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POLYAKOV STRING**

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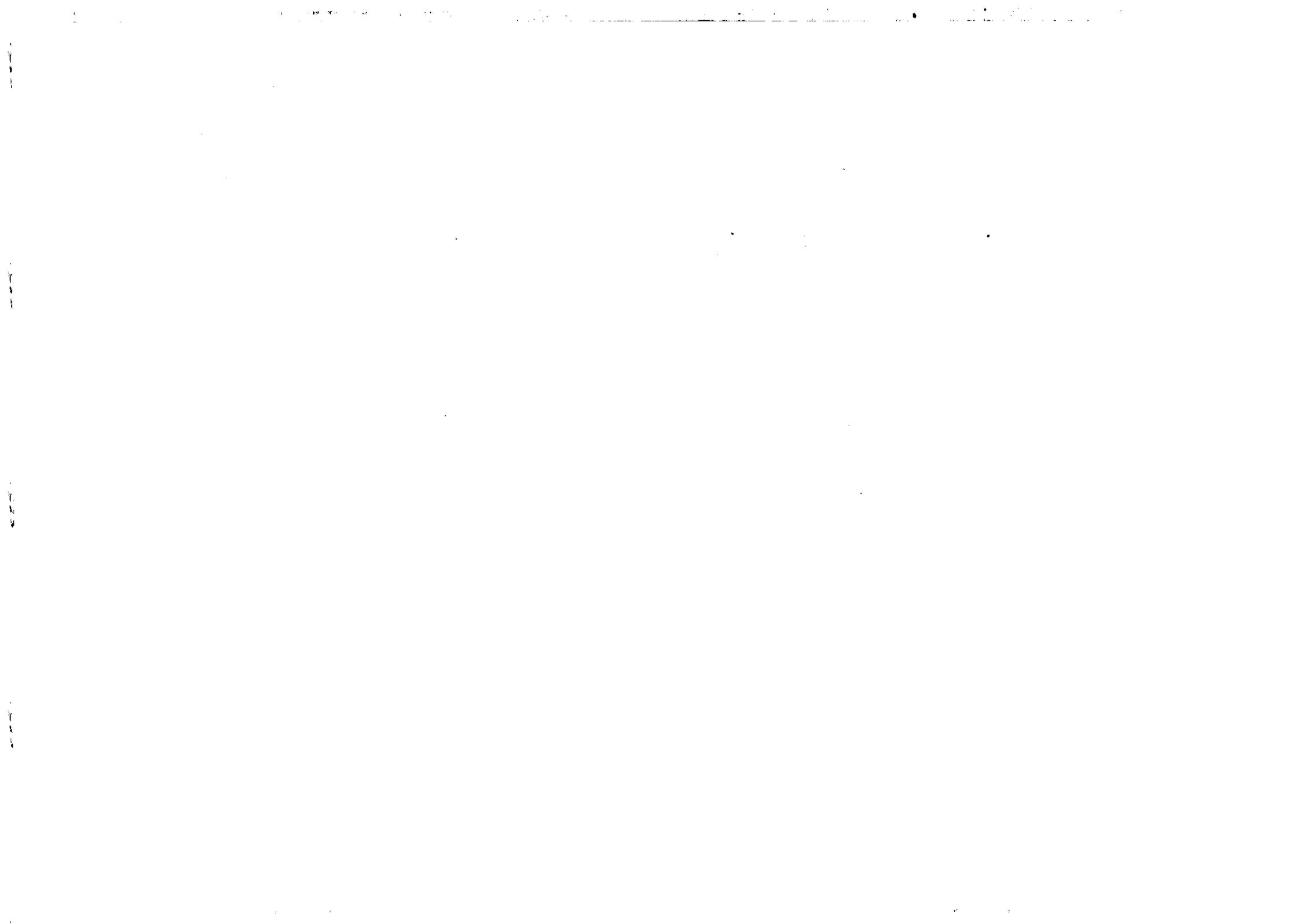


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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

**FIRST QUANTIZED NONCRITICAL RELATIVISTIC  
POLYAKOV STRING**

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**Abstract**

The first quantization of the relativistic Brink-DiVecchia-Howe-Polyakov (BDHP) string in the range  $1 < d < 25$  is considered. It is shown that using the Polyakov sum over bordered surfaces in the Feynman path integral quantization scheme one gets a consistent quantum mechanics of relativistic 1-dim extended objects in the range  $1 < d < 25$ . In particular, the BDHP string propagator is exactly calculated for arbitrary initial and final string configurations and the Hilbert space of physical states of noncritical BDHP string is explicitly constructed. The resulting theory is equivalent to the Fairlie-Chodos-Thorn massive string model. In contrast to the conventional conformal field theory approach to noncritical string and random surfaces in the Euclidean target space the path integral formulation of the Fairlie-Chodos-Thorn string obtained in this paper does not rely on the principle of conformal invariance. Some consequences of this feature for constructing a consistent relativistic string theory based on the "splitting-joining" interaction are discussed.

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## 1 Introduction

In the present paper we address the question whether the Polyakov sum over random surfaces [1] yields in the range  $1 < d < 25$  a consistent relativistic quantum mechanics of 1-dim extended objects. Within the Feynman path integral formulation the problem is to compute the Polyakov sum over all string trajectories starting and ending at prescribed string configurations and then to analyse the quantum mechanical content of the object obtained. Our motivation for considering the problem of the first quantized noncritical Polyakov string in this form is twofold. First the problem has its own interest as a nontrivial example of application of the covariant functional quantization techniques in the case of anomalous gauge theory. Secondly a well developed first quantized string in the functional formulation may shed new light on the long-standing problem of the interacting string theory in the physical dimension. The latter problem has recently received a considerable attention partly stimulated by the great progress which has been achieved over the last few years in the noncritical Polyakov string in dimensions  $d \leq 1$  and  $d = 2$  [2]. Although there are some interesting attempts [3, 4] to go beyond the  $c = 1$  barrier still very little is known about the Polyakov model in the most interesting range  $1 < d < 25$ .

Since the existence of the consistent quantum models of the free noncritical relativistic string is an old and well known fact [5, 6] it is probably in order to explain what kind of new insight one can get considering the path integral quantization of the free Polyakov string in the position representation. For that purpose we shall briefly consider two possible strategies of constructing noncritical string theory based on two equivalent but conceptually different formulations of the critical theory. For the sake of simplicity in all further considerations the open bosonic string in the flat Minkowski space is assumed.

In the so called Polyakov covariant approach the interacting critical string theory is formulated in terms of on-shell Euclidean amplitudes given by the Polyakov sum over surfaces with appropriate vertex insertions. The relativistic S-matrix in this approach is defined by analytic continuation in the momenta of external states. In the conformal gauge the amplitudes can be expressed as correlators of the 2-dim conformal field theory (the tensor product of 26 copies of the free scalar field) integrated over moduli spaces of corresponding Riemann surfaces [7]. In this form the Polyakov formulation can be seen

as a modern covariant version of the old dual model construction [8] and allows for the straightforward generalization to arbitrary 2-dim conformal field theory with the central charge  $c = 26$ . In contrast to the commonly used terminology we will call this formulation the *critical Polyakov dual model*.

The second independent approach starts with the quantum mechanics of the relativistic free string. The full interacting theory is based on the simple picture of local "joining-splitting" interaction. For example, the open strings can interact by joining their end points and merging into a single one or by splitting a single string into two. It is assumed that no particular interaction occurs at joining or splitting points. The string amplitude for a given process is defined as a sum over all possible evolutions of the system. Each evolution in this sum is represented by a world-sheet in the Minkowski target space describing causally ordered processes of joining and splitting and contributing the factor  $e^{iS}$  where  $S$  is the classical string action. In contrast to the critical Polyakov dual model we call this formulation the *critical relativistic string theory*.

The second approach is in fact known only in the case of the Nambu-Goto critical string in the light-cone gauge [9]. The advantage of this gauge is that all restrictions imposed by the causality and locality principles of the relativistic quantum theory can be easily implemented in the path integral representation of the amplitudes and the unitarity of the S-matrix is manifest. The string amplitudes obtained within this approach are Lorentz covariant at the critical dimension and reproduce the amplitudes of the old dual models.

The main difference in these approaches consists in their fundamental organizing principles. While in the case of the Polyakov dual model this is the principle of conformal invariance, in the case of the relativistic string theory one starts with the fundamental principles of the quantum mechanics of 1-dim extended relativistic systems.

The equivalence between the critical Polyakov dual model and the Mandelstam light cone critical relativistic string theory, conjectured for a long time, has been proved few years ago [10, 11]. This is one of the deepest and probably not fully appreciated results of the modern string theory. Strictly speaking this equivalence is the only known way by which one can give the stringy interpretation to the critical Polyakov dual model and prove the unitarity of its relativistic S-matrix.

Let us stress that in spite of the suggestive picture of Riemann surfaces in the Euclidean target space (emphasized in almost every introductory text on string theory) the relation

with the world-sheets of relativistic 1-dim extended objects interacting by joining and splitting is far from being obvious. The equivalence of the two conceptually different methods of constructing amplitudes in the critical theory is based on two facts. First of all due to the conformal invariance, the light cone diagram can be regarded as a special uniformization of the corresponding punctured Riemann surface. This in particular means that the singularities of the world-sheet in the Minkowski target space corresponding to the joining or splitting points are inessential. Secondly the parameters of this diagram yield a unique cover of the corresponding moduli space [10]. Note that the construction of the light cone diagram as well as the range of its characteristic parameters are uniquely determined by the causal propagation of joining and splitting strings in the Minkowski target space. It is one of the fundamental and nontrivial features of the critical string theory that the basic postulates of the relativistic quantum theory can be cast in a compact form of modular invariance in the Euclidean formulation.

With the two equivalent formulation of the critical string theory there are two possible ways to go beyond  $d=26$ .

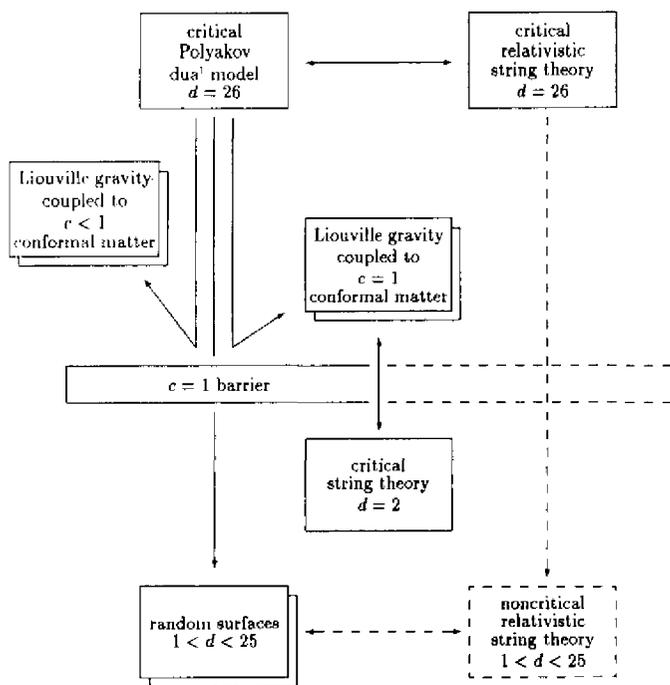
The first one is to construct an appropriate generalization of the critical Polyakov dual model taking into account the conformal anomaly. In the covariant continuous formulation developed in [12] it yields the noncritical Polyakov dual model given by some 2-dim conformal field theory with the central charge  $c$  coupled to the conformal Liouville theory with the central charge  $26 - c$ . The scheme of constructing dual amplitudes is the same as in the critical theory - they are given by correlators of vertex operators with the conformal weight 1. Vertices with this property are built from the conformal operators of the matter sector by gravitational dressing. By construction these amplitudes are conformally invariant.

The basic problem of the conventional approach sketched above is the famous  $c = 1$  barrier which manifests itself in the appearance of complex critical exponents in the range  $1 < c < 25$ . This is commonly interpreted as a manifestation of the tachyon instability [13, 14], although a precise mechanism of this phenomenon in the continuum approach is still unknown. In the case of the matter sector given by  $d$ -copies of the free scalar fields the noncritical Polyakov dual model can be seen as a theory of random surfaces in the Euclidean  $d$ -dim target space. With this interpretation it can be analysed by random triangulation techniques [15]. The numerical simulations suggest the branched

polymer phase [4] which partly justifies the results of the continuum approach. Also the matrix model constructions designed to capture the range  $1 < c < 25$  [3] indicate the polymerization of surfaces.

The second possible way to construct the interacting string theory in the range  $1 < d < 25$  is to follow the basic idea of the critical relativistic string theory (in the sense assumed in this paper). According to the brief description given above it consists of two steps: the relativistic quantum mechanics of 1-dim extended objects and the derivation of scattering amplitudes from the simple geometrical picture of joining-splitting interaction. Up to our knowledge this possibility remains completely unexplored. In fact all the recent achievements in noncritical string theory rely entirely on the noncritical Polyakov dual model.

Taking the risk of missing important results in the rapidly developing field of research one can summarize the current state of affairs in the following diagram.



The part of the diagram drawn in solid lines corresponds to the existing and well developed models and relations. Shadows of some boxes indicate the matrix model versions of the corresponding continuous models. We have also included the so called critical 2-dim string models arising from the interpretation of the Liouville field in the  $c = 1$  noncritical Polyakov dual models as the space-time coordinate in the target space [16]. With this interpretation they could be placed somewhere between the noncritical Polyakov dual model and the noncritical relativistic string theory.

There is a number of interesting questions making the problem of constructing the missing part of the diagram above worth pursuing. One of them is whether the results indicating strong instability of the noncritical Polyakov dual model (or branched polymer nature of random surfaces) apply to the noncritical relativistic string theory. Clearly the answer depends on whether the equivalence between the Polyakov dual model and the relativistic string theory holds in noncritical dimensions. The necessary condition for the positive answer is the conformal invariance of the noncritical string amplitudes. If this condition is not satisfied or more generally if the equivalence does not hold there is still room for a consistent relativistic string theory in the physical dimensions, although it is hard to expect that the amplitudes of such theory will be dual. On the other hand if the equivalence holds it would provide the relativistic string interpretation of the noncritical Polyakov dual model, justifying commonly used stringy terminology which up to now is merely based on the equivalence in the critical dimension.

Our main motivation to the present paper was to provide the first step toward the construction of the noncritical relativistic string theory. The path integral quantization of the Polyakov string in the position representation is especially convenient for this purpose. In fact it is the most suitable formalism for implementing the idea of "joining-splitting" interaction. As we shall see this approach allows for constructing the quantum theory without the assumption of the conformal symmetry and the equivalence mentioned above is a nontrivial and well posed problem.

The first quantized noncritical BDHP string is essentially the problem of quantization of the anomalous theory. Within the Feynman path integral approach the idea is to formulate the quantum theory entirely in terms of the path integral over trajectories in the configuration space without referring to the canonical phase space analysis. In the case of model without anomaly such formulation must of course reproduce other methods

of quantization. Applying this scheme to the model with anomaly one may hope that the resulting path integrals still can be given some meaning so it will make sense to ask about the consistency of the quantum theory derived in this way. In particular, in the case under consideration the conformal anomaly manifests itself in the appearance of the effective action for the conformal factor. The functional measure in the resulting path integral can be dealt with in a similar way as in the noncritical Polyakov dual model [17]. The difference (and also the complication) is that we are considering the sum over rectangular-like surfaces connecting prescribed string configurations in the  $d$ -dim target space which brings new effects related with the boundary conditions.

The main result of the present paper is that in the case of the open bosonic string described by the BDHP action the method sketched above yields a consistent relativistic quantum mechanics of 1-dim extended objects. The resulting theory coincides with the 20 years old Fairlie-Chodos-Thorn (FCT) model of the free massive string [6]. In the radial gauge the massive string is given by the following realization of the Virasoro algebra

$$\begin{aligned} L_n &= \frac{1}{2} \sum_{m=-\infty}^{\infty} : \alpha_m \cdot \alpha_{n-m} : + \frac{1}{2} \sum_{m=-\infty}^{\infty} : \beta_m \beta_{n-m} : + i(n+1)Q\beta_n & (1) \\ [\beta_m, \beta_n] &= m\delta_{m,-n} \quad , \quad [\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m,-n} \quad , \quad \eta^{\mu\nu} = \text{diag}(-1, +1, \dots, +1) \end{aligned}$$

In the Liouville sector this is the standard imaginary background charge realization commonly used in the noncritical Polyakov dual models [18]. The main difference consist in the hermiticity properties of the operators involved. In contrast to the standard construction of the Feigin-Fuchs modules [19] one has

$$\begin{aligned} (\alpha_k^\mu)^\dagger &= \alpha_{-k}^\mu \quad , \quad (\beta_k)^\dagger = \beta_{-k} \quad k \neq 0 \quad , \\ (\alpha_0^\mu)^\dagger &= \alpha_0^\mu \quad , \quad (\beta_0 + iQ)^\dagger = \beta_0 + iQ \quad . & (2) \end{aligned}$$

As a consequence the structure of the tachyonic states is similar to that of the critical string - there are no excited tachyonic states in the spectrum of the free noncritical Polyakov string.

There are some points in our derivation we would like to emphasize. First of all, in contrast to the noncritical Polyakov dual model, the derivation is independent of the techniques of 2-dim conformal field theory and does not rely on the principle of the

conformal invariance. The key point of our derivation is the exact calculation of the transition amplitude between two arbitrary string configurations. This is done by an appropriate extension of the space of states and then by translating the problem into an operator language. The Virasoro algebra of constraints arises as a set of consistency conditions of this method. As far as the theory of random surfaces is concerned there is no reason for introducing the full set of constraints. However if we assume the relativistic interpretation of the model at hand the additional constraints acquire a physical meaning - they can be seen as a consequence of the general kinematical requirement which must be satisfied by wave functionals of any quantum mechanics of relativistic 1-dim extended system. This indicates a difference between random surfaces and relativistic string theory. The general (Euclidean) path integral over surfaces with fixed boundaries is given by the model with hopelessly complicated boundary interaction. In a sense the relativistic string theory requires a very special type of this integral. The general theory of random surfaces with fixed boundaries is much more complicated and yet to be solved problem.

The organization of the paper is as follows. In Section 2 we review the full scheme of the covariant functional quantization of the critical string in the Schrödinger representation. The material included has introductory character and serves as an illustration of the methods used in the following. The reason for a rather lengthy form of this section is twofold. First of all, although the main idea of Feynman's quantization is well known, we are not aware of a self-contained presentation of this technique in the case of gauge models with reparameterization invariance, in particular in the form suitable for the quantization of the relativistic string. By self-contained we mean not only path integral representation of the transition amplitude (which is well known in the case of the critical string propagator [20, 21]) but also the construction of the Hilbert space of states and the derivation of the physical state conditions from the "classical data": the space of trajectories and the variational principle given by the classical action.

Secondly a large part of the analysis given in Sect.2 applies without changes in the case of noncritical string which is of our main interest. This concerns in particular the geometry of the space of trajectories (Subsect.2.1), the construction of the space of states (Subsect.2.2) and the proper choice of boundary conditions in the matter and in the conformal factor sectors. The last issue is crucial for the consistent path integral representation of the transition amplitude (the open string propagator) constructed in Subsect.2.3.

In Subsect.2.4 we derive the part of physical state conditions related to the constraints linear in momenta. Within the presented approach they are given by generators of the unitary realization of the residual gauge symmetry. In Subsect.2.5 the transition amplitude between the states satisfying constraints linear in momenta is calculated and the second part of the physical state conditions related to the constraints quadratic in momenta is derived. One of these constraints - the on-mass-shell condition is encoded in the transition amplitude. The rest can be regarded as a consequence of the general kinematical requirement mentioned above. Finally the full set of the physical state conditions can be expressed in terms of the familiar Virasoro constraints of the old covariant formulation of the first quantized critical string.

In Section 3 the functional formalism developed in Section 2 is applied in the case of noncritical Polyakov string in the range  $1 < d < 25$ . The lower bound results from the relativistic interpretation of the string which breaks down for  $d < 2$  while the upper one from the coefficient in front of the effective Liouville action proportional to  $(25 - d)$ . As it was mentioned above the first few steps of the quantization procedure proceed as in the case of the critical string. The main difference consists in the different symmetry properties of the noncritical string, briefly described in Subsect.3.1.

In Subsect.3.2 the transition amplitude for the noncritical string is constructed. As a result of the conformal anomaly and our choice of boundary conditions it involves the path integral over conformal factor satisfying the homogeneous Neumann boundary conditions. As a simple consequence of the Gauss-Bonnet theorem, the resulting theory is stable in the case of rectangle-like world-sheets if the bulk and the boundary cosmological constants vanish. The Liouville sector couples to the "matter" sector via boundary conditions for the  $x$ -variables which depend in a complicated way on the boundary value of the conformal factor. Even with the vanishing cosmological constant the resulting path integral cannot be directly calculated. Our method to overcome this difficulty is to express the transition amplitude as a matrix element of a simple operator in a suitably extended space of states. This is done by means of the generalized Forman formula.

In Subsect.3.3 we derive the full set of the physical state conditions in the extended space. It consists of the constraints linear in momenta related to the extension itself, the on-mass-shell condition encoded in the transition amplitude and the set of quadratic in momenta constraints arising by the mechanism similar to that of the critical string theory.

The resulting algebra of constraints yields the FCT massive string model [6]. Finally in Subsect.2.4 we provide the explicit DDF construction of the physical states of this model, which gives a simple proof of the no-ghost theorem.

Section 4 contains the discussion of the results obtained. A comparison with the noncritical Polyakov dual model is given and the open problems of the first quantized noncritical relativistic string are reviewed. We conclude this section by a brief discussion of the choice of vanishing cosmological constant in the interacting theory.

The paper contains three appendices. In Appendix A we gather some basic facts concerning the corner conformal anomaly. In Appendix B the 1-dim "conformal anomaly" is calculated. The proof of the generalized Forman formula is given in Appendix C.

## 2 Functional quantization of critical Polyakov string

### 2.1 Space of trajectories

In the Euclidean formulation a trajectory of the open Polyakov string is given by a triplet  $(M, g, x)$  where  $M$  is a rectangle-like 2-dim manifold with distinguished "initial"  $\partial M_i$  and "final"  $\partial M_f$  opposite boundary components,  $g$  is a Riemannian metric on  $M$ , and  $x$  is a map from  $M$  into the Euclidean target space  $\mathbf{R}^d$  satisfying the boundary conditions

$$n_g^a \partial_a x|_{\partial M_i} = 0 \quad (3)$$

along the "timelike" boundary components  $\partial M_i = \partial M \setminus (\partial M_i \cup \partial M_f)$ . In the formula above  $n_g$  denotes the normal direction along  $\partial M$  with respect to the metric  $g$ . In this section  $d$  will be equal to 26 but will be sometimes left as  $d$  to emphasize the dependence of quantities on the number of dimensions.

The BDHP action functional

$$S[M, g, x] = \int_M \sqrt{g} d^2 z g^{ab} \partial_a x^\mu \partial_b x_\mu$$

is invariant with respect to the Weyl rescaling of the internal metric as well as the general diffeomorphisms  $f : M \rightarrow M'$  preserving the initial and final boundary components and their orientations. The latter invariance can be partly restricted by fixing a model manifold  $M$  and a normal direction along the boundary  $\partial M$ . This can be seen as a partial

gauge fixing and has important consequences for all further constructions. Let us note that other gauge fixings are also possible, although they are much more difficult to deal with [22].

Let  $\mathcal{M}_M^n$  be the space of all Riemannian metrics on  $M$  with the normal direction  $n_g = n$  and  $\mathcal{E}_M^n$  the space of all maps from  $M$  into the target space satisfying the boundary conditions (3). In the  $(M, n)$ -gauge the space of trajectories is the Cartesian product  $\mathcal{M}_M^n \times \mathcal{E}_M^n$ . The gauge transformations form the semidirect product  $\mathcal{W}_M \odot \mathcal{D}_M^n$  of the additive group  $\mathcal{W}_M$  of scalar functions on  $M$  and the group  $\mathcal{D}_M^n$  of diffeomorphisms of  $M$  preserving corners and the normal direction  $n$ . The action of  $\mathcal{W}_M \odot \mathcal{D}_M^n$  on  $\mathcal{M}_M^n \times \mathcal{E}_M^n$  is given by

$$\mathcal{M}_M^n \times \mathcal{E}_M^n \ni (g, x) \xrightarrow{(\varphi, f) \in \mathcal{W}_M \odot \mathcal{D}_M^n} (e^\varphi f^* g, f^* x) \in \mathcal{M}_M^n \times \mathcal{E}_M^n$$

## 2.2 Space of states

Within the covariant functional approach to quantization in the Schrödinger representation the space of states consists of wave functionals defined on the Cartesian product  $\mathcal{C} \times \mathbf{R}$  where  $\mathcal{C}$  is a suitably chosen space of boundary conditions (half of the Cauchy data for the classical trajectory in the case of nondegenerate Lagrangian) and  $\mathbf{R}$  is the time axis. For gauge systems with the reparameterization invariance the inner time evolution is generated by a constraint quadratic in momenta and in the subspace of physical states it is simply given by the identity operator. In the covariant functional approach this feature manifests itself in the inner time independent formulation of variational principle. In consequence one can describe the space of states in terms of inner time independent wave functionals. The choice of  $\mathcal{C}$  itself is slightly more complicated. In order to explain the intricacies involved we consider the space related to the reduced "position" representation in the  $(M, n)$ -gauge [23].

For every string trajectory  $(g, x) \in \mathcal{M}_M^n \times \mathcal{E}_M^n$  we define the initial  $(e_i, x_i)$  and the final  $(e_f, x_f)$  boundary conditions

$$\begin{aligned} e_i^2 &= g_{ab} t^a t^b (dt)_{|\partial M_i}^2, \quad (i \rightarrow f), \\ x_i &= x_{|\partial M_i}, \quad (i \rightarrow f); \end{aligned} \quad (4)$$

where  $t$  denotes a vector tangent to the boundary and  $e_i, e_f$  are einbeins induced on the initial and the final boundary component respectively. All possible boundary conditions form the space

$$\mathcal{P}_i = \mathcal{M}_i \times \mathcal{E}_i;$$

where  $\mathcal{M}_i$  consists of all einbeins on  $\partial M_i$  and  $\mathcal{E}_i$  is the space of maps  $x_i : \partial M_i \rightarrow \mathbf{R}^d$  satisfying the Neumann boundary conditions at the ends of  $\partial M_i$ . Similarly the final boundary conditions form the space

$$\mathcal{P}_f = \mathcal{M}_f \times \mathcal{E}_f.$$

The interpretation of the transition amplitude as an integral kernel of some operator requires an identification of the spaces  $\mathcal{P}_i$  and  $\mathcal{P}_f$ . It can be done by introducing a model interval  $L$  and the space

$$\mathcal{P}_L = \mathcal{M}_L \times \mathcal{E}_L \quad (5)$$

together with the isomorphisms

$$\Gamma_i : \mathcal{P}_i \longrightarrow \mathcal{P}_L, \quad (i \rightarrow f), \quad (6)$$

induced by some arbitrary chosen diffeomorphisms

$$\gamma_i : L \longrightarrow \partial M_i, \quad (i \rightarrow f).$$

In the covariant functional approach, the part of the canonical analysis concerning constraints linear in momenta can be recovered by considering classes of gauge equivalent boundary conditions. We say that  $p, p' \in \mathcal{P}_L$  are gauge equivalent if there exist two  $\mathcal{W}_M \odot \mathcal{D}_M^n$  equivalent string trajectories with a common final boundary condition and starting at  $p$  and  $p'$  respectively. In our case the equivalence classes can be described as orbits of the group  $\mathcal{W}_L \odot \mathcal{D}_L$ , where  $\mathcal{W}_L$  is the additive group of real functions on  $L$  satisfying, as we shall see in the following, the Neumann boundary conditions at the ends of  $L$ , and  $\mathcal{D}_L$  is the group of orientation preserving diffeomorphisms of  $L$ . The action of  $\mathcal{W}_L \odot \mathcal{D}_L$  on  $\mathcal{P}_L$  is given by

$$\mathcal{P}_L \ni (e_i, x_i) \xrightarrow{(\tilde{\varphi}, \gamma) \in \mathcal{W}_L \odot \mathcal{D}_L} (e_i^{\tilde{\varphi}} \gamma^* e_i, x_i \circ \gamma) \in \mathcal{P}_L.$$

The first factor in  $\mathcal{W}_L \odot \mathcal{D}_L$  corresponds to the Weyl invariance in the space of trajectories while the second one - to the  $\mathcal{D}_M^n$ -invariance.

In the space of states  $\mathcal{H}(\mathcal{P}_L)$  consisting of string wave functionals defined on  $\mathcal{P}_L$  the "physical" states are  $\mathcal{W}_L \odot \mathcal{D}_L$  invariant functionals. The comparison with the canonical quantization shows that the generators of this symmetry form an operator realization of the constraints linear in momenta. In particular the choice of the quotient

$$\mathcal{K}_L = \frac{\mathcal{P}_L}{\mathcal{W}_L \odot \mathcal{D}_L}$$

corresponds to a formulation in which all constraints linear in momenta are solved.

As far as the constraints linear in momenta are concerned one can construct the space of states on an arbitrary quotient between  $\mathcal{P}_L$  and  $\mathcal{K}_L$ . It turns out however that the consistency requirements concerning the path integral representation of the transition amplitude yield strong restrictions on possible choices [23]. In general these requirements depend on the gauge fixing used to calculate (=define) the transition amplitude and for each particular choice of the space  $\mathcal{C}$  boil down to some regularity conditions concerning the gauge group action on the space  $\mathcal{T}[c_i, c_f]$  of trajectories starting and ending at two fixed points  $c_i, c_f \in \mathcal{C}$ . The analysis of the geometry of this action together with the Faddeev-Popov method of calculating path integral can be seen as a covariant counterpart of the symplectic reduction in the phase space approach.

In the case of the open Polyakov string in the conformal gauge there are only two admissible choices [23]

$$\begin{aligned} \mathcal{C}_L &= \frac{\mathcal{P}_L}{\mathcal{D}_L} , \\ \mathcal{C}'_L &= \frac{\mathcal{P}_L}{\mathbf{R}_+ \times \mathcal{D}_L} , \end{aligned} \quad (7)$$

where  $\mathbf{R}_+$  denotes the 1-dim group of constant rescalings acting on  $\mathcal{M}_L$ . Note that in the both cases above the identifications (6) factor out to the canonical  $\gamma$ -independent isomorphisms

$$\mathcal{C}_i = \mathcal{C}_f = \mathcal{C}_L \quad , \quad \mathcal{C}'_i = \mathcal{C}'_f = \mathcal{C}'_L \quad .$$

The quotients  $\mathcal{C}_L, \mathcal{C}'_L$  correspond to the situation in which the constraints related to the Weyl invariance are represented on the quantum level by generators of some symmetry group acting on  $\mathcal{C}_L$  ( $\mathcal{C}'_L$ ) while all the constraints related to the  $\mathcal{D}_M^n$ -invariance are completely solved. The only difference between the spaces  $\mathcal{C}_L$  and  $\mathcal{C}'_L$  consist in a different treatment of the constraint related to the Weyl rescaling of metric by conformal factor

constant along the (initial) boundary. It is more convenient to realize this constraint on the quantum level which corresponds to the choice of  $\mathcal{C}_L$ .

Our final task is to introduce an inner product in the space  $\mathcal{H}(\mathcal{C}_L)$  of string wave functionals on  $\mathcal{C}_L$ . To this end let us observe that the space  $\mathcal{M}_L \times \mathcal{E}_L$  carries the ultralocal  $\mathcal{D}_L$ -invariant structure and the scalar product in  $\mathcal{H}(\mathcal{C}_L)$  is given by the path integral

$$\langle \Psi | \Psi' \rangle = \int_{\mathcal{M}_L \times \mathcal{E}_L} \mathcal{D}^\alpha e \mathcal{D}^\alpha \tilde{x} (\text{Vol}_\alpha \mathcal{D}_L)^{-1} \overline{\Psi} \Psi' \quad . \quad (8)$$

In order to parameterize the quotient (7) it is convenient to use the 1-dim conformal gauge  $\hat{e} = \text{const}$  which yields the isomorphism

$$\mathcal{C}_L = \overline{\gamma} \mathbf{R}_+ \times \mathcal{E}_L \quad .$$

Using the Faddeev-Popov method in this gauge one gets

$$\langle \Psi | \Psi' \rangle = \int_0^\infty d\alpha \int_{\mathcal{E}_L} \mathcal{D}^{\alpha \tilde{x}} \overline{\Psi}[\alpha, \tilde{x}] \Psi'[\alpha, \tilde{x}] \quad . \quad (9)$$

### 2.3 Transition amplitude

The central object of the covariant functional quantization is the path integral representation of the transition amplitude. In the case of critical Polyakov string and with the choice of  $\mathcal{C}_L$  as a space of boundary conditions it takes the following form

$$P[c_i, c_f] = \int_{\mathcal{F}[c_i, c_f]} \mathcal{D}^g g \mathcal{D}^g x (\text{Vol}_g \mathcal{W}_M)^{-1} (\text{Vol}_g \mathcal{D}_M^n)^{-1} \exp\left(-\frac{1}{\alpha'} S[g, x]\right) \quad . \quad (10)$$

$\mathcal{F}[c_i, c_f] \subset \mathcal{M}_M^n \times \mathcal{E}_M^n$  in the formula above consists of all string trajectories starting and ending at  $c_i, c_f \in \mathcal{C}_L$  respectively, i.e. satisfying the boundary conditions

$$[(e_i, x_i)] = c_i \quad , \quad (i \rightarrow f) \quad , \quad (11)$$

where  $e_i, x_i$  are given by (4) and  $[(e_i, x_i)]$  denotes the  $\mathcal{D}_L$ -orbit of the element  $(e_i, x_i) \in \mathcal{M}_L \times \mathcal{E}_L; (i \rightarrow f)$ .

For every  $g \in \mathcal{M}_M^n$  the conditions (11) are  $g$ -dependent  $\mathcal{D}_M^n$ -invariant Dirichlet boundary conditions for  $x$  on  $\partial \mathcal{M}_f \cup \partial \mathcal{M}_i$ . With the boundary conditions (3),(11) the integration

over  $x$  in (10) is Gaussian and yields  $\mathcal{D}_M^n$ -invariant functional on  $\mathcal{M}_M^n$ . This allows for application of the F-P procedure with respect to the group  $\mathcal{D}_M^n$ . The consistency conditions for this method uniquely determine boundary conditions for the metric part of a string trajectory [24]. The space  $\mathcal{M}_M^{n*}$  of all metrics  $g \in \mathcal{M}_M^n$  satisfying these conditions can be described as follows. Let  $\mathcal{M}_M^{n0}$  be the space of all metrics from  $\mathcal{M}_M^n$  with the scalar curvature  $R_g = 0$ , such that all boundary components are geodesic and meet orthogonally. Then  $\mathcal{M}_M^{n*}$  consists of all metrics of the form  $\exp(\varphi)g_0$ , where  $g_0 \in \mathcal{M}_M^{n0}$  and  $\varphi$  satisfies the homogeneous Neumann boundary condition

$$n^a \partial_a \varphi = 0 \quad (12)$$

on all boundary components of  $M$ . An important consequence of this result is that  $\mathcal{C}_L$  is the largest space of boundary conditions for which there exists a consistent path integral representation of the transition amplitude. Note that in every conformal gauge in  $\mathcal{M}_M^{n*}$  the conformal factor satisfies the boundary condition (12). It follows that the gauge group  $\mathcal{W}_M \odot \mathcal{D}_M^n$  must be restricted to the group  $\mathcal{W}_M^n \odot \mathcal{D}_M^n$ , where  $\mathcal{W}_M^n$  consists of all  $\varphi \in \mathcal{W}_M$  satisfying the boundary conditions (12). This justifies our definition of the induced gauge transformations given in the previous subsection.

The boundary conditions (11) yield the restrictions on the internal length of the initial and final boundary components which can be implemented by appropriate delta function insertions in the path integral representation (10).

With the space  $\mathcal{F}[k_i, k_f]$  consisting of all string trajectories  $(g, x) \in \mathcal{M}_M^{n*} \times \mathcal{E}_M^d$  such that  $x$  satisfies the boundary conditions (3),(11) the calculations of the integral (10) proceeds along the standard lines. In the conformal gauge

$$(\widehat{M}_t, \widehat{g}_t) = ([0, t] \times [0, 1], \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \quad , \quad (13)$$

$$(\widehat{L} = \partial \widehat{M}_t, \widehat{e}) = ([0, t], 1) \quad , \quad (14)$$

the  $x$ -integration yields

$$\int \mathcal{D}^{n*} x \exp(-\frac{1}{4\pi\alpha'} S[e^{\varphi} \widehat{g}_t, x]) = (\det \mathcal{L}_{e^{\varphi} \widehat{g}_t})^{-\frac{d}{2}} \exp(-\frac{1}{4\pi\alpha'} S[\widehat{g}_t, \varphi, \widehat{x}_i, \widehat{x}_f])$$

where

$$S[\widehat{g}_t, \varphi, \widehat{x}_i, \widehat{x}_f] = \int_0^t dz^0 \int_0^1 dz^1 ((\partial_0 x_{cl})^2 + (\partial_1 x_{cl})^2) \quad (15)$$

and  $x_{cl} : \widehat{M}_t \rightarrow R^d$  is the solution of the boundary value problem

$$\begin{aligned} (\partial_0^2 + \partial_1^2) x_{cl} &= 0 \quad , \\ x_{cl}(0, z^1) &= \widehat{x}_i \circ \gamma[\widehat{\varphi}_i](z^1) \quad , \\ x_{cl}(t, z^1) &= \widehat{x}_f \circ \gamma[\widehat{\varphi}_f](z^1) \quad , \\ \partial_1 x_{cl}(z^0, 0) &= \partial_1 x_{cl}(z^0, 1) = 0 \quad . \end{aligned} \quad (16)$$

The functions  $\widehat{x}_i, \widehat{x}_f : [0, 1] \rightarrow R^d$  are representants of  $c_i, c_f$  in the 1-dim conformal gauge (14) (i.e.  $[(\alpha_i, \widehat{x}_i)] = c_i, [(\alpha_f, \widehat{x}_f)] = c_f$ ), and the diffeomorphisms  $\gamma[\widehat{\varphi}_i], \gamma[\widehat{\varphi}_f] : [0, 1] \rightarrow [0, 1]$  are uniquely determined by the equations

$$\begin{aligned} \frac{d}{dz^1} \gamma[\widehat{\varphi}_i](z^1) &\propto \exp \frac{1}{2} \widehat{\varphi}_i(z^1) \quad , \quad \widehat{\varphi}_i(z^1) \equiv \varphi(0, z^1) \quad , \\ \frac{d}{dz^1} \gamma[\widehat{\varphi}_f](z^1) &\propto \exp \frac{1}{2} \widehat{\varphi}_f(z^1) \quad , \quad \widehat{\varphi}_f(z^1) \equiv \varphi(t, z^1) \quad . \end{aligned} \quad (17)$$

Solving the boundary value problem (16) and inserting solution into the action (15) one has

$$\begin{aligned} S[\widehat{g}_t, \varphi, \widehat{x}_i, \widehat{x}_f] &= \frac{(X_{i0} - X_{f0})^2}{t} \\ &+ \frac{1}{2} \sum_{m>0} \frac{\pi m}{\sinh \pi m t} [(X_{im}^2 + X_{fm}^2) \cosh \pi m t - 2X_{im} X_{fm}] \quad . \end{aligned}$$

where

$$\begin{aligned} X_{i0}^\mu &= \int_0^1 dz^1 \widehat{x}_i \circ \gamma[\widehat{\varphi}_i](z^1) \quad (i \rightarrow f) \quad , \\ X_{im}^\mu &= 2 \int_0^1 dz^1 \widehat{x}_i \circ \gamma[\widehat{\varphi}_i](z^1) \cos \pi m z^1 \quad (i \rightarrow f) \quad . \end{aligned}$$

Note that the functional  $S[\widehat{g}_t, \varphi, \widehat{x}_i, \widehat{x}_f]$  depends only on the boundary values  $\widehat{\varphi}_i, \widehat{\varphi}_f$  of the conformal factor.

Applying the F-P method to the resulting path integral over  $\mathcal{M}_M^{n*}$  [23] and using the heat kernel method [17] to find out the  $\varphi$ -dependence hidden in the functional measure and in the volume of  $\mathcal{W}_M^n$  one gets

$$P[\alpha_f, \widehat{x}_f; \alpha_i, \widehat{x}_i] = \int_0^\infty dt \eta(t)^{1-\frac{d}{2}} t^{-\frac{d}{2}} \int_{\mathcal{W}^n} \mathcal{D}^{\widehat{g}_t} \varphi (\text{Vol}_{\widehat{g}_t} \mathcal{W}_M^n)^{-1}$$

$$\begin{aligned}
& \times \exp\left(-\frac{26-d}{48\pi}S_L[\hat{g}_t, \varphi]\right) \\
& \times \exp\left(\frac{6-d}{32}\sum_{\text{corners}}\varphi(z_i)\right) \\
& \times \exp\left(-\frac{1}{4\pi\alpha'}S[\hat{g}_t, \varphi, \tilde{x}_i, \tilde{x}_f]\right) , \\
& \times \delta\left(\alpha_f - \int_0^1 dz^1 e^{\frac{1}{2}\tilde{\varphi}_f}\right) \delta\left(\alpha_i - \int_0^1 dz^1 e^{\frac{1}{2}\tilde{\varphi}_i}\right) ,
\end{aligned} \tag{18}$$

where the Liouville action is given by

$$S_L[g, \varphi] = \int_M \sqrt{g} d^2z \left( \frac{1}{2} g^{ab} \partial_a \varphi \partial_b \varphi + R_g \varphi + \frac{\mu}{2} e^\varphi \right) + \lambda \int_{\partial M} e ds e^{\frac{\tilde{\varphi}}{2}} , \tag{19}$$

and

$$\eta(t) = e^{-\frac{\mu}{2}} \prod_{n=1}^{\infty} (1 - e^{-2\pi n t}) .$$

In contrast to the expression for an on-shell open string amplitude the conformal factor does not decouple in  $d = 26$ . As we shall see, the decoupling takes place for the transition amplitudes between states in  $\mathcal{H}(\mathcal{C}_L)$  satisfying the constraints linear in momenta.

## 2.4 Constraints linear in momenta

The next step in the quantization procedure is to determine the subspace of physical states in  $\mathcal{H}(\mathcal{C}_L)$ . There are three groups of interrelated physical state conditions: the constraints linear in momenta given by the generators of the residual induced gauge symmetry in  $\mathcal{C}_L$ , the constraint quadratic in momenta encoded in the transition amplitude, and the kinematical constraints following from the interpretation of the string as a 1-dim extended relativistic system. In this subsection we will discuss the first group of physical state conditions consisting of constraints linear in momenta.

The residual gauge symmetry in the space  $\mathcal{C}_L$  can be described by

$$\mathcal{C}_L \ni [(e, \tilde{x})] \xrightarrow{\tilde{\varphi} \in \mathcal{W}_L} [(e^{\frac{\tilde{\varphi}}{2}} e, \tilde{x})] \in \mathcal{C}_L . \tag{20}$$

In the 1-dim conformal gauge (14) the transformation (20) takes the form

$$\mathbf{R}_+ \times \mathcal{E}_L \ni (\alpha, \tilde{x}) \xrightarrow{(\lambda, \gamma) \in \mathbf{R}_+ \times \mathcal{D}_L} (\lambda[\tilde{\varphi}]\alpha, \tilde{x} \circ \gamma[\tilde{\varphi}]) \in \mathbf{R}_+ \times \mathcal{E}_L , \tag{21}$$

where

$$\lambda[\tilde{\varphi}] = \int_0^1 e^{\frac{\tilde{\varphi}}{2}} dz^1 ,$$

and  $\gamma[\tilde{\varphi}] : [0, 1] \rightarrow [0, 1]$  is uniquely determined by the equation

$$\frac{d}{dz^1} \gamma[\tilde{\varphi}] = (\lambda[\tilde{\varphi}])^{-1} e^{\frac{\tilde{\varphi}}{2}} .$$

It follows that all induced gauge transformations form the group  $\mathbf{R}_+ \times \mathcal{D}_L$  acting on  $\mathcal{C}_L$  by (21). Note that this group structure as well as the group action are consequences of the  $\mathcal{D}_M^\eta$ -invariant formulation and are independent of a gauge fixing used to parameterize the quotient (7). This remark also applies to all further considerations where for the sake of simplicity the conformal gauges (13,14) will be used. According to the discussion in Subsect.1.2 the wave functionals corresponding to physical states are invariant with respect to the induced gauge transformations represented in  $\mathcal{H}(\mathbf{R}_+ \times \mathcal{E}_L)$  by

$$\Psi[\alpha, \tilde{x}] \xrightarrow{(\lambda, \gamma) \in \mathbf{R}_+ \times \mathcal{D}_L} \Psi[\lambda\alpha, \tilde{x} \circ \gamma] \tag{22}$$

The problem with the representation above is that the scalar product (9) is not invariant with respect to (21). As a result the subspace  $\mathcal{H}_{\text{inv}}(\mathbf{R}_+ \times \mathcal{E}_L)$  of invariant wave functionals does not have a well defined scalar product and, as explained below, the transformation (22) should be modified.

As far as the  $\mathbf{R}_+$  symmetry is concerned this noninvariance is not important due to nondynamical nature of the  $\alpha$  variable. As we shall see the ambiguity in the choice of the scalar product on the subspace  $\mathcal{H}(\mathcal{E}_L)$  of  $\alpha$ -independent wave functionals can be hidden in the overall normalization factor. For this reason we can restrict our considerations to the  $\mathcal{D}_L$  gauge symmetry in the space  $\mathcal{H}^{\alpha\tilde{x}}(\mathcal{E}_L)$  defined as the space  $\mathcal{H}(\mathcal{E}_L)$  endowed with the scalar product

$$\langle \Psi | \Psi' \rangle = \int_{\mathcal{E}_L} \mathcal{D}^{\alpha\tilde{x}} \tilde{x} \overline{\Psi}[\tilde{x}] \Psi'[\tilde{x}] . \tag{23}$$

Let us recall that the functional measure  $\mathcal{D}^{\alpha\tilde{x}}$  in (23) is formally defined as the Riemannian volume element related to the following (weak) Riemannian structure on  $\mathcal{E}_L$

$$E_{\tilde{x}}^{\alpha\tilde{x}}(\delta\tilde{x}, \delta\tilde{x}') = \int_0^1 e ds \delta\tilde{x}^\mu(s) \delta\tilde{x}'_\mu(s) .$$

The pull-back of the Riemannian structure  $E^\alpha$  by the gauge transformation

$$F_\gamma : \mathcal{E}_L \ni \tilde{x} \longrightarrow \tilde{x} \circ \gamma \in \mathcal{E}_L ,$$

is given by

$$F_\gamma^* E_{\tilde{x}}^{\alpha\tilde{e}}(\delta\tilde{x}, \delta\tilde{x}') = \int_0^1 e ds \delta\tilde{x} \circ \gamma(s) \delta\tilde{x}' \circ \gamma(s) = E^{(\gamma^{-1})^* e_{\tilde{x}}(\delta\tilde{x}, \delta\tilde{x}') .$$

For  $e = \alpha\tilde{e}$  where  $\tilde{e}$  is a constant (as it is in the 1-dim conformal gauge (14)) we have

$$(\gamma^{-1})^* e = (\gamma^{-1})' \alpha\tilde{e} .$$

Then, calculating the 1-dim "conformal anomaly" (Appendix B, (64)) and choosing the (1-dim) bulk renormalization constant equal zero one gets

$$\mathcal{D}^{\alpha\tilde{e}} \tilde{x} \circ \gamma = e^{-\frac{d}{16}(\log \gamma'(0) + \log \gamma'(1))} \mathcal{D}^{\alpha\tilde{e}} \tilde{x} .$$

It follows that in order to get the unitary  $\mathcal{D}_L$ -action on  $\mathcal{H}(\mathcal{E}_L)$  the "naive" representation (22) of the induced gauge transformations must be replaced by

$$\mathcal{H}(\mathcal{E}_L) \ni \Psi[\tilde{x}] \xrightarrow{\gamma \in \mathcal{D}_L} \rho[\gamma] \Psi[\tilde{x} \circ \gamma] \in \mathcal{H}(\mathcal{E}_L) , \quad (24)$$

where

$$\rho[\gamma] \equiv e^{-\frac{d}{16}(\log \gamma'(0) + \log \gamma'(1))} , \quad \rho[\gamma \circ \delta] = \rho[\gamma] \rho[\delta] : \quad \gamma, \delta \in \mathcal{D}_L .$$

Indeed in this case we have

$$\begin{aligned} \int_{\mathcal{E}_L} \mathcal{D}^{\alpha\tilde{e}} \tilde{x} \rho[\gamma] \overline{\Psi[\tilde{x} \circ \gamma]} \rho[\gamma] \Psi'[\tilde{x} \circ \gamma] &= \int_{\mathcal{E}_L} \mathcal{D}^{\alpha\tilde{e}} \tilde{x} \circ \gamma^{-1} \rho[\gamma]^2 \overline{\Psi[\tilde{x}]} \Psi'[\tilde{x}] \\ &= \int_{\mathcal{E}_L} \mathcal{D}^{\alpha\tilde{e}} \tilde{x} \overline{\Psi[\tilde{x}]} \Psi'[\tilde{x}] . \end{aligned}$$

Let us note that the modified  $\mathcal{D}_L$ -action is independent of  $\alpha$ . The fully covariant with respect to the choice of  $\tilde{e}$  description of the modified action requires more general geometrical framework and will not be discussed here.

According to (24) the space  $\mathcal{H}_{\text{inv}}(\mathcal{E}_L)$  consists of all functionals satisfying

$$\Psi[\tilde{x}] = e^{-\frac{d}{16}(\log \gamma'(0) + \log \gamma'(1))} \Psi[\tilde{x} \circ \gamma] , \quad \gamma \in \mathcal{D}_L .$$

The space  $\mathcal{H}_{\text{inv}}(\mathcal{E}_L)$  can be also characterized in terms of constraints linear in momenta given by the generators of the representation (24) ( $k \geq 1$ )

$$\begin{aligned} V_k^x &\equiv -i \int_0^1 ds \sin \pi k s (\tilde{x}^\mu)'(s) \frac{\delta}{\delta \tilde{x}^\mu(s)} - i \int_0^1 ds \sin \pi k s \frac{\delta}{\delta \gamma(s)} \rho(\gamma) \Big|_{\gamma = \text{id}_L} \quad (25) \\ &= -\frac{\pi}{2} \sum_{n=1}^k n x_n^\mu p_{\mu(k-n)} + \frac{\pi}{2} \sum_{n=1}^{\infty} (n x_k^\mu p_{\mu(n+k)} - (n+k) x_{n+k}^\mu p_{\mu n}) + i\pi \frac{d}{8} p(k) k ; \end{aligned}$$

where

$$\begin{aligned} p_\mu(s) &\equiv -i \frac{\delta}{\delta \tilde{x}^\mu(s)} = p_{\mu 0} + 2 \sum_{k=1}^{\infty} p_{\mu k} \cos \pi k s , \\ \tilde{x}^\mu(s) &= x_0^\mu + \sum_{k=1}^{\infty} x_k^\mu \cos \pi k s , \end{aligned}$$

and

$$p(k) = \begin{cases} 1 & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd} \end{cases}$$

Let us note that the constraints  $V_k^x$  are formally self-adjoint operators in the Hilbert space  $\mathcal{H}^{\alpha\tilde{e}}(\mathcal{E}_L)$  which is in agreement with the path integral derivation of the unitary  $\mathcal{D}_L$ -action given above. Another interesting property is that  $V_k^x$  are normally ordered, a feature which is required in the canonical quantization on different grounds.

## 2.5 Constraints quadratic in momenta

In the covariant formulation of the first quantized relativistic particle the Euclidean transition amplitude is interpreted as a matrix element of the constraint quadratic in momenta. Inverting this operator and performing the Wick rotation on  $\tau$  gets the on-mass-shell condition simply given by the Klein-Gordon wave equation. As we shall see a similar interpretation is valid in the case of Polyakov string.

Due to the presence of residual induced gauge symmetry it is enough to consider the transition amplitude between states  $|\Psi\rangle \in \mathcal{H}(C_L)$  described by  $\alpha$ -independent,  $\mathcal{D}_L$ -invariant string wave functionals  $\Psi[\tilde{x}]$ . According to (9) and (18), for  $d = 26$  the transition amplitude between states  $\Psi, \Psi' \in \mathcal{H}_{\text{inv}}(\mathcal{E}_L) \subset \mathcal{H}(C_L)$  is given by

$$\begin{aligned} \langle \Psi | P | \Psi' \rangle &= \int_0^\infty d\alpha_f \int_{\mathcal{E}_L} \mathcal{D}^{\alpha, \tilde{e}} \tilde{x}_f \int_0^\infty d\alpha_i \int_{\mathcal{E}_L} \mathcal{D}^{\alpha, \tilde{e}} \tilde{x}_i \overline{\Psi[\tilde{x}_f]} P[\alpha_f, \tilde{x}_f; \alpha_i, \tilde{x}_i] \Psi'[\tilde{x}_i] \\ &= \int_0^\infty dt \eta(t) \int_{\mathcal{W}^n} \mathcal{D}^{\tilde{e}} \varphi (\text{Vol}_{\mathcal{G}_t} \mathcal{W}_M^n)^{-1} e^{\frac{d-d}{32} \sum_{\text{corners}} \varphi(\epsilon_i)} \\ &\times \int_0^\infty d\alpha_f \delta(\alpha_f - \int e^{\frac{1}{2} \tilde{\varphi}_f}) \int_{\mathcal{E}_L} \mathcal{D}^{\alpha, \tilde{e}} \tilde{x}_f \int_0^\infty d\alpha_i \delta(\alpha_i - \int e^{\frac{1}{2} \tilde{\varphi}_i}) \int_{\mathcal{E}_L} \mathcal{D}^{\alpha, \tilde{e}} \tilde{x}_i \\ &\times \overline{\Psi[\tilde{x}_f]} (\tilde{x}_f \circ \gamma[\tilde{\varphi}_f]) e^{-iH_d^0[\tilde{x}_i \circ \gamma[\tilde{\varphi}_i]]} \Psi'[\tilde{x}_i] ; \end{aligned}$$

where

$$H_0^z = \frac{\pi}{2} \left[ \frac{1}{2M_x} p_0^2 + \sum_{k=1}^{\infty} \left( \frac{1}{M_x} p_k^2 + M_x k^2 x_k^2 \right) \right] ; \quad M_x = \frac{1}{4\alpha'} . \quad (26)$$

Changing variables

$$\tilde{x}_i \longrightarrow \tilde{x}_i \circ \gamma[\tilde{\varphi}_i]^{-1} \quad , \quad (i \rightarrow f) \quad ,$$

and using the relations (valid for  $\alpha_i = \int e^{\frac{1}{2}\tilde{\varphi}_i}$ ; see Appendix B, (64))

$$\mathcal{D}^{\alpha_i} \tilde{x}_i^\mu \circ \gamma[\tilde{\varphi}_i]^{-1} = e^{\frac{d}{i\tilde{\alpha}_i}(\tilde{\varphi}_i(0)+\tilde{\varphi}_i(1))} \mathcal{D}^{\tilde{\alpha}_i} \tilde{x}_i^\mu \quad , \quad (i \rightarrow f) \quad , \quad (27)$$

one gets for  $\mathcal{D}_L$ -invariant states

$$\begin{aligned} \langle \Psi | P | \Psi' \rangle &= \int_0^\infty dt \eta(t) \int_{\mathcal{W}_M^n} \mathcal{D}^{\tilde{\alpha}_i} \varphi \left( \text{Vol}_{\tilde{\varphi}_i}^n \mathcal{W}_M^n \right)^{-1} e^{\frac{\tilde{\alpha}_i}{i\tilde{\alpha}_i} \sum_{\text{corners}} \varphi(z_i)} \left( \int e^{\frac{1}{2}\tilde{\varphi}_i} \int e^{\frac{1}{2}\tilde{\varphi}_i} \right)^{\frac{13}{4}} \\ &\times \int_{\tilde{\mathcal{E}}_L} \mathcal{D}^{\tilde{\alpha}_f} \tilde{x}_f \int_{\tilde{\mathcal{E}}_L} \mathcal{D}^{\tilde{\alpha}_i} \tilde{x}_i \overline{\Psi}[\tilde{x}_f] \langle \tilde{x}_f | e^{-tH_{\tilde{\alpha}_i}^*} | \tilde{x}_i \rangle \Psi'[\tilde{x}_i] \quad . \end{aligned}$$

In the formula above the integration over conformal factor decouples yielding an overall divergent factor independent of the states  $\Psi, \Psi' \in \mathcal{H}_{\text{inv}}(\mathcal{C}_L)$ . It follows that one can restrict oneself to the space  $\mathcal{H}(\mathcal{E}_L) \subset \mathcal{H}(\mathcal{C}_L)$  of  $\alpha$ -independent states endowed with the scalar product (23) with  $\epsilon = \tilde{\epsilon}$ . Then the (regularized) transition amplitude between  $\mathcal{D}_L$ -invariant states in  $\mathcal{H}^{\tilde{\epsilon}}(\mathcal{E}_L)$  takes the following simple form

$$\langle \Psi | P_R | \Psi' \rangle = \langle \Psi | \int_0^\infty dt \eta(t) e^{-tH_{\tilde{\alpha}_i}^*} | \Psi' \rangle \quad . \quad (28)$$

Let us note that the path integral representation (18) of the transition amplitude is well defined only on the subspace  $\mathcal{H}_{\text{inv}}(\mathcal{E}_L) \subset \mathcal{H}^{\tilde{\epsilon}}(\mathcal{E}_L)$ . In order to get a well defined representation in the whole Hilbert space  $\mathcal{H}^{\tilde{\epsilon}}(\mathcal{E}_L)$  one has to restrict the space of trajectories in (10) such that the conformal factor already decouples in the formula (18). This restriction depends on some additional geometrical data which in the conformal gauges (13,14) consist of fixed parameterizations of the initial and final boundary components. Any particular choice of this data leads to some extension of the formula (28) to the space  $\mathcal{H}^{\tilde{\epsilon}}(\mathcal{E}_L)$  and can be regarded as a choice of gauge in the second quantized theory [23, 25]. For the sake of simplicity we will use the simplest extension

$$P_R[\tilde{x}_f, \tilde{x}_i] = \langle \tilde{x}_f | P_R | \tilde{x}_i \rangle = \langle \tilde{x}_f | \int_0^\infty dt \eta(t) e^{-tH_{\tilde{\alpha}_i}^*} | \tilde{x}_i \rangle \quad . \quad (29)$$

It should be stressed that all further considerations are independent of this choice.

In order to derive the on-mass-shell condition one would like to invert the operator  $P_R$ . The problem is that, due to the  $\eta$ -function insertion,  $P_R$  is not invertible on the whole

space  $\mathcal{H}^{\tilde{\epsilon}}(\mathcal{E}_L)$ . Moreover the formula (29) does not describe any operator on the subspace  $\mathcal{H}_{\text{inv}}(\mathcal{E}_L)$  of  $\mathcal{D}_L$ -invariant states. The largest subspace of  $\mathcal{H}_{\text{inv}}(\mathcal{E}_L)$  on which  $P_R[\tilde{x}_f, \tilde{x}_i]$  can be regarded as an integral kernel of a well defined operator is characterized by the equations

$$H_k^x |\Psi\rangle = 0 \quad , \quad V_k^x |\Psi\rangle = 0 \quad ; \quad k = 1, \dots \quad , \quad (30)$$

where

$$\begin{aligned} H_k^x &\equiv -\frac{i}{\pi k} [H_0^x, V_k^x] \\ &= \frac{\pi}{2} \left[ \frac{1}{2M_x} p_k \cdot p_0 + \frac{1}{2} \sum_{n=1}^k \left( \frac{1}{M_x} p_n \cdot p_{k-n} - M_x n(k-n) x_n \cdot x_{k-n} \right) \right. \\ &\quad \left. + \sum_{n=1}^\infty \left( \frac{1}{M_x} p_n \cdot p_{k+n} + M_x n(k+n) x_n \cdot x_{k+n} \right) \right] \quad . \end{aligned} \quad (31)$$

In order to analyse the integrability conditions of the equations (30) it is convenient to introduce the operators

$$L_{\pm k}^x \equiv \frac{1}{\pi} (H_k^x \pm iV_k^x) \quad , \quad k = 1, \dots \quad ;$$

which are just the standard representations of the Virasoro generators :

$$\begin{aligned} L_k^x &= \frac{1}{2} \sum_{n=-\infty}^{+\infty} \alpha_{-n} \cdot \alpha_{k+n} \quad , \quad k = \pm 1, \pm 2, \dots \\ \alpha_0^\mu &\equiv \frac{1}{\sqrt{2M_x}} p_{\mu 0} \quad , \\ \alpha_n^\mu &\equiv \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{M_x}} p_{\mu n} - i\sqrt{M_x} n x_n^\mu \right) \quad , \quad n = \pm 1, \pm 2, \dots \quad ; \\ [\alpha_n^\mu, \alpha_m^\nu] &= n \delta^{\mu\nu} \delta_{n,-m} \quad , \quad \alpha_n^\mu = \alpha_{-n}^\mu \quad . \end{aligned} \quad (32)$$

In terms of  $L_k^x$  the integrability conditions take the form

$$[L_n^x, L_m^x] = (n-m)L_{n+m}^x + \frac{26}{12} \delta_{n,-m} (n^3 - n) \quad ,$$

where

$$L_0^x \equiv \frac{1}{2} \alpha_0 \cdot \alpha_0 + \sum_{n=1}^\infty \alpha_{-n} \cdot \alpha_n \quad . \quad (33)$$

In order to obtain nontrivial solutions to the equations (30) one has to relax their strong form. Since  $L_k^{x+} = L_{-k}^x$ , this leads to the familiar conditions for the off-mass-shell physical states

$$L_k^x |\Psi\rangle = 0 \quad , \quad k \geq 1 \quad . \quad (34)$$

The derivation of the physical state conditions presented above requires an explanation. The additional constraints  $H_k^\pm$  have been introduced for technical reasons. There is however a physical motivation for these constraints stemming from the fact that the string is an extended relativistic system. This implies that the intersection of the string world sheet by an equal time hyperplane provides a half of the Cauchy data for string trajectory. It follows that for each particular choice of reference system in the Minkowski space-time the string wave functional should be independent of string fluctuations in the time direction. This kinematical requirement can be regarded as a manifestation of the general locality and causality principles of relativistic quantum theory and indicates a fundamental difference between the theory of relativistic string and the theory of random surfaces. Note that in the canonical quantization the kinematical requirement simply means that all negative norm states must decouple.

The way in which the  $H_k^\pm$  constraints yield exactly the missing part of the physical state conditions necessary and sufficient to satisfy the kinematical requirement is not quite straightforward. The rough counting of the degrees of freedom shows that two sets of constraints  $\{V_k\}$  and  $\{H_k\}$  reduce by 2 the number of physical direction. However, as it was mentioned above one can impose only a half of these constraints as conditions for physical states, which explicitly removes only one direction. The mechanism which ensures the decoupling of the second direction is that of the null states which are among solutions to the equations (34) [26, 27]. As the null states decouple from all other solutions the space of physical states is effectively given by the space of equivalence classes. One possible way to describe the corresponding quotient is to introduce the Euclidean "quantum" version of the light-cone gauge conditions

$$\alpha_k^\pm |\Psi\rangle \equiv (i\alpha_k^0 + \alpha_k^{25}) |\Psi\rangle = 0 \quad , \quad k \geq 1 \quad . \quad (35)$$

The conditions above can be seen as an explicit implementation of the kinematical requirement in the limit case of the light-like direction. It is an interesting open question whether the Virasoro algebra of constraints is the only solution to the kinematical requirement of the quantum theory of 1-dim extended relativistic systems.

The subspace  $\mathcal{H}_{\text{ph}}^{\text{off}}(\mathcal{E}_L)$  of solutions to the equations (34,35) can be explicitly constructed by means of the (extended) DDF method [28]. One can check that the operators  $\{\tilde{x}^\pm(s)\} \equiv \{\tilde{x}^k(s)\}_{k=1,\dots,d-2}$  form in  $\mathcal{H}_{\text{ph}}^{\text{off}}(\mathcal{E}_L)$  a complete set of commuting "observables".

In particular every state  $|\Psi_{\text{ph}}\rangle \in \mathcal{H}_{\text{ph}}^{\text{off}}(\mathcal{E}_L)$  is completely described by the wave functional

$$\Psi_{\text{ph}}[\tilde{x}^\pm] = \langle \tilde{x}^\pm | \Psi_{\text{ph}} \rangle \quad . \quad (36)$$

The transition amplitude between off-mass-shell physical states  $\Psi_{\text{ph}}, \Psi'_{\text{ph}} \in \mathcal{H}_{\text{ph}}^{\text{off}}(\mathcal{E}_L)$  described by the wave functionals (36) takes the following form

$$\begin{aligned} \langle \Psi_{\text{ph}} | P_R | \Psi'_{\text{ph}} \rangle &= \int_{\mathcal{E}_L} \mathcal{D}^d \tilde{x}_f \int_{\mathcal{E}_L} \mathcal{D}^d \tilde{x}_i \overline{\Psi_{\text{ph}}[\tilde{x}_f]}(\tilde{x}_f) \int_0^\infty dt \eta(t) e^{-iH\tilde{\sigma}} |\tilde{x}_i\rangle \Psi'_{\text{ph}}[\tilde{x}_i^\pm] \\ &= \langle \Psi_{\text{ph}} | \int_0^\infty dt e^{-i\pi(K_0^\pm - 1)} | \Psi'_{\text{ph}} \rangle_{\text{ph}}^{\text{off}} \quad , \end{aligned} \quad (37)$$

where  $K_0^\pm$  denotes the restriction of the operator  $L_0^\pm$  to the subspace  $\mathcal{H}_{\text{ph}}^{\text{off}}(\mathcal{E}_L)$  and  $\langle \dots | \dots \rangle_{\text{ph}}^{\text{off}}$  is the scalar product in  $\mathcal{H}_{\text{ph}}^{\text{off}}(\mathcal{E}_L)$  regarded as a Hilbert subspace of  $\mathcal{H}^{\pm}(\mathcal{E}_L)$ . It follows that the operator defined by the matrix elements (37) can be inverted on the subspace  $\mathcal{H}_{\text{ph}}^{\text{off}}(\mathcal{E}_L)$ . Then the on-mass-shell condition in  $\mathcal{H}_{\text{ph}}^{\text{off}}(\mathcal{E}_L)$  is given by

$$(K_0^\pm - 1) |\Psi\rangle = 0 \quad ,$$

which is equivalent to the condition

$$(L_0^\pm - 1) |\Psi\rangle = 0 \quad . \quad (38)$$

in the space  $\mathcal{H}^{\pm}(\mathcal{E}_L)$ .

Performing the Wick rotation in the physical state conditions (34,38) one gets the familiar equations of the so called old covariant approach. All further steps of quantization proceed along the standard lines [26, 27].

### 3 The noncritical Polyakov string

#### 3.1 Gauge symmetry

In this section we will present the covariant functional quantization of the Polyakov string in the flat target space of dimension  $d$  in the range  $1 < d < 25$ . The first steps of this quantization procedure – the description of the space of trajectories in the configuration space, the construction of the space of states, and the path integral representation of the

transition amplitude – are almost the same as in the case of the critical string. The main difference consists in the symmetry requirements imposed on the quantum theory.

Since, in the range  $1 < d < 25$ , the conformal anomaly breaks the Weyl invariance completely, the gauge symmetry in the space of trajectories reduces to the group  $\mathcal{D}_M^n$  acting on  $\mathcal{M}_M^n \times \mathcal{E}_M^n$  by

$$\mathcal{M}_M^n \times \mathcal{E}_M^n \ni (g, x) \xrightarrow{f \in \mathcal{D}_M^n} (f^*g, f^*x) \in \mathcal{M}_M^n \times \mathcal{E}_M^n .$$

Accordingly, the induced gauge transformations in the space  $\mathcal{P}_L$  (5) of boundary conditions take the form

$$\mathcal{P}_L \ni (\epsilon_i, x_i) \xrightarrow{\gamma \in \mathcal{D}_L} (\gamma^*\epsilon_i, x_i \circ \gamma) \in \mathcal{P}_L .$$

Repeating the reasoning of Subsect.2.2 one gets the space of states  $\mathcal{H}(\mathcal{C}_L)$  endowed with the scalar product (8).

Because of the restricted gauge group there is no residual gauge invariance in  $\mathcal{H}(\mathcal{C}_L)$ .

### 3.2 Transition amplitude

According to the different symmetry requirements, the path integral representation (10) for the transition amplitude gets slightly modified

$$P[c_f, c_i] = \int_{\mathcal{F}[c_f, c_i]} \mathcal{D}^d g \mathcal{D}^d x (\text{Vol}_g \mathcal{D}_M^n)^{-1} \exp\left(-\frac{1}{4\pi\alpha'} S[g, x]\right)$$

In the conformal gauges (13,14) one has

$$\begin{aligned} P[\alpha_f, \bar{x}_f; \alpha_i, \bar{x}_i] &= \int_0^\infty dt \eta(t)^{1-\frac{d}{2}} t^{-\frac{d}{2}} \int_{\mathcal{W}_M^n} \mathcal{D}^{\hat{g}_t} \varphi \\ &\times \exp\left(-\frac{25-d}{48\pi} S_L[\hat{g}_t, \varphi]\right) \\ &\times \exp\left(\frac{7-d}{32} \sum_{\text{corners}} \varphi(z_i)\right) \\ &\times \exp\left(-\frac{1}{4\pi\alpha'} S[\hat{g}_t, \varphi, \bar{x}_i, \bar{x}_f]\right) , \\ &\times \delta\left(\alpha_f - \int_0^1 dz^1 e^{\frac{1}{2}\varphi_f}\right) \delta\left(\alpha_i - \int_0^1 dz^1 e^{\frac{1}{2}\varphi_i}\right) \end{aligned} \quad (39)$$

Lets us note that in contrast to the formula (18) the  $\varphi$ -dependence of the functional measure  $\mathcal{D}^{\hat{g}_t} \varphi$  is not canceled by the similar  $\varphi$ -dependence in the volume factor  $\text{Vol}_{e^{\varphi}} \mathcal{W}_M^n$ .

As a result, one gets the different coefficients in front of the Liouville action and the corner anomaly term.

Some remarks concerning the formula (39) are in order. First of all one has to choose some values of the renormalization constants  $\mu, \lambda$  appearing in the Liouville action (19). In the following we will restrict ourselves to the simplest choice

$$\mu = \lambda = 0 . \quad (40)$$

This is well justified in the free theory. First, let us observe that the nonvanishing boundary cosmological constant is incompatible with the boundary conditions (12). Secondly, using the Gauss-Bonnet theorem on the rectangle one can easily show that for  $\hat{g}_t \in \mathcal{M}_M^{n*}$  the Liouville equation

$$\Delta_{\hat{g}_t} \varphi + \mu e^\varphi = 0 ,$$

does not have any solution in the space  $\mathcal{W}_M^n$ . It follows that the variational problem given by the Liouville action (19) is well posed and has a solution in  $\mathcal{W}_M^n$  if and only if the equations (40) are satisfied. Let us stress that the conclusion above is not necessarily valid for more complicated world sheet topologies. In particular in the case of hyperbolic hexagon the classical solution exists only for  $\mu > 0$ . An independent motivation for the choice (40) in the free string theory stems from the semiclassical calculations of the static potential [29], where one obtains the same result for the vanishing and for the positive bulk cosmological constant.

Under the assumption (40) and in the conformal gauge (13) the Liouville action is just the free field action

$$S[\hat{g}_t, \varphi] = \frac{1}{2} \int_0^1 dz^0 \int_0^1 dz^1 \left( (\partial_0 \varphi)^2 + (\partial_1 \varphi)^2 \right) .$$

Even with this simplification the formula (39) still contains the complicated nonlocal interaction which prevents calculations of the functional integral in all but few special cases of  $x$ -boundary conditions [29]. Our idea to overcome this difficulty is to regard the transition amplitude (39) as a matrix element of some simple operator between special states in an extended space. The method to find such extension of  $\mathcal{H}(\mathcal{C}_L)$  is based on the simple observation that the interaction terms depend only on the boundary values of the conformal factor. Thus it should be possible to replace the integration over fields satisfying the homogeneous Neumann boundary condition by the integral over fields with

a fixed nonhomogeneous Dirichlet condition (which is Gaussian) and then an integral over all possible boundary values. In the case under consideration one can expect the following formula

$$\int_{\mathcal{W}_0^d} \mathcal{D}^d \varphi e^{-\frac{25-d}{48} S_L[\hat{g}_t, \varphi]} F[\tilde{\varphi}_f, \tilde{\varphi}_i] = \eta^{-\frac{1}{2}} t^{-\frac{1}{2}} \int_{\mathcal{W}_L} \mathcal{D}^d \tilde{\varphi}_f \int_{\mathcal{W}_L} \mathcal{D}^d \tilde{\varphi}_i e^{-\frac{25-d}{48} S_L[\tilde{g}_t, \tilde{\varphi}_f, \tilde{\varphi}_i]} F[\tilde{\varphi}_f, \tilde{\varphi}_i] , \quad (41)$$

where

$$S_L[\tilde{g}_t, \tilde{\varphi}_f, \tilde{\varphi}_i] \equiv S_L[\hat{g}_t, \varphi_d] ,$$

and  $\varphi_d: \tilde{M}_t \rightarrow R^d$  is the solution of the boundary value problem

$$\begin{aligned} (\partial_0^2 + \partial_1^2) \varphi_d &= 0 , \\ \varphi_d(0, z^1) &= \tilde{\varphi}_i(z^1) , \\ \varphi_d(t, z^1) &= \tilde{\varphi}_f(z^1) , \\ \partial_1 \varphi_d(z^0, 0) &= \partial_1 \varphi_d(z^0, 1) = 0 . \end{aligned}$$

The relation (41) is well known for Gaussian integrals (as a formula for determinants [30]) and is supposed to be valid in general situation [31]. Since we are not aware of any proof in the case of boundary interactions, a simple derivation of the formula (41) is given in the Appendix B.

Using (41) the transition amplitude between arbitrary states  $|\Psi\rangle, |\Psi'\rangle \in \mathcal{H}(\mathbb{R}_+ \times \mathcal{E}_L) \subset \mathcal{H}(\mathcal{C}_L)$  can be rewritten in the following form

$$\begin{aligned} \langle \Psi | P | \Psi' \rangle &= \int_0^\infty d\alpha_f \int_{\mathcal{E}_L} \mathcal{D}^d \tilde{x}_f \int_0^\infty d\alpha_i \int_{\mathcal{E}_L} \mathcal{D}^d \tilde{x}_i \overline{\Psi}[\alpha_f, \tilde{x}_f] P[\alpha_f, \tilde{x}_f; \alpha_i, \tilde{x}_i] \Psi'[\alpha_i, \tilde{x}_i] \\ &= \int_{\mathcal{W}_L} \mathcal{D}^d \tilde{\varphi}_f \int_{\mathcal{E}_L} \mathcal{D}^d \tilde{x}_f \int_{\mathcal{W}_L} \mathcal{D}^d \tilde{\varphi}_i \int_{\mathcal{E}_L} \mathcal{D}^d \tilde{x}_i \\ &\times \overline{\Psi}[\alpha_f, \tilde{x}_f] e^{\frac{25-d}{32}(\tilde{\varphi}_f(0) + \tilde{\varphi}_f(1))} \\ &\times (\tilde{\varphi}_f \otimes \langle \tilde{x}_f \circ \gamma[\tilde{\varphi}_f] | \int_0^\infty dt \eta(t) e^{-i(H_0^x + H_0^p)} | \tilde{\varphi}_i \rangle \otimes | \tilde{x}_i \circ \gamma[\tilde{\varphi}_i] \rangle \\ &\times \Psi'[\alpha_i, \tilde{x}_i] e^{\frac{25-d}{32}(\tilde{\varphi}_i(0) + \tilde{\varphi}_i(1))} , \end{aligned}$$

where  $\alpha_i = \int e^{\frac{1}{2} \tilde{\varphi}_i}$ , ( $i \rightarrow f$ ).

In the formula above  $H_0^x + H_0^p$  is regarded as an operator on the space  $\mathcal{H}(\mathcal{W}_L \times \mathcal{E}_L)$  with the  $x$ -part defined by (26) and with the  $\varphi$ -part given by

$$H_0^x \equiv \frac{\pi}{2} \left[ \frac{1}{2M_\varphi} \pi_0^2 + \sum_{k=1}^\infty \left( \frac{1}{M_\varphi} \pi_k^2 + M_\varphi k^2 \varphi_k^2 \right) \right] \quad M_\varphi = \frac{25-d}{96} ;$$

where

$$\begin{aligned} \pi(s) &\equiv -i \frac{\delta}{\delta \tilde{\varphi}(s)} = \pi_0 + 2 \sum_{k=1}^\infty \pi_k \cos \pi k s , \\ \tilde{\varphi}(s) &= \varphi_0 + \sum_{k=1}^\infty \varphi_k \cos \pi k s . \end{aligned} \quad (42)$$

Changing variables

$$\tilde{x}_i \longrightarrow \tilde{x}_i \circ \gamma[\tilde{\varphi}_i]^{-1} , \quad (i \rightarrow f) ,$$

and using the relations (27) one gets the representation required

$$\langle \Psi | P | \Psi' \rangle = \langle \tilde{\Psi} | \int_0^\infty dt \eta(t) e^{-i(H_0^x + H_0^p)} | \tilde{\Psi}' \rangle , \quad (43)$$

where for each state  $|\Psi\rangle \in \mathcal{H}(\mathbb{R}_+ \times \mathcal{E}_L)$  the state  $|\tilde{\Psi}\rangle \in \mathcal{H}(\mathcal{W}_L \times \mathcal{E}_L)$  is given by the wave functional

$$\tilde{\Psi}[\tilde{\varphi}, \tilde{x}] \equiv e^{\frac{25-d}{32}(\tilde{\varphi}(0) + \tilde{\varphi}(1))} \Psi[\int e^{\frac{1}{2} \tilde{\varphi}}, \tilde{x} \circ \gamma[\tilde{\varphi}]^{-1}] . \quad (44)$$

and the scalar product

$$\langle \tilde{\Psi} | \tilde{\Psi}' \rangle = \int_{\mathcal{W}_L} \mathcal{D}^d \tilde{\varphi} \int_{\mathcal{E}_L} \mathcal{D}^d \tilde{x} \overline{\tilde{\Psi}[\tilde{\varphi}, \tilde{x}]} \tilde{\Psi}'[\tilde{\varphi}, \tilde{x}] \quad (45)$$

is used on the r.h.s of (43).

### 3.3 Physical state conditions

Due to the simple representation (43) of the transition amplitude it is convenient to analyse the physical state conditions in the extended space  $\mathcal{H}(\mathcal{W}_L \times \mathcal{E}_L)$ . We start with the discussion of the conditions related to the extension itself. Their role is to select in the extended space the image of the original space of states under the extension map

$$\text{Ext} : \mathcal{H}(\mathbb{R}_+ \times \mathcal{E}_L) \ni \Psi \longrightarrow \tilde{\Psi} \in \mathcal{H}(\mathcal{W}_L \times \mathcal{E}_L) ,$$

where  $\tilde{\Psi}$  is given by the formula (44). Using the equations (17) one can show that the functionals  $\Phi[\tilde{\varphi}, \tilde{x}] \in \mathcal{H}(\mathcal{W}_L \times \mathcal{E}_L)$  of the form (44) can be uniquely characterized as functionals invariant with respect to the following  $\mathcal{D}_L$  action

$$\mathcal{H}(\mathcal{W}_L \times \mathcal{E}_L) \ni \Phi[\tilde{\varphi}, \tilde{x}] \xrightarrow{\gamma \in \mathcal{D}_L} \tilde{\rho}[\gamma] \Phi[\tilde{\varphi} \circ \gamma + 2 \log \gamma', \tilde{x} \circ \gamma] \in \mathcal{H}(\mathcal{W}_L \times \mathcal{E}_L) , \quad (46)$$

where

$$\tilde{\rho}[\gamma] \equiv e^{-\frac{1+d}{16}(\log \gamma'(0) + \log \gamma'(1))} , \quad \tilde{\rho}[\gamma \circ \delta] = \tilde{\rho}[\gamma] \tilde{\rho}[\delta] ; \quad \gamma, \delta \in \mathcal{D}_L . \quad (47)$$

Within the old covariant approach the  $\mathcal{D}_L$ -action (46) has to be modified in order to meet the requirement of unitary realization of the residual symmetry. The analysis of the transformation properties of the scalar product (45) with respect to the transformations

$$\mathcal{W}_L \times \mathcal{E}_L \ni (\tilde{\varphi}, \tilde{x}) \xrightarrow{\gamma \in \mathcal{D}_L} (\tilde{\varphi} \circ \gamma + 2 \log \gamma', \tilde{x} \circ \gamma) \in \mathcal{W}_L \times \mathcal{E}_L ,$$

leads to the following unitary  $\mathcal{D}_L$ -action

$$\mathcal{H}(\mathcal{W}_L \times \mathcal{E}_L) \ni \Phi[\tilde{\varphi}, \tilde{x}] \xrightarrow{\gamma \in \mathcal{D}_L} \rho[\gamma] \Phi[\tilde{\varphi} \circ \gamma + 2 \log \gamma', \tilde{x} \circ \gamma] \in \mathcal{H}(\mathcal{W}_L \times \mathcal{E}_L) , \quad (48)$$

with

$$\rho[\gamma] \equiv e^{-\frac{1+d}{16}(\log \gamma'(0) + \log \gamma'(1))} , \quad \rho[\gamma \circ \delta] = \rho[\gamma] \rho[\delta] ; \quad \gamma, \delta \in \mathcal{D}_L . \quad (49)$$

The discrepancy between  $\tilde{\rho}$  (47) derived from the representation (43) and  $\rho$  (49) obtained from the unitarity requirement is related to the fact that in the old covariant approach one disregards the ghost sector. The detailed discussion of this point requires the full BRST formulation which is beyond the scope of the present paper. Let us only mention that  $\tilde{\rho}$  given by the formula (47) leads to a unitary realization of the  $\mathcal{D}_L$ -symmetry in the BRST extended space.

The generators of the representation (48) take the form

$$\begin{aligned} V_k &\equiv -i \int_0^1 ds \sin \pi k s (\tilde{x}^\mu)'(s) \frac{\delta}{\delta \tilde{x}^\mu(s)} \\ &\quad - i \int_0^1 ds \sin \pi k s (\tilde{\varphi})'(s) \frac{\delta}{\delta \tilde{\varphi}(s)} - i 2\pi k \int_0^1 ds \cos \pi k s \frac{\delta}{\delta \tilde{\varphi}(s)} \\ &\quad - i \int_0^1 ds \sin \pi k s \frac{\delta}{\delta \gamma(s)} \rho[\gamma] \Big|_{\gamma=id_L} , \quad k \geq 1 \\ &= V_k^z + V_k^\varphi ; \end{aligned}$$

where  $V_k^z$  is given by (25) and

$$\begin{aligned} V_k^\varphi &= -\frac{\pi}{2} \sum_{n=1}^k n \varphi_n \pi_{(k-n)} + \frac{\pi}{2} \sum_{n=1}^{\infty} (n \varphi_n \pi_{n+k} - (n+k) \varphi_{n+k} \pi_n) \\ &\quad + 2\pi k \pi_k + i\pi \frac{1}{8} \rho(k) k , \quad k \geq 1 . \end{aligned}$$

Note that  $V_k$  are hermitian with respect to the scalar product (45) and normally ordered. The subspace  $\mathcal{H}_{\text{inv}}(\mathcal{W}_L \times \mathcal{E}_L)$  of states invariant with respect to the action (48) is determined by the equations

$$V_k |\Phi\rangle = 0 , \quad k \geq 1 .$$

The rest of the physical state conditions can be derived using the method discussed in Subject.2.4. The subspace of  $\mathcal{H}_{\text{inv}}(\mathcal{W}_L \times \mathcal{E}_L)$  on which

$$\int_0^\infty dt \eta(t) e^{-i(H_0^z + H_0^\varphi)}$$

reduces to a well defined operator is given by the equations

$$H_k |\Psi\rangle = 0 , \quad V_k |\Psi\rangle = 0 ; \quad k = 1, \dots , \quad (50)$$

where

$$H_k \equiv -\frac{i}{\pi k} [H_0^z + H_0^\varphi, V_k] = H_k^z + H_k^\varphi$$

and

$$\begin{aligned} H_k^\varphi &\equiv -\frac{i}{\pi k} [H_0^\varphi, V_k] \\ &= \frac{\pi}{2} \left[ \frac{1}{2M_\varphi} \pi_k \pi_0 + \frac{1}{2} \sum_{n=1}^k \left( \frac{1}{M_\varphi} \pi_n \pi_{k-n} - M_\varphi n(k-n) \varphi_n \varphi_{k-n} \right) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \left( \frac{1}{M_\varphi} \pi_n \pi_{k+n} + M_\varphi n(k+n) \varphi_n \varphi_{k+n} \right) + 4M_\varphi k^2 \varphi_k \right] . \end{aligned}$$

The analysis of the integrability conditions of the equations (50) can be simplified by introducing

$$L_{\pm k} \equiv \frac{1}{\pi} (H_k \pm i V_k) = L_{\pm k}^z + L_{\pm k}^\varphi , \quad k = 1, \dots$$

where the operators  $L_k^z$  are given by (32) and  $L_k^\varphi$  are just the Virasoro generators in the FCT representation [6, 32] with the central charge  $c = 26 - d$

$$L_k^\varphi \equiv \frac{1}{2} \sum_{n=-\infty}^{+\infty} \beta_{-n} \beta_{k+n} + ikQ \beta_k , \quad k = \pm 1, \pm 2, \dots$$

$$Q \equiv 2\sqrt{2M_\varphi} = \sqrt{\frac{25-d}{12}} ,$$

$$\beta_0^\varphi \equiv \frac{1}{\sqrt{2M_\varphi}} \pi_0 ,$$

$$\beta_n^\varphi \equiv \frac{1}{\sqrt{2}} \left( \frac{1}{M_\varphi} \pi_n - i\sqrt{M_\varphi} n \varphi_n \right) , \quad n = \pm 1, \pm 2, \dots ;$$

$$[\beta_n, \beta_m] = n\delta_{n,-m} \quad , \quad \beta_n^+ = \beta_{-n} \quad .$$

The integrability conditions take the following form

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{26}{12}\delta_{n,-m}(n^3-n) \quad ,$$

where

$$\begin{aligned} L_0 &\equiv L_0^x + L_0^y \quad , \\ L_0^y &\equiv \frac{1}{2}\beta_0^2 + \sum_{n=1}^{\infty} \beta_{-n}\beta_n + \frac{Q^2}{2} \quad , \end{aligned}$$

and  $L_0^x$  is given by (33).

As in the case of critical string relaxing the strong form of the equations (50) one gets the conditions for the off-mass-shell physical states

$$L_k|\Psi\rangle = 0 \quad , \quad k \geq 1 \quad . \quad (51)$$

The structure of the space  $\mathcal{H}_{\text{ph}}^{\text{off}}(\mathcal{W}_L \times \mathcal{E}_L)$  of solutions to the equations (51) is similar to that of critical string theory. As we shall see in the next subsection, the quotient space of the off-mass-shell physical states modulo null states can be uniquely characterized as the space of solutions of (51) satisfying the "quantum" light-cone gauge conditions

$$\alpha_k^+|\Psi\rangle \equiv (i\alpha_k^0 + \alpha_k^{d-1})|\Psi\rangle = 0 \quad , \quad k \geq 1 \quad .$$

In particular repeating the considerations presented in Subsect.2.4. one can derive the on-mass-shell condition which in the space  $\mathcal{H}(\mathcal{W}_L \times \mathcal{E}_L)$  takes the following form

$$(L_0 - 1)|\Psi\rangle = 0 \quad . \quad (52)$$

### 3.4 DDF construction and no-ghost theorem

In this subsection we present an explicit construction of physical states of the relativistic theory. Performing the Wick rotation in the conditions (51), (52) one gets

$$L_k|\Psi\rangle = 0 \quad , \quad k \geq 1 \quad , \quad (L_0 - 1)|\Psi\rangle = 0 \quad , \quad (53)$$

where

$$\begin{aligned} L_n &= \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_m \cdot \alpha_{n-m} + \frac{1}{2} \sum_{m=-\infty}^{\infty} \beta_m \beta_{n-m} + imQ\beta_n \quad n \neq 0 \\ L_0 &= \frac{1}{2}\alpha_0 \cdot \alpha_0 + \frac{1}{2}\beta_0^2 + \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \sum_{m=1}^{\infty} \beta_{-m}\beta_m + \frac{Q^2}{2} \end{aligned}$$

and

$$[\beta_m, \beta_n] = m\delta_{m,-n} \quad , \quad \{\alpha_m^\mu, \alpha_n^\nu\} = m\eta^{\mu\nu}\delta_{m,-n} \quad , \quad \eta^{\mu\nu} = \text{diag}(-1, +1, \dots, +1) \quad .$$

Following the DDF approach [28] we introduce the operators of "position" and "momentum":

$$\begin{aligned} X^\mu(\theta) &= x_0^\mu + \alpha_0^\mu \theta + \sum_{k \geq 1} \frac{i}{k} (\alpha_k^\mu e^{-ik\theta} - \alpha_k^{\mu+} e^{ik\theta}) \quad , \\ P^\mu(\theta) &= \alpha_0^\mu + \sum_{k \geq 1} (\alpha_k^\mu e^{-ik\theta} + \alpha_k^{\mu+} e^{ik\theta}) \quad , \\ \Phi(\theta) &= \varphi_0 + \beta_0 \theta + \sum_{k \geq 1} \frac{i}{k} (\beta_k e^{-ik\theta} - \beta_k^+ e^{ik\theta}) \quad , \\ \Pi(\theta) &= \beta_0 + \sum_{k \geq 1} (\beta_k e^{-ik\theta} + \beta_k^+ e^{ik\theta}) \quad . \end{aligned}$$

Using the relations

$$\begin{aligned} [L_n, \alpha_m^\mu] &= -m\alpha_{m+n}^\mu \quad , \\ [L_n, \beta_m] &= -m\beta_{m+n} + iQn^2\delta_{n,-m} \quad , \\ [L_n, L_m] &= (n-m)L_{n+m} + \frac{26}{12}(n^3-n)\delta_{n,-m} \quad , \end{aligned}$$

one gets

$$\begin{aligned} [X^\mu(\theta), X^\nu(\theta')] &= -i\pi\eta^{\mu\nu}\text{sgn}(\theta - \theta') \quad , \\ [P^\mu(\theta), P^\nu(\theta')] &= 2i\pi\eta^{\mu\nu}\delta'(\theta - \theta') \quad , \\ [L_n, P^\mu(\theta)] &= -i\frac{d}{d\theta} (P^\mu e^{in\theta}) \quad , \\ [L_n, \Pi(\theta)] &= -i\frac{d}{d\theta} (\Pi e^{in\theta}) + iQn^2 e^{in\theta} \quad . \end{aligned} \quad (54)$$

Let us consider the state  $|p_L, p^\mu\rangle$  satisfying

$$\begin{aligned} \alpha_n^\mu |p_L, p^\mu\rangle &= \delta_{n0} p^\mu |p_L, p^\mu\rangle \quad , \\ \beta_n |p_L, p^\mu\rangle &= \delta_{n0} p_L |p_L, p^\mu\rangle \quad , \\ (L_0 - 1) |p_L, p^\mu\rangle &= 0 \quad . \end{aligned} \quad (55)$$

For  $p^\mu \neq 0$  there exists a vector  $k$  such that  $k^\mu k_\mu = 0$  and  $k^\mu p_\mu = 1$ .

The construction of vertex operators, which acting on the state (55) generate positive norm physical states with transverse excitations is the same as in the case of Nambu-Goto

string. One gets the operators

$$A_n^i = \frac{1}{2\pi} \int_0^{2\pi} d\theta P^i(\theta) e^{ink \cdot X(\theta)} \quad , \quad i = 1, \dots, d-2 \quad , \quad n \geq 1 \quad ,$$

satisfying the relations

$$\begin{aligned} [A_m^i, A_n^j] &= m\delta^{ij}\delta_{m,-n} \quad , \\ [L_k, A_m^i] &= 0 \quad , \quad k \geq 0 \\ A_n^{i+} &= A_{-n}^i \quad . \end{aligned} \quad (56)$$

Due to the  $n^2$ -term in the commutation relation (54) the construction of the vertex operator generating states with excitation in the Liouville direction is slightly more complicated. In order to compensate this term one can use a modification introduced by Brower in his construction of the vertex generating longitudinal excitations in the Nambu-Goto string [5] and write

$$A_n^L = \frac{1}{2\pi} \int_0^{2\pi} d\theta \left( \Pi(\theta) - Q(k \cdot \dot{P}(\theta))(k \cdot P(\theta))^{-1} \right) e^{ink \cdot X(\theta)} \quad , \quad n \geq 1 \quad .$$

In contrast to Brower's longitudinal vertex the operator above satisfies the relations analogous to  $A_n^i$ :

$$\begin{aligned} [A_m^L, A_n^L] &= m\delta_{m,-n} \quad , \\ [L_k, A_m^L] &= 0 \quad , \quad k \geq 0 \quad , \\ [A_m^i, A_n^L] &= 0 \quad , \\ A_n^{L+} &= A_{-n}^L \quad . \end{aligned} \quad (57)$$

All the states generated by the operators  $A_n^i, A_m^L$  from the states  $|p_L, p^\mu\rangle$  with  $p^\mu \neq 0$  we call the DDF states. The commutation relations (56),(57) imply that the DDF states are physical states with positive norm. The inverse statement can be formulated as follows

**Theorem.** *Any solution of the equations (53) in the Hilbert space  $\mathcal{H}(\mathcal{W}_L \times \mathcal{E}_L)$  is of the form*

$$|\Phi\rangle = |\Psi\rangle + |ns\rangle \quad ,$$

where  $|\Psi\rangle$  is either a DDF state or one of the states

$$|p_L = \pm Q', p^\mu = 0\rangle \quad , \quad Q' = \sqrt{\frac{d-1}{12}} \quad , \quad (58)$$

and  $|ns\rangle$  is a null spurious state, i.e.  $|ns\rangle$  is orthogonal to all physical states and is a linear combination of states of the form  $L_{-n}|\chi\rangle, n \geq 1$ .

A counterpart of the theorem above for the standard free field realization of the Virasoro algebra with the central charge  $c = 26$  and  $\alpha_0 = 1$  has been proved long time ago by Goddard and Thorn [33]. Since the first of the two proofs given in [33] is based only on the algebraic properties of the DDF operators it applies without modification in the present case. In fact for a fixed kinematical configuration given by a state (55) with  $p^\mu \neq 0$  one can introduce the operators

$$K_n = k \cdot \alpha_n$$

satisfying the algebra

$$[K_m, K_n] = 0 \quad [L_m, K_n] = -nK_{n+m} \quad K_n^+ = K_{-n}$$

The algebra of  $A$ 's,  $K$ 's and  $L$ 's is exactly the same as in [33]. The only difference is the number of positive-norm directions ( $d-1$  instead of 24) which however does not alter the reasoning given in [33]. To complete the proof of the present version of the theorem let us observe that in the space  $\mathcal{H}(\mathcal{W}_L \times \mathcal{E}_L)$  the operator  $\beta_0$  is self-adjoint and therefore has a real spectrum. Consequently the only physical states with all components of the spacetime momenta equal zero are the lowest states given by (58) and all excited positive norm physical states can be achieved by the DDF construction.

As a simple consequence of the theorem above and the algebra (56), (57), one gets the no-ghost theorem for the model given by the equations (53). The physical content of the model can be easily inferred using the DDF construction. One can show that the positive norm physical states could be uniquely characterized in terms of the space-time spin and momenta along with an additional internal quantum number represented by the operator  $\beta_0$ . For each particular eigenvalue  $p_L$  of  $\beta_0$  the physical states satisfying  $\beta_0|\psi\rangle = p_L|\psi\rangle$  form the Hilbert space  $\mathcal{H}(p_L)$  describing a free noncritical string with the intercept  $\alpha_0 = \frac{d-1}{24} - \frac{p_L^2}{2}$ . For  $p_L^2 < \frac{d-1}{24}$  the lowest states in the space  $\mathcal{H}(p_L)$  are tachyons.

Let us note that the theory described by the Hilbert space  $\mathcal{H}(p_L)$  differs from that with the same intercept and obtained by the dimension reduction from the critical Nambu-Goto string [27]. In fact, if  $T^d(N)$  is the number of states on the level  $N$  generated by the  $d$ -component oscillators, then the numbers of the positive norm physical states on the

level  $N$  is  $T^{d-1}(N)$  in the Polyakov string while in the reduced Nambu-Goto string one gets  $T^{d-1}(N) - T^{d-1}(N-1)$  [5].

## 4 Conclusions

The main result of the present paper is that the Polyakov path integral over surfaces does lead to a free quantum theory of 1-dim extended relativistic system in the range  $1 < d < 25$ . The resulting theory is equivalent to the FCT "massive" string model. As far as the free theory is concerned this model can be directly compared with the noncritical Polyakov dual model in the range  $1 < d < 25$ . In the commonly used radial gauge the "massive" string is given by the realization (1),(2) of the Virasoro algebra. From this point of view the FCT string can be regarded as a special version of the 2-dim Liouville model coupled to  $d$ -copies of the free scalar conformal field theory, characterized by the equations

$$\mu = 0, \quad (59)$$

$$(\beta_0 + iQ)^+ = \beta_0 + iQ. \quad (60)$$

Within the Polyakov dual model approach the equations (59, 60) are just a special (in a sense trivial) choices of free "parameters": the cosmological constant and the scalar product in the space of states. In the present approach the first equation is an assumption partly justified by the requirement of stability while the second one uniquely follows from the interpretation of the model as a quantum mechanics of 1-dim extended relativistic system. Since the equations are crucial for the physical interpretation of the model we shall briefly discuss their origin within the present approach.

First of all let us stress that our whole derivation is based on the particular choice of boundary conditions for the string trajectories. In the "matter" sector these boundary conditions have been first introduced in [34]. More recently it was shown that they are relevant for constructing the off-shell-critical string amplitudes [23]. The derivation of the boundary conditions in the metric sector, based on the geometry of the space of trajectories has been presented in [24]. The outcome of this analysis is that as far as the interpretation of the Polyakov path integral as a sum over bordered surfaces is assumed

the boundary conditions in this sector are uniquely determined. Finally the relevance of these boundary conditions in the noncritical Polyakov string theory has been confirmed in our previous paper [29] concerning the quasiclassical calculation of the static potential in the range  $1 < d < 25$ . All these results along with the considerations of Sect.2 show that within the Feynman functional quantization scheme in the  $(M, n)$ -gauge the choice of boundary conditions is unique.

The second important point in our paper is the assumption concerning the vanishing cosmological constant. As it was discussed in Subsect.3.2 in the case of rectangle and for  $\mu \neq 0$  there is no classical solutions of the Liouville equation of motion in the space of conformal factors over which one has to integrate in the path integral representation of the transition amplitude. If we interpret the absence of classical extremum as an indication of instability of the system the only consistent choice is  $\mu = 0$ . Whether or not this conclusion is fully justified is still an open problem. One possible approach is to assume that the generalized Forman formula still holds in the case of the bulk exponential interaction (which can be justified to some extent by means of the perturbation expansion [31]) and then to analyse the resulting theory in the extended space. Whatever the final understanding of the model with  $\mu > 0$  would be, the simplest case  $\mu = 0$  yields a consistent free theory and it is a nontrivial and interesting problem to investigate the "joining-splitting" interaction in this model.

While the equation (59) is an assumption more or less justified by our choice of boundary conditions, the second equation (60) uniquely follows from them. In fact the central technical point of our approach - the generalized Forman formula - yields not only the simple expression for the transition amplitude in the extended space but also the inner product in the conformal factor sector.

The relation of the FCT realization of the Virasoro algebra with the Polyakov path integral over surfaces has been known since the first attempts to quantize the Liouville theory [35]. More recently it became a standard tool in analysing the physical states [18] and correlators [36, 37] in the Liouville gravity coupled to the conformal matter. The question arises what we have learned from lengthy derivation of the particular realization characterized by the equation (60).

The most important lesson is the path integral formulation of the FCT string. As it was emphasized in the introduction it paves a way for analysing the "joining-splitting"

interaction in this string model. Note that the lack of the path integral formulation was a basic obstacle for developing the interacting theory of the "massive" string twenty years ago [6].

A related issue is the target space interpretation of the free Polyakov model in the range  $1 < d < 25$ . As far as this interpretation is concerned there are no physical states with imaginary (or complex) Liouville momentum in the model. This is in contrast with the Polyakov noncritical dual model interpreted as the 2-dim Euclidean gravity coupled to the conformal matter. This interpretation concentrates on the world sheet physics bringing all the questions of the theory of 2-dim statistical systems. In particular one of the prominent observables in these framework is the area operator getting complex for the central charge of the matter sector in the range  $1 < d < 25$ . The physical states with the imaginary Liouville momenta are therefore indispensable within this interpretation and lead to the "unstable" critical behaviour [12, 13].

The derivation of the FCT free string from the Polyakov path integral over bordered surfaces given in Sect.3 sheds new light on the role the conformal factor plays in the Polyakov string model. The noncritical string model we have started with was originally described in terms of the variables  $\{\alpha, x^\mu(\sigma)\}_{\sigma \in [0,1]}$ . Roughly speaking the dynamics is given by an on-mass-shell condition (the string wave equation) and the kinematical requirement. It means that in a fixed frame in the Minkowski target space all nonzero modes of the  $x^+$  variable are unphysical and there is a relation for the momenta conjugate to the zero modes  $\{x_0^\mu\}_{\mu=0,\dots,d-1}$ . It follows that the set  $\{\alpha, x_0^\mu, x_k^-, x_k^i\}_{k=1,\dots}^{\mu=0,\dots,d;i=1,\dots,d-1}$  is a complete system of commuting physical micro-observables. We have used the prefix micro- in order to distinguish them from the "true" physical macro-observables which are given by the generators of the Poincare group in the Minkowski target space. Actually the spectra of the macro-observables are of the main interest in the free theory as they provide a relativistic particle interpretation of the string physical states.

In terms of the physical micro-observables  $\{\alpha, x_0^\mu, x_k^-, x_k^i\}_{k=1,\dots}^{\mu=0,\dots,d;i=1,\dots,d-1}$  the geometrical content of the model is clear - it is a theory of the free parametrized string with internal length. In this formulation however the macro-observables are not diagonal. Moreover we do not know how to derive the on-mass-shell condition in these variables from the complicated form of the transition amplitude. The idea of extension we applied to deal with this problem was to introduce an auxiliary variable  $\varphi(\sigma)$  along with some constraints

ensuring the equivalence with the original theory. This allows for expressing the original set of physical micro-observables in terms of a new one  $\{\varphi_0, x_0^\mu, \varphi_k, x_k^i\}_{k=1,\dots}^{\mu=0,\dots,d;i=1,\dots,d-1}$ . As follows from the DDF construction given in Subsect.3.4 in the new variables the macro-observables are diagonal and the relativistic particle content of the model can be easily inferred. The role of the conformal factor is therefore to express in a convenient way the influence of the nontrivial dynamics of the internal length  $\alpha$  and the longitudinal excitations  $\{x_k^-\}_{k \geq 1}$  on the particle spectrum of the noncritical relativistic string model. In this sense the Liouville theory describes the dynamics of the longitudinal modes.

Although the FCT model satisfies all the consistency conditions of formal relativistic quantum mechanics its physical content is not quite satisfactory. First of all it contains an internal quantum number entering the on-mass-shell condition which entails an undesirable continuous range of intercepts. Secondly for some values of this quantum number one gets tachyonic states on the lowest level.

In the free string theory the zero mode of the "Liouville momenta" is conserved and the theory can be truncated at any real value of  $p_L$ . On the other hand the appearance of this additional internal degree of freedom is a consequence of our choice of boundary conditions for string trajectories involving the internal string length  $\alpha$ . The relation between  $\alpha$  and  $p_L$  is a part of the relation between two sets of physical micro-observables discussed above

$$\{\alpha, x_0^\mu, x_k^-, x_k^i\}_{k=1,\dots}^{\mu=0,\dots,d;i=1,\dots,d-1} \longleftrightarrow \{\varphi_0, x_0^\mu, \varphi_k, x_k^i\}_{k=1,\dots}^{\mu=0,\dots,d;i=1,\dots,d-1} \quad (61)$$

Using the constraint equations one can easily express the set of micro-observables  $\{\alpha, x_0^\mu, x_k^-, x_k^i\}_{k=1,\dots}^{\mu=0,\dots,d;i=1,\dots,d-1}$  in terms of  $\{\varphi_0, x_0^\mu, \varphi_k, x_k^i\}_{k=1,\dots}^{\mu=0,\dots,d;i=1,\dots,d-1}$ . The opposite relation is however very complicated and we have not found any convincing method of removing the  $\alpha$ -dependence within the first quantized theory. It seems that the truncation is essentially the problem of the interacting theory where  $\alpha$  plays the role similar to that of the "length" parameter in the "covariantized" light cone formulation of the critical string field theory [38]. Let us only mention that the results concerning noncritical Polyakov string with fixed ends [29] and the naive consideration of the "joining-splitting" interaction suggest a consistent truncation at  $p_L = 0$ .

The second problem with the physical interpretation of the FCT massive string is the presence of tachyons in its spectrum. Since the structure of the model is similar to

that of the critical string, one may expect that the problem can be solved in the fermionic noncritical Polyakov string by a counterpart of GSO projection [39]. The crucial issue here is an appropriate choice of boundary conditions for the fermionic string trajectories. Since the geometry of the fermionic path integral is far less understood than that of the bosonic one (the infinite-dimensional supergeometry virtually does not exist [40]) the methods we have used to determine the boundary conditions in the bosonic case are not available. This makes the problem of a "super" generalization of our approach more difficult than the construction of the supersymmetric noncritical Polyakov dual model [41].

The considerations of the present paper are entirely devoted to the so called "old" covariant formulation of the free open bosonic string. This leaves a number of interesting questions concerning the first quantized theory.

*Closed string.* In contrast to the open string there is no 1-dim conformal anomaly and the natural scalar product is diff-invariant. The issue of the normal ordering of the Virasoro generators appears if one tries to separate the left and right movers. The additional complication is that this decomposition is not invariant with respect to the residual  $S^1$ -symmetry. For these reasons the free closed noncritical string is an interesting problem and would lead to a better insight into the relation between the geometry of the scalar product and the ordering problem.

*BRST formulation.* Within the path integral approach of this paper the idea is in a sense opposite to that of the "old" covariant formulation. One starts with the same path integral representation of the transition amplitude in the extended space of boundary conditions. However instead of restricting oneself to a subspace of states on which the corresponding operator is well defined and invertible, we are looking for yet another (BRST) extension which allows to represent the transition amplitude as a special matrix element of an invertible operator. In the case of the critical string this idea can be easily realized [25] leading to the well known covariant BRST formulation. In the case of noncritical string the problem gets complicated due to nontrivial coupling of the conformal factor to the zero modes of the ghost sector. In particular the two extension procedures do not commute. Note that the complete BRST formulation of the free theory is interesting at least for two reasons. First it should provide a clarification of the discrepancy between the realization of the residual  $\mathcal{D}_L$ -symmetry calculated from the extension procedure and the unitary one in the "old" covariant approach. Secondly, it paves the way for the

field theory formulation which gives new tools for investigating the interacting theory. In particular the problem of the joining-splitting interaction vertex can be posed as the problem of the BRST-invariant extension of the corresponding functional delta function in the variables  $\{\alpha, x^\mu(\sigma)\}_{\sigma \in [0,1]}$ . Note that the role of  $\alpha$  in the interacting theory is especially clear in this formulation. The values of  $\alpha$  determine the way in which two parametrized strings form a third one.

*Light-cone formulation.* The idea of this approach is to explicitly implement the basic kinematical requirement of the relativistic quantum mechanic of 1-dim extended objects by constructing the transition amplitude in a fixed reference system as a sum over causal string trajectories in the Minkowski target space. The main difficulty with respect to the critical string consists in the fact that the constraints appear in the process of quantization and one cannot describe relevant string trajectories in terms of "true" classical variables. A related problem is an appropriate choice of the space of internal metrics corresponding to causal string trajectories in the Minkowski target space. The form of the DDF-states given in Subsect.3.4 suggest however that this formulation has the structure very similar to that of the critical string theory.

*Operator - states correspondence.* The basic tool in calculating the on-shell critical string amplitudes is the so called operator formalism based on the possibility of reproducing string wave functionals corresponding to the physical states by functional integral over half-disc with a local operator insertion. In the present case this equivalence still holds, as can be inferred from the construction of DDF states. The operator-state correspondence along with the solution of the previous problem form basic ingredients of the Mandelstam method [9] of constructing the on-mass-shell "massive" string amplitudes.

As it was emphasized in the introduction the most interesting open question is whether the "joining-splitting" interaction leads to a consistent interacting theory of the FCT string. We have already mentioned two possible approaches to solve this problem: the BRST and the light-cone formulations. We conclude this section by a brief discussion of only one aspect of the interacting theory which is to some extent independent of a particular formulation.

The path integral formulation of the free FCT string derived in this paper involves the assumption that the cosmological constant vanishes. The question arises whether this assumption can be maintained also in the interacting theory. To analyse this problem let

us consider the three point off-shell noncritical string amplitude. As in the case of the critical string theory [23] it is given by the path integral over string trajectories connecting three prescribed string configurations. On the tree level one has to sum over trajectories of the topology of hexagon. The construction of the path integral representation of the corresponding off-shell amplitude is exactly the same as in the case of rectangle-like trajectories relevant for the string propagator. Choosing the hyperbolic hexagon as a model manifold one gets in the conformal gauge an expression involving the path integral over all conformal factors satisfying the homogeneous Neumann boundary conditions. The reasoning we have used in the case of rectangle to derive the condition  $\mu = 0$  now leads to a positive cosmological constant. This makes the resulting path integral prohibitively complicated. Up to now the only method to deal with the resulting theory is to impose the requirement of the conformal invariance and then to translate the problem into the language of 2-dim conformal field theory. This leads however to the complex Liouville action - yet another manifestation of the  $c = 1$  barrier.

The disappointing conclusion above is based on the assumption that the model manifold for a string trajectory describing the elementary process of joining or splitting is given by a hexagon with smooth "timelike" boundaries. If we consider such a process in the Minkowski target space the boundary of the corresponding world sheet has a corner at the interaction point. It follows that if we choose the hyperbolic hexagon (Fig.1.a) as a model manifold the causal string trajectories will be described by singular (at the interaction point) functions. An equivalent description in terms of regular  $x$ -functions can be achieved if we choose the "light-cone" model manifold (Fig.1.b). Note that in the "light-cone" model manifold the internal angle of the "interaction" corner equals  $2\pi$ . This is related to the assumption that no particular interaction occurs at the joining point. Applying the reasoning based on the Gauss-Bonnet theorem in the case of "light-cone" model manifold one gets as in the free theory the vanishing cosmological constant.

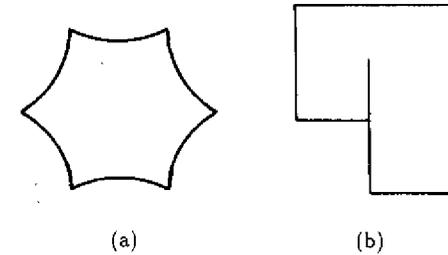


Fig.1.

It should be stressed that as far as the critical string theory is concerned the choice of the model manifold is irrelevant. In fact due to the decoupling of the conformal factor the critical on-shell amplitude is invariant with respect to the conformal transformations of the model manifold. As it was mentioned in the introduction this is one of the features crucial for the equivalence of the Polyakov dual model with the relativistic string theory in the critical dimension.

The situation in the noncritical string theory is different. Roughly speaking due to the fact that the boundary conditions for the conformal factor are fixed the Liouville action "hears" the shape of the model manifold and the theories based on the hyperbolic hexagon and on the "light-cone" model manifold are different.

It follows from the considerations above that the condition  $\mu = 0$  can be consistently imposed in the interacting theory provided that we restrict ourselves to the "light-cone-like" model manifolds. There are some interesting consequences of this choice. First of all due to the corner conformal anomaly one gets the operator insertion at the interaction point of the form  $\exp \gamma \phi(z_i)$ . Note that in contrast to the noncritical Polyakov dual model the operator  $:\exp \gamma \phi(z_i):$  has the conformal weight different from 1. In fact the coefficient  $\gamma$  is a finite constant uniquely determined by the corner conformal anomaly. Let us also stress that within the present approach there is no reason to interpret the integral of this operator over the world-sheet as the "volume of the universe".

Assuming the same general form of the transition from the off-shell to the on-shell amplitudes as in the critical string theory one can expect that the on-shell amplitudes can be expressed in terms of correlators of the 2-dim conformal field theory involving

the vertex operators corresponding to the DDF states and the insertions. This is a very promising feature of the model – one can use the techniques of the conformal field theory to analyse the FCT string amplitudes.

The complete analysis of the interacting FCT string theory requires solutions of a number of technical and conceptual problems, which are far beyond the scope of the present paper. We hope however that the covariant functional integral formulation of the free FCT string makes the program of constructing the noncritical relativistic string theory more promising than twenty years ago and still worth pursuing.

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## Appendix A

In this Appendix we gather the results for the corner conformal anomaly in the case of Laplace-Beltrami operator acting on scalar functions with Dirichlet and Neumann boundary conditions.

The corner anomaly appears in the expansion ( $f$  is a scalar function)

$$\text{Tr}(e^{t\Delta} f) = \frac{k_{-1}}{t} + \frac{k_{-1/2}}{t^{1/2}} + k_0 + O(t^{1/2})$$

in the  $t$  independent part (functional  $k_0$ ) and sums the values of the function  $f$  in the corners with the appropriate coefficient. As it was shown by Kac [42] the contribution from each corner is independent of the global geometry of the surface. It can be estimated by mapping a neighborhood of the corner to the wedge on the plain with the same opening angle. In the case of Dirichlet boundary conditions on both arms of the corner with opening angle  $\gamma$  Kac derived the following formula for the corner conformal anomaly

$$A_{DD}(\gamma) = \frac{\pi^2 - \gamma^2}{24\pi\gamma} f(0)$$

Using this result one can easily infer the corner conformal anomaly for the Dirichlet-Neumann and the Neumann-Neumann boundary conditions. Doubling the corner one gets the following relations

$$\begin{aligned} A_{DD}(\gamma) + A_{DN}(\gamma) &= A_{DD}(2\gamma) , \\ A_{DN}(\gamma) + A_{NN}(\gamma) &= A_{NN}(2\gamma) . \end{aligned}$$

Using the result of Kac one has

$$\begin{aligned} A_{DN}(\gamma) &= -\frac{\pi^2 + 2\gamma^2}{48\pi\gamma} f(0) , \\ A_{NN}(\gamma) &= \frac{\pi^2 - \gamma^2}{24\pi\gamma} f(0) . \end{aligned}$$

In the problems discussed in this paper we have only right angles so we quote the results for  $\gamma = \pi/2$ :

$$\begin{aligned} A_{DD}(\pi/2) &= A_{NN}(\pi/2) = \frac{f(0)}{16} , \\ A_{DN}(\pi/2) &= -\frac{f(0)}{16} . \end{aligned}$$

As far as the rectangle with the standard flat metric is concerned the results above are enough to derive the corner conformal anomaly for the operators  $P^+P, PP^+$  acting on the vector fields and symmetric traceless tensors, respectively. In this case the corresponding operators act separately on every component of vector or tensor fields. Since the components satisfy independent boundary conditions the problem can be reduced to the scalar one. The corner conformal anomaly for these operators has been first derived in [21].

## Appendix B

In this appendix we shall calculate the 1-dim conformal anomaly. Let  $e$  be an einbein on the interval  $[0, 1]$  and

$$\Delta_e \equiv -\frac{1}{e} \frac{d}{dt} \frac{1}{e} \frac{d}{dt} ,$$

the 1-dim Laplace operator acting on the space  $\mathcal{S}_N$  of scalar functions  $\psi(t)$  on  $[0, 1]$  satisfying Neumann boundary conditions at the ends of  $[0, 1]$ . Let us denote by  $\mathcal{D}'\psi$  the functional measure related to the scalar product on  $\mathcal{S}_N$ :

$$\langle \psi | \psi' \rangle = \int_0^1 e dt \overline{\psi(t)} \psi'(t) .$$

The 1-dim conformal anomaly  $J[\varphi, \hat{e}]$  is defined by the relation

$$\mathcal{D}^{e^{\frac{\alpha}{2}} \hat{e}} \psi = J[\varphi, \hat{e}] \mathcal{D}^{\hat{e}} \psi \quad (62)$$

between the functional measures corresponding to the einbeins  $e = e^{\frac{\alpha}{2}} \hat{e}$  and  $e = \hat{e}$ . Using the method proposed in the context of the Liouville measure in 2 dimensions [17] one can derive the following regularized formula for variation

$$\delta \log J_N[\varphi, \hat{e}] = \frac{1}{4} \lim_{\epsilon \rightarrow 0} \int_0^1 e dt e^{-\epsilon \Delta_e(t, t)} \delta \psi(t) ,$$

where  $e = e^{\frac{\alpha}{2}} \hat{e}$ . In terms of normalized eigenfunctions  $\{\psi_m\}_{m \geq 0}$  of the operator  $\Delta_e$  the formula above takes the form

$$\delta \log J_N[\varphi, \hat{e}] = \frac{1}{4} \lim_{\epsilon \rightarrow 0} \int_0^1 dt \left( \sum_{m \geq 0} e^{-\epsilon \frac{m^2 \pi^2}{\alpha^2}} \psi_m(t)^2 \delta \varphi(t) \right) ,$$

where  $\alpha = \int_0^1 e^{\frac{\alpha}{2}} \hat{e} dt$ .

Proceeding to the parameterization  $s = s(t)$  of [0.1] in which  $e^{\frac{x}{2}} \hat{e} = \text{const} = \alpha$  and using the expansion

$$\delta\varphi(s) = \sum_{n \geq 0} \delta\varphi_n \cos \pi n s \quad ,$$

one gets

$$\begin{aligned} \delta \log J_N[\varphi, \hat{e}] &= \frac{1}{4} \lim_{\epsilon \rightarrow 0} \int_0^1 ds \left( \sum_{\substack{n \geq 0 \\ m \geq 1}} e^{-\epsilon \frac{n^2 + m^2}{2}} \delta\varphi_n 2 \cos \pi n s \cos^2 \pi m s + \sum_{n \geq 0} \delta\varphi_n \cos \pi n s \right) \\ &= \frac{1}{16} (\delta\varphi(0) + \delta\varphi(1)) + \lim_{\epsilon \rightarrow 0} \frac{1}{8\sqrt{\pi\epsilon}} \int_0^1 e^{\frac{x}{2}} \hat{e} dt \delta\varphi(t) \quad . \end{aligned}$$

Integrating with respect to  $\varphi$  one has

$$\log J_N[\varphi, \hat{e}] = \frac{1}{16} (\varphi(0) + \varphi(1)) + \frac{1}{4\sqrt{\pi\epsilon}} \int_0^1 e^{\frac{x}{2}} \hat{e} dt \quad . \quad (63)$$

The corresponding result in the case of Dirichlet boundary conditions takes the form

$$\log J_D[\varphi, \hat{e}] = -\frac{1}{16} (\varphi(0) + \varphi(1)) + \frac{1}{4\sqrt{\pi\epsilon}} \int_0^1 e^{\frac{x}{2}} \hat{e} dt \quad .$$

Inserting (63) into (62) one gets the formula used in the main text

$$\mathcal{D}^{\frac{x}{2}} \hat{e} \psi = \exp \left[ +\frac{1}{16} (\varphi(0) + \varphi(1)) + \frac{1}{4\sqrt{\pi\epsilon}} \int_0^1 e^{\frac{x}{2}} \hat{e} dt \right] \mathcal{D} \hat{e} \psi \quad . \quad (64)$$

## Appendix C

In this appendix we shall prove the formula (41) of Subsect.3.2. Consider the mode expansion of the conformal factor

$$\varphi = \frac{2}{\sqrt{t}} \sum_{m, n \geq 0} \varphi_{nm} \cos \frac{\pi n z^0}{t} \cos \pi m z^1$$

and the change of variables

$$\begin{aligned} \psi_{00} &= \varphi_{00} \quad ; \\ \psi_{k0}^+ &= \sum_{l \geq k} \varphi_{(2l)0} \quad , \quad \psi_{k0}^- = \sum_{l \geq k} \varphi_{(2l-1)0} \quad , \quad k \geq 1 \quad ; \\ \psi_{km}^- &= \sum_{l \geq k} \varphi_{(2l+1)m} \quad , \quad \psi_{km}^+ = \sum_{l \geq k} \varphi_{(2l+1)m} \quad , \quad k \geq 0, m \geq 1 \quad . \end{aligned} \quad (65)$$

The modes  $\tilde{\varphi}_{im}, \tilde{\varphi}_{jm}$  of the boundary values  $\tilde{\varphi}_i, \tilde{\varphi}_j$  of  $\varphi$  (42) can be expressed in terms of the variables (65) as follows

$$\begin{aligned} \tilde{\varphi}_{i0} &= \frac{2}{\sqrt{t}} (\psi_{00} + \psi_{10}^+ + \psi_{10}^-) \quad , \\ \tilde{\varphi}_{j0} &= \frac{2}{\sqrt{t}} (\psi_{00} + \psi_{10}^+ - \psi_{10}^-) \quad , \\ \tilde{\varphi}_{im} &= \frac{2}{\sqrt{t}} (\psi_{0m}^+ + \psi_{0m}^-) \quad , \quad m \geq 1 \quad , \\ \tilde{\varphi}_{jm} &= \frac{2}{\sqrt{t}} (\psi_{0m}^+ - \psi_{0m}^-) \quad , \quad m \geq 1 \quad . \end{aligned}$$

In terms of the variables (65) the l.h.s. of the formula (41) can be written as the iterated integral

$$\begin{aligned} Z &= \text{const} \int d\psi_{00} \int \prod_{k \geq 1} d\psi_{k0}^+ d\psi_{k0}^- \\ &\times \exp \left[ -\frac{b}{2t^2} \sum_{k \geq 1} 4k^2 (\psi_{k0}^+ - \psi_{(k+1)0}^+)^2 \right] \\ &\times \exp \left[ -\frac{b}{2t^2} \sum_{k \geq 1} (2k+1)^2 (\psi_{k0}^- - \psi_{(k+1)0}^-)^2 \right] \times Z_1 \quad , \end{aligned} \quad (66)$$

where

$$\begin{aligned} Z_1 &= \int \prod_{\substack{k \geq 0 \\ m \geq 1}} d\psi_{km}^+ d\psi_{km}^- \exp \left[ -b \sum_{m \geq 1} m^2 (\psi_{0m}^+ - \psi_{1m}^+)^2 \right] \\ &\times \exp \left[ -\frac{b}{2} \sum_{\substack{k \geq 1 \\ m \geq 1}} \left( \frac{4k^2}{t^2} + m^2 \right) (\psi_{km}^+ - \psi_{(k+1)m}^+)^2 \right] \end{aligned} \quad (67)$$

$$\begin{aligned} &\times \exp \left[ -\frac{b}{2} \sum_{\substack{k \geq 0 \\ m \geq 1}} \left( \frac{(2k+1)^2}{t^2} + m^2 \right) (\psi_{km}^- - \psi_{(k+1)m}^-)^2 \right] \\ &\times F[\tilde{\varphi}_i, \tilde{\varphi}_j] \quad , \end{aligned} \quad (68)$$

end

$$b = \frac{25-d}{48} \pi \quad .$$

Let us note that the functional  $F[\tilde{\varphi}_i, \tilde{\varphi}_j]$  is independent of the modes  $\{\psi_{km}^+, \psi_{km}^-\}_{k \neq 0}$  and  $Z_1 \equiv Z_1(\tilde{\varphi}_{i0}, \tilde{\varphi}_{j0})$  is a function only of the zero modes of  $\tilde{\varphi}_i, \tilde{\varphi}_j$ .

The integrals  $Z, Z_1$  involve chains of Gaussian integrals which can be exactly calculated by means of the formula ( $\lambda_{N+1} = 0$ )

$$\int \prod_{k \geq 1}^N \exp \left[ -\frac{\beta}{2} \lambda_0 (x_0 - x_1)^2 - \frac{\beta}{2} \sum_{k \geq 1}^N \lambda_k (x_k - x_{k+1})^2 \right] = \left( \prod_{k \geq 0}^N \frac{\beta \lambda_k}{2\pi} \right)^{-\frac{1}{2}} \left( \frac{\beta}{2\pi A} \right)^{\frac{1}{2}} \exp \left( -\frac{\beta}{2\pi A} x_0^2 \right), \quad (69)$$

where

$$A = \sum_{k \geq 0}^N \frac{1}{\lambda_k}.$$

Applying the formula (69) to the integral  $Z$  one obtains

$$Z = \text{const} \left[ \prod_{k \geq 1} \frac{4k^2 (2k+1)^2}{t^2} \right]^{-\frac{1}{2}} \int d\psi_{00} \int \frac{d\psi_{10}^+}{t\sqrt{A_0^+}} \int \frac{d\psi_{10}^-}{t\sqrt{A_0^-}} \times \exp \left[ -\frac{b}{t^2 A_0^+} \psi_{10}^{+2} - \frac{b}{t^2 A_0^-} \psi_{10}^{-2} \right] \times Z_1 \left( \frac{2}{\sqrt{t}} (\psi_{00} + \psi_{10}^+ + \psi_{10}^-), \frac{2}{\sqrt{t}} (\psi_{00} + \psi_{10}^+ - \psi_{10}^-) \right),$$

where

$$A_0^+ = \sum_{k \geq 1} \frac{1}{4k^2} = \frac{\pi^2}{24},$$

$$A_0^- = \sum_{k \geq 0} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

Changing variables

$$\tilde{\varphi}_{i0} = \frac{2}{\sqrt{t}} (\psi_{00} + \psi_{10}^+ + \psi_{10}^-),$$

$$\tilde{\varphi}_{j0} = \frac{2}{\sqrt{t}} (\psi_{00} + \psi_{10}^+ - \psi_{10}^-),$$

$$\varphi_{00} = \psi_{00},$$

and integrating over  $\varphi_{00}$  one finally gets

$$Z = \text{const} t^{-\frac{1}{2}} \int d\tilde{\varphi}_{i0} d\tilde{\varphi}_{j0} \exp \left[ -\frac{b}{2\pi^2} \frac{(\tilde{\varphi}_{j0} - \tilde{\varphi}_{i0})^2}{t} \right] Z_1(\tilde{\varphi}_{i0}, \tilde{\varphi}_{j0}). \quad (70)$$

In the case of the integral  $Z_1$  the formula (69) yields

$$Z_1 = \text{const} \left[ \prod_{\substack{k \geq 0 \\ m \geq 1}} \left( \frac{4k^2}{t^2} + m^2 \right) \left( \frac{(2k+1)^2}{t^2} + m^2 \right) \right]^{-\frac{1}{2}} \int \prod_{m \geq 1} \frac{d\psi_{0m}^+}{t\sqrt{A_m^+}} \frac{d\psi_{0m}^-}{t\sqrt{A_m^-}} \times \exp \left[ -\frac{b}{2} \sum_{m \geq 1} \left( \frac{\psi_{0m}^{+2}}{A_m^+} + \frac{\psi_{0m}^{-2}}{A_m^-} \right) \right] \times F[\tilde{\varphi}_i, \tilde{\varphi}_j],$$

where

$$A_m^+ = \frac{1}{2m^2} + \sum_{k \geq 1} \frac{t^2}{4k^2 + t^2 m^2} = \frac{\pi t}{4m} \coth \frac{\pi m t}{2},$$

$$A_m^- = \sum_{k \geq 1} \frac{t^2}{(2k+1)^2 + t^2 m^2} = \frac{\pi t}{4m} \tanh \frac{\pi m t}{2}.$$

Changing variables

$$\tilde{\varphi}_{im} = \frac{2}{\sqrt{t}} (\psi_{0m}^+ + \psi_{0m}^-),$$

$$\tilde{\varphi}_{jm} = \frac{2}{\sqrt{t}} (\psi_{0m}^+ - \psi_{0m}^-),$$

and using the formula

$$\eta(t) = \prod_{\substack{k \geq 0 \\ m \geq 1}} \left( \frac{k^2}{t^2} + m^2 \right),$$

one has

$$Z_1 = \text{const} \eta(t)^{-\frac{1}{2}} \int \prod_{m \geq 1} d\tilde{\varphi}_{im} d\tilde{\varphi}_{jm} \times \exp \left[ -\frac{b}{4\pi^2} \sum_{m \geq 1} \frac{\pi m}{\sinh \pi m t} [(\tilde{\varphi}_{im}^2 + \tilde{\varphi}_{jm}^2) \cosh \pi m t - 2\tilde{\varphi}_{im} \tilde{\varphi}_{jm}] \right] \times F[\tilde{\varphi}_i, \tilde{\varphi}_j]. \quad (71)$$

Substituting (71) to (70) one gets the generalized Forman formula (41).