eta = i\eta_5, \tag{24}$$

$$J_2 \chi = -i\eta, \quad J_2 \eta = -i\chi, \quad {}^*J_2 \chi = -i\eta_5, \quad {}^*J_2 \eta = -i\chi_5, \tag{25}$$

$$J_3 \chi = -i\eta_5, \quad J_3 \eta = i\chi_5, \quad {}^*J_3 \chi = i\eta, \quad {}^*J_3 \eta = -i\chi. \tag{26}$$

Now we are ready to proceed further to the construction of the generalized phase transformations.

[3]The reason to avoid the light-cone tetrad in some cases becomes clear from formulae (21)-(23). If the tetrad is a light-cone one the squares of the matrices J_k would be zero in two of the free cases. However the existence of such squares would be essential for us in order to construct nontrivial exponentials of these matrices.

3 The Dirac–Maxwell spinor equation and the generalized phase transformations

Let us consider the Dirac–Maxwell classical spinor equation

$$i \not\partial\psi + m\psi + e \not A\psi = 0, \tag{27}$$

where A_μ is the electromagnetic field[4] interacting with the Dirac spinor field ψ. Now let us decompose this equation into two equations for the real and the imaginary parts of the Dirac spinor (χ and η, respectively). So we have

$$i \not\partial\chi = -m\chi - e\{\widetilde{A}\chi - [B + B_5\gamma_5]\eta\}, \tag{28}$$

$$i \not\partial\eta = -m\eta - e\{A\eta - [B - B_5\gamma_5]\chi\}, \tag{29}$$

where we have introduced the notations

$$A = j^\mu A_\mu, \quad \widetilde{A} = \widetilde{j}^\mu A_\mu, \quad B = h^\mu A_\mu, \quad B_5 = h_5^\mu A_\mu. \tag{30}$$

To obtain the right-hand sides of eqs. (28) and (29) we have made use of formulae (18) and (19).

Now let us consider a phase matrix $\Phi(x) \in GL(4,C)/GL(4,R)$. Following the analysis of paper [1] we shall impose the additional condition[5]

$$\Phi^* = \Phi^{-1} \tag{31}$$

where "*" stands for complex (not Hermitian) conjugation. To ensure this condition it is sufficient to ask that all four generators of the considered four-parametric phase transformation commute with each other. As pointed in the previous section a particular choice of a set of such generators is given by 1, γ_5, iJ_k and i^*J_k ($k = 1,2,3$). In what follows we shall consider just these three cases.

3.1 Case A. Phase matrix generators 1, γ_5, iJ_1 and i^*J_1

Let us consider a phase matrix of the form

$$\Phi = \exp\{ie(\phi + \phi_5\gamma_5) + e(\omega + \omega_5\gamma_5)J_1\}, \tag{32}$$

where e is the electric charge and ϕ, ϕ_5, ω, and ω_5 are some real functions of the space-time coordinates, which are so far arbitrary. The field equations shall impose some constraints on these functions, which we shall call consistency conditions. Now having in mind formulae (24) the following explicit form of the transformed spinor field is obtained

$$\Psi' = e^{ie(\phi+\phi_5\gamma_5)}\left\{e^{-ie(\omega+\omega_5\gamma_5)}\chi + ie^{ie(\omega+\omega_5\gamma_5)}\eta\right\} \tag{33}$$

[4] In what follows the space-time argument will be omited for simplicity.

[5] Actually this condition is needed in order to ensure that such a matrix can transform a Majorana spinor into a Dirac one.

So we see that two independant phases $\phi \pm \omega$ for the spinors χ and η appear explicitly. The remaining two parameters $\phi_5 \pm \omega_5$ are to be related to the phases of χ_5 and η_5 although this is not explicit.

What we can see, however, is that now the matrix structure of the exponents is extremely simplified and the matrix part of the operators $i\gamma^\mu \partial_\mu$ and $A^\mu \gamma_\mu$ will readily commute with these exponents (just changing the sign of the γ_5 coefficients) to come and act on the Majorana spinors, on which its action is already known. So we have

$$
\begin{aligned}
e \not{\mathcal{A}} \Psi' &= \epsilon\, \epsilon^{ie(\phi - \phi_5 \gamma_5)} \\
&\times \left\{ \left[\widetilde{\mathcal{A}} e^{ie(\omega - \omega_5\gamma_5)} - i(\mathcal{B} - \mathcal{B}_5 \gamma_5) e^{-ie(\omega - \omega_5 \gamma_5)} \right] \chi \right. \\
&\left. + i \left[\mathcal{A} e^{-ie(\omega - \omega_5\gamma_5)} + i(\mathcal{B} + \mathcal{B}_5\gamma_5) e^{ie(\omega - \omega_5\gamma_5)} \right] \eta \right\}
\end{aligned} \tag{34}
$$

where $\mathcal{A}$, $\widetilde{\mathcal{A}}$, $\mathcal{B}$ and $\mathcal{B}_5$ are the projections of the transformed electromagnetic field $\mathcal{A}_\mu$ on the directions of the light-cone tetrad vectors (11), which are defined according to eq. (30).

Now we shall act on the transformed spinor field Ψ' with the differential matrix operator $i \not{\partial}$. We have

$$
\begin{aligned}
i \not{\partial} \Psi' &= -\epsilon^{ie(\phi - \phi_5)} \\
&\times \left\{ \epsilon^{-ie(\omega - \omega_5\gamma_5)} \right] \left[\epsilon\, \partial^\mu (\phi - \omega - (\phi_5 - \omega_5)\gamma_5) \gamma_\mu \chi - i \not{\partial} \chi \right] \\
&+ i e^{ie(\omega - \omega_5\gamma_5)} \left[\epsilon \partial^\mu (\phi + \omega - (\phi_5 + \omega_5)\gamma_5) \gamma_\mu \eta - i \not{\partial} \eta \right] \right\} .
\end{aligned} \tag{35}
$$

We recall that the initial Majorana spinors χ and η are supposed to satisfy the Dirac–Maxwell equations (28)–(29). This allows to replace their derivatives in the above formula with the corresponding r.h.s of eqs. (28)–(29). Then we use again formulae (18)–(19) to obtain the coefficients of χ and η. Finally we substitute the expressions (34) and (35) in eq. (27) and determine again the coefficients of χ and η. Once these final coefficients obtained they should be equated to zero[6]. So we obtain the set of equations

$$
\begin{aligned}
& i(\mathcal{B} - \mathcal{B}_5\gamma_5) + \widetilde{\mathcal{A}} - i(\nabla - \nabla_5\gamma_5)\left[\phi - \omega - (\phi_5 - \omega_5)\gamma_5\right] + \frac{m}{e}\left[1 - e^{2ie(\phi_5 - \omega_5)\gamma_5}\right] \\
& - \; e^{2ie(\omega - \omega_5\gamma_5)} \left\{ \widetilde{\mathcal{A}} + i(\mathcal{B} - \mathcal{B}_5\gamma_5) - \widetilde{\Delta}(\phi + \omega - (\phi_5 + \omega_5)\gamma_5) \right\} = 0,
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
& i(\mathcal{B} + \mathcal{B}_5\gamma_5) - \mathcal{A} - i(\nabla + \nabla_5\gamma_5)\left[\phi + \omega - (\phi_5 + \omega_5)\gamma_5\right] - \frac{m}{e}\left[1 - e^{2ie(\phi_5 + \omega_5)\gamma_5}\right] \\
& + \; e^{-2ie(\omega - \omega_5\gamma_5)} \left\{ \mathcal{A} - i(\mathcal{B} + \mathcal{B}_5\gamma_5) - \Delta(\phi - \omega - (\phi_5 - \omega_5)\gamma_5) \right\} = 0,
\end{aligned} \tag{37}
$$

[6]The spinors χ and η are linearly independant according to condition (3) and after the use of formulae (28)–(29) and (18)–(19) the matrix structure of eqs. (34) and (35) would contain only the unit matrix and γ_5, which are incapable of course to itertwin between χ and η.

where

$$\Delta = j^\mu \partial_\mu, \quad \widetilde{\Delta} = \widetilde{j}^\mu \partial_\mu, \quad \nabla = h^\mu \partial_\mu, \quad \nabla_5 = h_5^\mu \partial_\mu. \tag{38}$$

At this point a small comment concerning the use of formulae (18)–(19) and (24)–(26) is necessary. If formulae (18)–(19) did not exist we could not derive eqs. (28)–(29) and then in order to treat further formula (35) we would be inforced to suppose that the initial spinor is free and massless. Then it is not only the massive case that would drop out, but the idea of generalized phase transformations would be questionable, because it would refer to the specific case only of a transition from an initial free spinor to a fixed interacting spinor. Thus generality is lost. On the other hand, if we did not use formulae (24)–(26) to obtain eq. (33) out of (32), we would have applied the operators $i\ \not\partial$ and $\not{\mathcal{A}}$ to the matrix (32) itself and equate the result as a matrix. However the matrix structure of the resulting equation contains the matrices γ_μ ans $\gamma_\mu\gamma_5$ only. So that the treatment of the massive case of equation (27) would be impossible at all. This makes clear that the use of the formulae obtained in the previous section not only simplify the calculations, but it also makes possible the treatment of the massive case and is necessary in order to formulate precisely the problem for the generalized phase transformations.

Now separating the real and imaginary parts and equating separately the coefficients of γ_5 and the unit matrix we obtain actually a system of eight real equations. Just four of them can be solved algebraically with respect to $\mathcal{A}$, $\widetilde{\mathcal{A}}$, $\mathcal{B}$ and $\mathcal{B}_5$ thus determining the corresponding transformation of the electromagnetic field. So we have

$$\mathcal{A}_\mu^{(a)} = \partial_\mu \phi + \frac{1}{2} j_{[\mu} \widetilde{j}_{\nu]} \partial^\nu \omega - h_{[\mu} h_{5\nu]} \partial^\nu \omega_5 + K_{\mu\nu}^{(a)} + \frac{m}{c} M_\mu^{(a)}, \tag{39}$$

where

$$K_{\mu\nu}^{(a)} = \frac{1}{(\cosh 4e\omega_5 + \cos 4e\omega)} \left\{ 2g_{\mu\nu} \cos 2e\omega \cosh 2e\omega_5 + \frac{1}{2}(j+\widetilde{j})_{[\mu} h_{\nu]} \sin 4e\omega - \frac{1}{2}(j-\widetilde{j})_{[\mu} h_{5\nu]} \sinh 4e\omega_5 \right\} \tag{40}$$

and

$$\begin{aligned} M_\mu^{(a)} = & \frac{1}{(\cosh 4e\omega_5 + \cos 4e\omega)} \\ & \times \Big\{ 2\Big[j_\mu (1 - \cosh 2e(\phi_5 - \omega_5)) + \widetilde{j}_\mu (1 - \cosh 2e(\phi_5 + \omega_5)) \Big] \cos 2e\omega \cosh 2e\omega_5 \\ & - h_\mu (1 - \cosh 2e\phi_5 \cosh 2e\omega_5) \sin 4e\omega + h_{5\mu} (1 + \cos 4e\omega) \sinh 2e\phi_5 \cosh 2e\omega_5 \Big\}. \end{aligned} \tag{41}$$

The superscript (a) just denotes that we are considering case A.

So, a compensating transformation of the electromagnetic field might really exist. However the Dirac–Maxwell equation is not automatically invariant, since the set of equations (36)–(37) contain four more equations, which actually provide the consistency conditions for the parameters ω, ω_5 and ϕ_5. Such a situation is already known from the literature (see f.e. paper [9]). These conditions can be written compactly in the form

$$\partial_\mu \phi_5 = -\frac{1}{2} j_{[\mu}\tilde{j}_{\nu]}\partial^\nu \omega_5 - h_{[\mu}h_{\nu]}\partial^\nu \omega + K^{5(a)}_{\mu\nu}A^\nu + \frac{m}{a}M^{5(a)}_\mu, \tag{42}$$

where

$$K^{5(a)}_{\mu\nu} = -\frac{1}{(\cosh 4\epsilon\omega_5 + \cos 4\epsilon\omega)}\{2g_{\mu\nu}\sin 2\epsilon\omega \sinh 2\epsilon\omega_5 + \frac{1}{2}(j+\tilde{j})_{[\mu}h_{\nu]}\sinh 4\epsilon\omega_5 + \frac{1}{2}(j-\tilde{j})_{[\mu}h_{5\nu]}\sin 4\epsilon\omega\} \tag{43}$$

and

$$\begin{aligned} M^{5(a)}_\mu = &-\frac{1}{(\cosh 4\epsilon\omega_5 + \cos 4\epsilon\omega)} \\ &\times\Big\{2\Big[j_\mu(1-\cosh 2\epsilon(\phi_5-\omega_5)) + \tilde{j}_\mu(1-\cosh 2\epsilon(\phi_5+\omega_5))\Big]\sin 2\epsilon\omega \sinh 2\epsilon\omega_5 \\ &-h_\mu[\sinh 4\epsilon\omega_5 - (1-\cos 4\epsilon\omega)\cosh 2\epsilon\phi_5 \sinh 2\epsilon\phi_5] \\ &+h_{5\mu}\sinh 2\epsilon\phi_5 \sinh 2\epsilon\omega_5 \sin 4\epsilon\omega\}. \end{aligned} \tag{44}$$

Evidently these constraints are very complicated at least in the general case. One might hope however to find enough simple examples (varying the initial conditions, i.e. the initial fields) for which they can be solved.

It should be noted however that these consistency conditions for the parameters of the generalized phase transformations ensure the conservation of the transformed current. The latter has the form

$$\begin{aligned} J^{(a)}_\mu = &R\{j_\mu \cosh 2\epsilon(\phi_5-\omega_5) + \tilde{j}_\mu \cosh 2\epsilon(\phi_5+\omega_5) - 2h_\mu \sin 2\epsilon\omega \cosh 2\epsilon\phi_5 \\ &-2h_{5\mu}\cos 2\epsilon\omega \sinh 2\epsilon\phi_5\}. \end{aligned} \tag{45}$$

Indeed, since the initial Majorana spinors satisfy the equations (28)-(29), the normalized initial currents would satisfy the continuity equations

$$\partial^\mu j_\mu = -2eB - j_\mu \partial^\mu \ln R. \qquad \partial^\mu \tilde{j}_\mu = 2eB - \tilde{j}_\mu \partial^\mu \ln R, \tag{46}$$

$$\partial^\mu h_\mu = e(A-\tilde{A}) - h_\mu \partial^\mu \ln R. \qquad \partial^\mu h_{5\mu} = -h_{5\mu}\partial^\mu \ln R. \tag{47}$$

Then we can easily show using eqs. (42)-(44) that

$$\partial^\mu J^{(a)}_\mu = 0.$$

However the current conservation does not imply in its turn the relations (42)-(44). They carry additional information. In the next section we shall try to clarify what the geometric sense of these consistency conditions might be for the case of some extreme choice of the initial fields.

3.2 Case B. Phase matrix generators 1, γ_5, iJ_2 and i^*J_2

The phase matrix now is

$$\Phi = \exp\{ie(\phi + \phi_5\gamma_5) + e(\omega + \omega_5\gamma_5)J_2\}. \tag{48}$$

Using the formula (25) we obtain the explicit expression for the action of Φ on the intitial spinor ψ

$$\begin{aligned} \Psi' &= e^{ie(\phi+\phi_5\gamma_5)}\{[\cos e(\omega + \omega_5\gamma_5) + \sin e(\omega + \omega_5\gamma_5)]\chi \\ &\quad + i[\cos e(\omega + \omega_5\gamma_5) - \sin e(\omega + \omega_5\gamma_5)]\eta\}. \end{aligned} \tag{49}$$

The same procedure as in case A can be applied just using the identities

$$\begin{aligned} \frac{\cos e(\omega - \omega_5\gamma_5) - \sin e(\omega - \omega_5\gamma_5)}{\cos e(\omega - \omega_5\gamma_5) + \sin e(\omega - \omega_5\gamma_5)} &= \frac{\cos 2e\omega + \gamma_5 \sinh 2e\omega_5}{\cosh 2e\omega_5 + \sin 2e\omega}, \\ \frac{\cos e(\omega - \omega_5\gamma_5) + \sin e(\omega - \omega_5\gamma_5)}{\cos e(\omega - \omega_5\gamma_5) - \sin e(\omega - \omega_5\gamma_5)} &= \frac{\cos 2e\omega - \gamma_5 \sinh 2e\omega_5}{\cosh 2e\omega_5 - \sin 2e\omega} \end{aligned}$$

We shall not consider the details, but we shall just write down the explicit form of the transformed electromagnetic field and the consistency conditions.

$$\mathcal{A}^{(b)}_\mu = \partial_\mu\phi - \frac{1}{2}(j + \tilde{j})_{[\mu}h_{\nu]}\partial^\nu\omega + \frac{1}{2}(j - \tilde{j})_{[\mu}h_{5\nu]}\partial^\nu\omega_5 + K^{(b)}_{\mu\nu}A^\nu + \frac{m}{e}M^{(b)} \tag{50}$$

where

$$\begin{aligned} K^{(b)}_{\mu\nu} &= \frac{1}{\cosh 4e\omega_5 + \cos 4e\omega}\{2g_{\mu\nu}\cos 2e\omega \cosh 2e\omega_5 \\ &\quad + \frac{1}{2}j_{[\mu}\tilde{j}_{\nu]}\sin 4e\omega - h_{[\mu}h_{5\nu]}\sinh 4e\omega_5\Big\} \end{aligned} \tag{51}$$

and

$$\begin{aligned} M^{(b)}_\mu &= \frac{1}{\cosh 4e\omega_5 + \cos 4e\omega}\{[j_\mu(\cosh 2e\omega_5 + \sin 2e\omega) \\ &\quad + \tilde{j}_\mu(\cosh 2e\omega_5 - \sin 2e\omega)]\cos 2e\omega(1 - \cosh 2e\phi_5) \\ &\quad + h_{5\mu}\sinh 2e\phi_5(\cosh 4e\omega_5 + \cos 4e\omega)\} \end{aligned} \tag{52}$$

Conversely the consistency conditions read

$$\partial_\mu\phi_5 = \frac{1}{2}(j + \tilde{j})_{[\mu}h_{\nu]}\partial^\nu\omega + \frac{1}{2}(j - \tilde{j})_{[\mu}h_{5\nu]}\partial^\nu\omega + K^{5(b)}_{\mu\nu}A^\nu + \frac{m}{e}M^{5(b)}_\mu \tag{53}$$

where

$$K^{5(b)}_{\mu\nu} = -\frac{1}{\cosh 4e\omega_5 + \cos 4e\omega}\{2g_{\mu\nu}\sin 2e\omega \sinh 2e\omega_5 + \frac{1}{2}j_{[\mu}\tilde{j}_{\nu]}\sinh 4e\omega_5 + h_{[\mu}h_{5\nu]}\sin 4e\omega\Big\} \tag{54}$$

$$M^{5(b)}_{\mu} = \frac{(1-\cosh 2e\phi_5)\sinh 2e\omega_5}{\cosh 4e\omega_5 + \cos 4e\omega}\{j_\mu(\cosh 2e\omega_5 - \sin 2e\omega) - \tilde{j}_\mu(\cosh 2e\omega_5 + \sin 2e\omega)\Big\}. \tag{55}$$

As in case A these consistency conditions ensure the conservation of the transformed current

$$J^{(b)}_\mu = R\Big\{\Big[j_\mu(\cosh 2e\omega_5 + \sin 2e\omega) + \tilde{j}_\mu(\cosh 2e\omega_5 - \sin 2e\omega)\Big]\cosh 2e\phi_5 - 2(h_\mu \sinh 2e\omega_5 + h_{5\mu}\cos 2e\omega)\sinh 2e\phi_5\}, \tag{56}$$

but although somewhat simplified, they still carry additional information.

3.3 Case C. Phase matrix generators 1, γ_5, iJ_3, and ${}^*J_3$

Now the "phase" matrix is

$$\Phi = \exp\{ie(\phi + \phi_5\gamma_5) + e(\omega + \omega_5\gamma_5)J_3\} \tag{57}$$

and the transformed spinor field looks as follows

$$\Psi' = e^{ie(\phi+\phi_5\gamma_5)+e(\omega_5-\omega\gamma_5)}(\lambda + i\eta). \tag{58}$$

We shall now just write down the corresponding expressions which are obtained following the procedure of case A. The electromagnetic field transformation has the form

$$\mathcal{A}^{(c)}_\mu = A_\mu + \partial_\mu\phi - \frac{1}{2}(j+\tilde{j})_{[\mu}h_{5\nu]}\partial^\nu\omega - \frac{1}{2}(j-\tilde{j})_{[\mu}h_{\nu]}\partial^\nu\omega_5 + \frac{m}{2}(j+\tilde{j})_\mu(1-\cosh 2e\phi_5\cos 2e\omega) + h_{5\mu}m\sinh 2e\phi_5\cos 2e\omega, \tag{59}$$

while the consistency conditions read

$$\partial_\mu\phi_5 = -\frac{1}{2}(j-\tilde{j})_{[\mu}h_{\nu]}\partial^\nu\omega + \frac{1}{2}(j+\tilde{j})_{[\mu}h_{5\nu]}\partial^\nu\omega_5 + m\left\{\frac{1}{2}(j+\tilde{j})_\mu\cosh 2e\phi_5\sin 2e\omega - h_{5\mu}\sinh 2e\phi_5\sin 2e\omega\right\} \tag{60}$$

Among the considered three cases this is obviously the simplest one, although all three of them have similar structure.

The conservation of the transformed current

$$J^{(e)}_{\mu} = Re^{2e\omega_5}\{(j+\tilde{j})_{\mu}\cosh 2e\phi_5 - 2h_{5\mu}\sinh 2e\phi_5\} \tag{61}$$

is again implicit from the consistency conditions (60).

At that point a small comment on the recognition of the parameters ω, ω_5 and ϕ_5 as real phases of some spinor components is necessary. In all three cases this is not explicit as seen from eqs. (33), (49) and (57). However this is not so striking, since we have determined the phase matrix following paper [1], i.e. so as to produce a Dirac spinor out of a Majorana one. To make things transparent we have to note that according to this procedure our initial Dirac field can be represented as

$$\psi \equiv \frac{1}{\sqrt{2}} e^{-\frac{\pi}{4}J_2}\chi .$$

Here J_2 plays the role of the imaginary unit ($J_2^* = -J_2$, $J_2^2 = -1$) and $\frac{\pi}{4}$ is a phase in that sence only. It is evident that, although it is related to the introduction of an imaginary part of the spinor, we can not recognize it to be a phase of any separate spinor component (from the basis of the Majorana spinor space which we use). The same thing, but more complicated, takes place in all three cases which we have considered. If we want that ω, ω_5 and ϕ_5 are explicitly recognized as phases of the separate spinor components we must further specify the phase matrix and use some other set of commuting generators. We have considered the above three cases just for their simplicity.

To conclude this section we shall point on the interesting possibility to consider a free initial spinor field and trivial electromagnetic field (i.e. $A_{\mu} = 0$). This is not only because the consistency conditions are essentially simplified in that case, but still because the phase transformation in that case would intertwin between the free and the interacting cases giving rise to a nontrivial electromagnetic field. In the last two sections we shall look from that point of view on the problem.

4 The consistency conditions for free initial spinors

In this section we shall just slightly touch the problem of the consistency conditions. We shall consider the case only when the initial electromagnetic field is zero and to simplify things maximally we shall consider the massless case.

4.1 Case A

Under the above assumptions the consistency conditions are considerably simplified and we have

$$\partial_{\mu}\phi_5 = -\frac{1}{2} j_{[\mu}\tilde{j}_{\nu]}\partial^{\nu}\omega_5 - h_{[\mu}h_{\nu]}\partial^{\nu}\omega . \tag{62}$$

This equation nicely looking so we can try to see what are his implications. Projecting on the tetrad vectors (j_{μ}, $\tilde{j}_{\mu}$, h_{μ} and $h_{5\mu}$) we obtain

$$\Delta(\varphi_5 - \omega_5) = 0, \qquad \tilde{\Delta}(\phi_5 + \omega_5) = 0, \tag{63}$$

$$\nabla\phi_5 = \nabla_5\omega, \qquad \nabla_5\phi_5 = -\nabla\omega, \tag{64}$$

where the notations (38) are used.

Equation (63) are showing that at any space-time point the surfaces defined by the equations

$$\phi_5 - \omega_5 = const, \qquad \phi_5 + \omega_5 = const$$

are locally orthogonal to the light-cone vectors j_μ and $\tilde{j}_\mu$, respectively. So that their geometrical sense is evident.

The second pair of conditions (64) in fact formally coincides with the Cauchy–Riemann analiticity conditions. So we could think of the matrix functions $\omega \pm \phi_5\gamma_5$ as of a sort of locally analytical function of the variables $z_1 = H^\mu x_\mu$ and $z_2 = h_5^\mu x_\mu$, i.e. depending only on the matrix variables $\hat{z} = z_1 \pm z_2\gamma_5$, respectively. Things are more complicated, however, since h_μ and $h_{5\mu}$ depend on x_μ and we should take care of that.

The striking thing about this geometrical picture is that it splits up (at least locally, i.e. in a small neighbourhood of any space-time point) the four-dimensional space-time into a two-dimensional hyperbolic space time and a two-dimensional Euclidean space. A similar situation has been already observed for the self-dual solutions of $SU(2)$ gauge theories in the Euclidean space-time (see paper [10]). In both cases, however, two privileged orthogonal planes are introduced in space-time. In the $SU(2)$ gauge theory case these two planes are implicit from the self-duality condition. And in our case these planes are determined by the bivectors $j_{[\mu}\tilde{j}_{\nu]}$ and ${}^*j_{[\mu}\tilde{j}_{\nu]} = h_{[\mu}h_{5\nu]}$ which define the generators J_1 and ${}^*J_1$ [7]. The four-dimensional space-time splits exactly into a direct sum of these two planes. It is natural that the same situation shall be repeated in the remaining two cases.

At the same time the electromagnetic field, which arises from the generalized phase transformations has the form

$$\mathcal{A}_\mu^{(a)} = \partial_\mu\phi + \frac{1}{2}j_{[\mu}\tilde{j}_{\nu]}\partial^\nu\omega - h_{[\mu}h_{5\nu]}\partial^\nu\omega_5. \tag{65}$$

As evident it is far from being trivial, so that a nonzero Maxwell field strength tensor emerges.

Going further and making the initial conditions still more extreme, we shall consider the case of just constant initial spinors (Bosonization ansatz). Then what was said for the analiticity becomes globally true.

Under this very restrictive assumption we can go somewhat further and look on the implications of eqs. (63)–(64). Actually we have

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[7] The appearance of the dual tensor ${}^*j_{[\mu}\tilde{j}_{\nu]}$ is evident from the identity

$$\sigma_{\mu\nu}\gamma_5 = \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}\sigma^{\lambda\rho}$$

and the equation ${}^*j_{[\mu}\tilde{j}_{\nu]} = h_{[\mu}h_{5\nu]}$ as well as the corresponding relations for the remaining four bivectors can be obtained using again the Fierz formula.

$$\Box_y \phi_5 = \Box_y \omega_5 = 0, \qquad \Box_y = \Delta\tilde{\Delta}, \tag{66}$$

$$\Delta_z \phi_5 = \Delta_z \omega = 0, \qquad \Delta_z = \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2}, \Box = \Delta\tilde{\Delta} - \Delta_z. \tag{67}$$

To specify further these functions we have to use the Maxwell equation, which we shall do in the next section.

4.2 Case B

The consistency condition now reads

$$\partial_\mu \phi_5 = \frac{1}{2}(j + \tilde{j})_{[\mu} h_{\nu]} \partial^\nu \omega + \frac{1}{2}(j - \tilde{j})_{[\mu} h_{5\nu]} \partial^\nu \omega. \tag{68}$$

To make things still more transparent, however, it is convenient to introduce a new orthonormal tetrad through the combinations

$$j^{\pm}_\mu = \frac{1}{2}(j + \tilde{j})_\mu \pm h_\mu, \quad h_\mu = \frac{1}{2}(j - \tilde{j})_\mu, \quad \bar{h}_{5\mu} = h_{5\mu}. \tag{69}$$

Then projecting the consistency conditions (65) on the directions of the new tetrad vectors we obtain

$$\Delta_+(\phi_5 - \omega_5) = 0, \qquad \Delta_-(\phi_5 + \omega_5) = 0, \tag{70}$$

$$\nabla_1 \omega = \nabla_2 \phi_5, \qquad \nabla_2 \omega = -\nabla_1 \phi_5, \tag{71}$$

where we have introduced the notations

$$\Delta_\pm = j^{\pm}_\mu \partial^\mu, \quad \nabla_1 = h_\mu \partial^\mu, \quad \nabla_2 = \bar{h}_{5\mu} \partial^\mu. \tag{72}$$

So we see that equations (70)–(71) formally coincide with equations (63)–(64) and everything said about the geometric sense of the first pair can be literally repeated here too.

The electromagnetic field in that case has the form

$$\mathcal{A}^{(b)}_\mu = \partial_\mu \phi - \frac{1}{2}(j + \tilde{j})_{[\mu} h_{\nu]} \partial^\nu \omega + \frac{1}{2}(j - \tilde{j})_{[\mu} h_{5\nu]} \partial^\nu \omega_5. \tag{73}$$

And evidently it is a nontrivial vector field (i.e. leading to a nonzero Maxwell field strength tensor).

4.3 Case C

Now the consistency condition is

$$\partial_\mu \phi_5 = -\frac{1}{2}(j - \tilde{j})_{[\mu} h_{\nu]} \partial^\nu \omega + \frac{1}{2}(j + \tilde{j})_{[\mu} h_{5\nu]} \partial^\nu \omega_5 \tag{74}$$

As in the previous case B we introduce a new light-cone tetrad, but now through the combinations

$$j^{\pm}_{\mu} = \frac{1}{2}(j + \tilde{j})_{\mu} \pm h_{5\mu}, \quad h_{\mu} = \frac{1}{2}(j - \tilde{j})_{\mu}, \quad \hbar_{5\mu} = h_{\mu}. \tag{75}$$

Projecting the consistency conditions on this new tetrad we obtain

$$\Delta_{+}(\varphi_5 - \omega_5) = 0, \qquad \Delta_{-}(\varphi_5 + \omega_5) = 0, \tag{76}$$

$$\nabla_1 \varphi_5 = \nabla_2 \omega, \qquad \nabla_2 \varphi_5 = -\nabla_1 \omega, \tag{77}$$

where

$$\Delta_{\pm} = j^{\pm}_{\mu}\partial^{\mu}, \quad \nabla_1 = h_{\mu}\partial^{\mu}, \quad \nabla_2 = \hbar_{5\mu}\partial^{\mu}. \tag{78}$$

Then since equations (76)–(77) formally coincide with (63)–(64) (and (71)–(72) respectively) everything said before on occasion of these equations is still true in case C too.

The electromagnetic field is now

$$\mathcal{A}^{(c)}_{\mu} = A_{\mu} + \partial_{\mu}\varphi - \frac{1}{2}(j + \tilde{j})_{[\mu}h_{5\nu]}\partial^{\nu}\omega - \frac{1}{2}(j - \tilde{j})_{[\mu}h_{\nu]}\partial^{\nu}\omega_5 \tag{79}$$

and it is evidently nontrivial.

Finally we should note that the electromagnetic fields defined by formulae (65), (73) and (79) remain still nontrivial even if the initial spinors and therefore their light-cone tetrad become constant.

5 The Maxwell equation in the Bosonization ansatz

In this section we shall consider the Maxwell equation in the Lorentz gauge, i.e.

$$\Box \mathcal{A}_{\mu} = \epsilon J'_{\mu} \tag{80}$$

$$\partial^{\mu}\mathcal{A}_{\mu} = 0, \tag{81}$$

where the J'_{μ} denotes the transformed current.

We shall discuss the case only, when a Bosonization ansatz is assumed (the initial Majorana spinors are constant), the spinor mass is zero and the initial electromagnetic field is zero. This would give us the possibility to use the results of the previous section.

Then the gauge fixing equation (81) just leads to the condition

$$\Box \varphi = 0.$$

One can readily see from formulae (65), (73) and (79) that under the assumptions made above the remaining parts of the corresponding expressions for the electromagnetic field automatically satisfy the Lorentz condition.

5.1 Case A and B

Having in mind the explicit form of the electromagnetic field and the current (see formulae (65) and (45), respectively) we can project eq. (79) on the directions of the tetrad vectors to write it by components

$$\tilde{\Delta}\Box_y\omega = 2Re\cosh 2e(\phi_5+\omega_5), \tag{82}$$

$$\Delta\Box_y\omega = -2Re\cosh 2e(\phi_5-\omega_5), \tag{83}$$

$$\nabla_5\Delta_z\omega_5 = -2Re\sin 2e\omega\cosh 2e\phi_5 \tag{84}$$

$$\nabla\Delta_z\omega_5 = 2Re\cos 2e\omega\sinh 2e\phi_5. \tag{85}$$

Here the notations (38) and (66) (67) are used. Now we can use the relations (63)–(64) and (66)–(67) to show that

$$\nabla\Box_y\omega = -\nabla_5\Box_y\phi_5 = 0, \quad \nabla_5\Box_y\omega = \nabla\Box_y\phi_5 = 0.$$

Then applying the differential operators ∇ and ∇_5 on the equations (82)–(83) we see that we must have

$$\nabla(\phi_5\pm\omega_5) = 0, \quad \nabla_5(\phi_5\pm\omega_5) = 0.$$

So neither ϕ_5 nor ω_5 depend on the variables $z_{1,2}$. Therefore the left hand sides of equations (84)–(85) are zero and therefore we have

$$\sin 2e\omega\cosh 2e\phi_5 = 0, \quad \cos 2e\omega\sinh 2e\phi_5 = 0.$$

The only solution of these two equations is $\phi_5 = 0$, $\omega = \pm k\pi$, $(k = 0, 1, 2, \ldots)$. But then equations (63) implies that ω_5 does not depend on the other two coordiantes too. So it is just a constant and so are ω and ϕ_5. Thus the only true function is the gauge function ϕ itself, but this is trivial.

So we see that the extreme initial conditions imposed by the Bosonization ansatz totally ruin the nice construction of eq. (65). The electromagnetic field becomes a pure gauge.

The same thing happens in case B. Here we can use the modified light-cone tetrad (69) to project the Maxwell equation (79) on the directions of the tetrad vectors. And since all four components of the current are still nonzero we come to the same conclusion.

5.2 Case C

In this case the current has only two nonzero components and we might hope that the catastrophy of the previous two cases can be at least partially avoided. We shall use the "rotated" tetrad (75) to write down the current in the explicitly two-component form

$$J^{(c)}_\mu = R\{j^+_\mu e^{2e(\omega_5-\phi_5)} + \tilde{j}^-_\mu e^{2e(\omega_5+\phi_5)}\}. \tag{86}$$

In these notations the electromagnetic field is written as

$$\mathcal{A}^{(c)}_\mu = \partial_\mu + j^+_{[\mu}j^-_{\nu]}\partial^\nu\omega - h_{[\mu}h_{5\nu]}\partial^\nu\omega_5. \tag{87}$$

As we can see its form formally coincides with that of eq. (65). So the same analysis as in case A can start to give us the conditions

$$\nabla_{1,2}\sigma_5 = \nabla_{1,2}\omega_5 = 0. \tag{88}$$

Thus the third term in (87) is zero. But the analogy breaks up from here on because the current does not contain the other two components corresponding to $\hbar_\mu$ and $\hbar_{5\mu}$. So that the remaining two equations are automatically satisfied due to eqs. (88). Thus the problem actually becomes two-dimensional (if we neglect the gauge function ϕ which is inessential). The relevant two components of the Maxwell equation read

$$\Delta_-^2\Delta_+\omega = \epsilon R e^{2\epsilon(\omega_5-\sigma_5)}, \tag{89}$$

$$\Delta_+^2\Delta_-\omega = -\epsilon R e^{2\epsilon(\omega_5+\sigma_5)}. \tag{90}$$

Now having in mind the consistency conditions (76)-(77) and the condition (88) it is clear that either of the fuctions $(\omega_5 \mp \sigma_5)$ depends on one variable only. Namely, we have

$$(\omega_5 \mp \sigma_5) = W_\mp(y_\mp),$$

where

$$y_\mp = j_\mu^\mp x^\mu.$$

Then evidently the solution of equations (89)-(90) is written in the form

$$\omega(y_+, y_-) = \epsilon R \left\{ y_+ \int_0^{y_-} ds \int_0^s dt e^{2\epsilon W_-(t)} - y_- \int_0^{y_+} ds \int_0^s dt e^{2\epsilon W_+(t)} \right\} + F_-(y_-) + F_+(y_+), \tag{91}$$

where $F_\mp(y_\mp)$ are "constants" of integration and as $W_\mp(y_\mp)$ are arbitrary functions of their arguments. So in this case the analysis is held to its very end.

Thus we have finally the following expressions for the electromagnetic and the spinor fields, respectively

$$\mathcal{A}_\mu^{(c)} = \partial_\mu\sigma + j^+_{[\mu}j^-_{\nu]}\partial^\nu\omega$$

and

$$\Psi' = \exp\{ie\phi - \epsilon W_+(1 - i\gamma_5) - \epsilon W_-(1 + i\gamma_5) - \epsilon\omega\gamma_5\}(\chi + i\eta),$$

where $\omega(y_+, y_-)$ is defined by eq. (91). $W_\mp(y_\mp)$ are arbitrary functions of their arguments and $\phi(x)$ satisfies the equation (81).

6 Conclusions

The considered three cases of generalized phase transformations have shown that such transformations can be introduced and might be usefull. The consistency conditions for that are in general nonlinear and there is practically no hope for them to be solved. However, when the massless case for initial free spinors is considered these conditions are simplified and give chance for further analysis in all three cases. The interesting thing is that in this simplified case the consistency conditions in a sense split up the four-dimensional space-time into two orthogonal two-dimensional subspaces.

The further simplification of the problem which is due to the use of a Bosonization ansatz leads to results, which can be interpreted twofold. On one hand the Maxwell equation leads finally to trivial (i.e. constant) phases in cases A and B. And from that point of view it seems that the Bosonization ansatz and the generalized phase transformations are incompatible. On the other hand in Case C an exact solution is obtained. It is unphysical, because it is two-dimensional and has infinite charge and energy. Yet it is a solution and that is already something. So we could think that, provided a more suitable initial conditions and more appropriate phase matrix are introduced, Bosonization and generalized phase transformations still can do a good job.

Finally we must note that the Bosonization ansatz can be treated a zero order term of a Taylor expansion of the initial spinor in a small neighbouhood of a fixed space-time point. Then taking some next order terms we can try to proceed with the same analysis.

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