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POSSIBILITY OF EXTENDING SPACE–TIME COORDINATES

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ABSTRACT

It has been shown that one coordinate system can describe a whole space–time region except some supersurfaces on which there are coordinate singularities. The conditions of extending a coordinate from real field to complex field are studied. It has been shown that many–valued coordinate transformations may help us to extend space–time regions and many–valued metric functions may make one coordinate region to describe more than one space–time regions.

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I. Introduction

In 1983 the author developed an imaginary coordinate method[1]. Using the method, the following two problems have been solved.

(1) In 1917 Weyl solutions were obtained[2] and many years later the relations between Weyl and Schwarzschild solutions were found. However, the space-time region inside the horizon cannot be described by Weyl coordinates. Using imaginary Weyl coordinates, the difficulty has been solved[1]. The space-time structures for several black holes have been described by imaginary Weyl coordinates[3,4].

(2) With the development of the soliton solution's generation method of the stationary axisymmetric Einstein field equations, many superposed black hole solutions have been obtained. Although several authors have analysed the space-time structures, their analyses are only confined within the exterior regions[5,6,7]. Using imaginary Weyl coordinates, the space-time structures of inner regions of these superposed black holes have been obtained[1,8,9]. It is very interesting to point out that if two superposed black holes intersect each other, they cannot form one big black hole and naked singularities emerge.

Although the imaginary coordinate method is effective and useful, it is short of theoretical bases. The present paper is devoted to find the theoretical bases.

II. Elementary Conditions of Space-Time Coordinates

Suppose $\{g_{\mu\upsilon}(x^\tau)\}$, μ, υ, τ =0,1,2,3 are solutions of Einstein field equations in region D_1 of the coordinates $\{x^\tau\}$ and $\{g'_{\alpha\beta}(y^\rho)\}$, $\alpha, \beta, \rho = 0, 1, 2, 3$ are solutions of the same field equations under same conditions in region D_2 of the coordinates $\{y^\rho\}$. How do we decide whether the coordinate regions D_1 and D_2 describe space-time regions or not? What relations are there between the two coordinates? It is well known, the mathematical model for space-time is a pair ($\mathcal{M}$,**g**) where $\mathcal{M}$ is a connected four-dimensional Hausdorff C^∞ manifold and **g** is a Lorentz metric on $\mathcal{M}$[10]. According to the definition, a space-time coordinate region must be of the following three elementary conditions.

1. The region D_1 must connect with given space-time coordinate regions because space-time is a connected manifold. In other words, if there is a space-time singularity between D_1 and given space-time coordinate regions, the region D_1 is not a space-time coordinate region.

2. The metric functions $\{g_{\mu\upsilon}(x^\tau)\}$ in region D_1 must be Lorentzian. In detail, $\{g_{\mu\upsilon}(x^\tau)\}$ in region D_1 must be continuous, differentiable and signature +2. In a more convenient way, the condition can be replaced by Mϕller coordinate conditions

$$g_{00} < 0, \quad g_{ll} > 0, \quad \begin{vmatrix} g_{ll} & g_{lk} \\ g_{kl} & g_{kk} \end{vmatrix} > 0, \quad \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} > 0$$
$$k, l = 1, 2, 3 \qquad . \tag{1}$$

That is to say, if a coordinate region is a space-time coordinate region, the metric functions in the coordinate region must satisfy (1). If the metric functions in the coordinate region do not satisfy (1), the coordinate region is not a space-time coordinate region.

3. Suppose D_1 and D_2 intersect each other and $D = D_1 \bigcap D_2$ is the intersect region. There must be coordinate transformations between the two coordinate systems in D. The coordinate transform functions $\{x^\tau = x^\tau(y^\rho)\}$ must be analytical functions and have inverse functions $\{y^\rho = y^\rho(x^\tau)\}$. The line element dS^2 in D must satisfy the following relation

$$dS^2 = g_{\mu\upsilon}(x^\tau)dx^\mu dx^\upsilon = g'_{\alpha\beta}(y^\rho)dy^\alpha dy^\beta. \tag{2}$$

III. A Coordinate System Can Describe a Whole Space-Time Region Except Some Supersurfaces Where There are Coordinate Singularities

The coordinate regions in a space-time ($\mathcal{M}$,g) are charts and intersect each other. Suppose two of the coordinate regions are $D_1\{x^\tau\}$ and $D_2\{y^\rho\}$ and the four dimensional volume of $D = D_1 \bigcap D_2$ is not equal to zero. $\{g_{\mu\upsilon}(x^\tau)\}$ and $\{g'_{\alpha\beta}(y^\rho)\}$ are metric functions defined on D_1 and D_2 respectively. According to the elementary condition 3 in section II, there are coordinate transformations $\{x^\tau = x^\tau(y^\rho)\}$ between the two coordinates in D and $\{x^\tau(y^\rho)\}$ are analytical functions .Therefore, $\{x^\tau(y^\rho)\}$ can analytically be continued to the region $D_2 - D_1$, $\{g_{\mu\upsilon}(x^\tau)\}$ in D_1 can be continued to $\{g_{\mu\upsilon}(x^\tau(y^\rho))\}$ in $D_2 - D_1$ and $\{g_{\mu\upsilon}(x^\tau(y^\rho))\}$ are analytical functions in $D_2 - D_1$ except some coordinate singularities. Using this way, one coordinate system $\{x^\tau\}$ can describe a whole space-time region except some coordinate singularities.

IV. Possibility of Extending a Coordinate Range from Real Field to Complex Field

Suppose the given ranges of space-time coordinates $\{x^\tau\}$ are maximum in real field. x^p is one of $\{x^\tau\}$ and its range is (a,b) or [a,b], (a,b], [a,b). If the metric functions $\{g_{\mu\upsilon}(x^\tau)\}$ are all independent on x^p , there is no singularity about x^p. For example, the metric functions $\{g_{\mu\upsilon}(r)\}$ in Schwarzschild solutions are independent on t, θ, φ and there is no singularity about them. If one or more than one of the metric functions $\{g_{\mu\upsilon}(x^\tau)\}$ are dependent on x^p, there usually are some coordinate singularities about x^p. For example, the metric functions $\{g_{\mu\upsilon}(r)\}$ in Schwarzschild solutions are dependent on r and there is a coordinate singularity r=2m in the solutions.

If the metric functions $\{g_{\mu\upsilon}(x^\tau)\}$ are all independent on x^p, the range of x^p can not be extended from real field to complex field. The reason is as follows. If the range of x^p could be extended to complex field, the complex coordinate region must connect with the real coordinate region because space-time coordinate is continuous. Therefore the coordinate x^p branches off. However, x^p is one of the four space-time coordinates and it cannot branch off in four dimensional space-time.

Suppose one or more than one of the metric functions $\{g_{\mu\upsilon}(x^\tau)\}$ are dependent on x^p. If there is no coordinate singularity in the real range of x^p, the range of x^p cannot be extended to complex field. The reason is the same as above. If there is a coordinate singularity c in the real range of x^p, it can be considered to extend the range of x^p to complex field. Since the coordinate must be continuous, the complex coordinate region must connect with c . In order to keep coordinate continuity and the four dimensional property of space-time coordinates, the complex coordinate x^p must take the form

$$c + w(d + ie), \tag{3}$$

where d and e are constants, w is a self-variable. In order to connect with c and to keep coordinate continuity, the range of w must be

$$[0, k); or[0, k]; (l, 0]; [l, 0], \tag{4}$$

where $k > 0$, $l < 0$ are all constants. If x^p takes form (3), the line element can be written as

$$dS^2 = g_{pp}(x^p, x^\tau)(d+ie)^2 dw^2 + g_{pv}(x^p, x^\tau)(d+ie)dwdx^v + g_{\mu v}(x^p, x^\tau)dx^\mu dx^v$$
$$\mu, v, \tau \neq p; \qquad p \;\; is \;\; fixed \qquad , \tag{5}$$

Obviously, it is very difficult that the metric functions $\{g_{pp}(x^p, x^\tau)(d+ie)^2, g_{pv}(x^p, x^\tau)(d+ie), g_{\mu v}(x^p, x^\tau)\}$ satisfy Mϕller coordinate conditions (1). Therefore, the cases of extending a space-time coordinate from real field to complex field are very rare.

It is very easy to see, if a real coordinate x^p can be extended from a coordinate singularity c to complex field, the real coordinate point c does not correspond to one space-time point (or supersurface). In fact, the complex coordinate region is lost by real coordinate description and two different space-time points (or supersurfaces) which connect with the lost complex coordinate region have the same coordinate c.

V. Many-Valued Coordinate Transformations and Many-Valued Metric Functions

Suppose there is a coordinate system $\{x^\tau\}$ and there are coordinate transformations from $\{x^\tau\}$ to another coordinate system $\{y^\rho\}$. If the transformations are many-valued functions and one given single-valued branch of the functions is known as space-time coordinates. It should be decided whether the other single-valued branches are space-time coordinates or not. If one or more than one of the other single-valued branches are space-time coordinates, some new space-time coordinate regions are obtained by the coordinate transformations. For example, the transformations from Schwarzschild to Kruskal coordinates are double-valued functions and the two single-valued branches are all space-time coordinates. Therefore new space-time coordinate regions are obtained.

Suppose the metric functions $\{g_{\mu\nu}(x^\tau)\}$ in a coordinate region are many-valued functions and more than one single-valued branches are space-time metric functions. Therefore the one coordinate region describe more than one space-time regions . For example, Weyl solution[1,2] is

$$dS^2 = -fdt^2 + \frac{1}{f}e^{2\nu}(d\rho^2 + dz^2) + \frac{1}{f}\rho^2 d\varphi^2 \tag{6}$$

$$f = \frac{R + R' - 2m}{R + R' + 2m} \tag{7}$$

$$R = \pm[\rho^2 + (z-m)^2]^{1/2}; \qquad R' = \pm[\rho^2 + (z+m)^2]^{1/2} \tag{8}$$

Obviously the metric functions are many- valued. In the real coordinate region ($0 \leq \rho < \infty, -\infty < z, t < \infty, 0 \leq \varphi < 2\pi$) only one single-valued branch ($R = +[\rho^2 + (z - m)^2]^{1/2}$, $R' = +[\rho^2 + (z+m)^2]^{1/2}$) is space-time metric . However, in the imaginary coordinate region ($\rho = i|\rho|, 0 \leq |\rho| \leq m - |z|, -m < z < m, -\infty < t < \infty, 0 \leq \varphi < 2\pi$), four single-valued branches are all space-time metric functions (ref. [1] in detail). Therefore, one coordinate region describe four space-time regions.

In conclusion, many-valued coordinate transformations may help us to extend space-time coordinate region and many-valued metric functions may make one coordinate region to describe more than one space-time regions.

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