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PULSE PILE-UP IV: BIPOLAR PULSES

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Abstract

The study of pulse pile-up is extended from the case of unipolar pulses, for which ruin theory is an excellent approximation, to the case of bipolar pulses for which ruin theory is not applicable to the effect of the back-kicks in reducing the pile-up: an appropriate solution is presented.

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1. Introduction

The pile-up of small pulses simulating a single large pulse has been treated in three previous papers (refs. [1], [2] and [3] here referred to as Parts I, II and III, respectively) which have introduced and evaluated an approximate analytical treatment known as ruin theory which was originally devised for estimating the probability of the bankruptcy of insurance companies. Parts I, II and III have considered only the case of individually unipolar pulses, but it is also important to consider bipolar pulses, *viz.* those whose (nominally positive) leading portion (call it the "fore-kick"), the maximum amplitude of which (call it the "height of the pulse") we take as transmitting the information that it is desired to encode, is followed by a *negative*-going portion (call it the "back-kick") whose area (pulse-amplitude integrated over time) may or may not balance the positive-going area of the leading portion of the pulse such that, in the former case, each complete pulse effects its own base-line restoration. Such bipolar pulses (in old-fashioned notation "double delay-line clipped") evidently reduce pile-up, although we must examine the sense in which this is usefully true, and so must be carefully considered in the context of our current enquiry.

2. General considerations

We continue the nomenclature of the earlier Parts, namely to consider the probability $\Omega(D)$ that, at an arbitrary instant, pulses of form $c(t)$ and of normalized pulse-height distribution $g(c_{max})$ build up to a height of greater than D.

It is our task, in this paper, to consider the influence on such pile-up of going from unipolar to bipolar pulses as described in sec. 1. There is a point of immediate importance to be made concerning the level from which such pile-up is relevant. All our calculations of $\Omega(D)$ refer D to the true base-line from which such pile-up takes place. However, as we noted in Part I, in the case of unipolar pulses the effective base-line from which a genuine isolated large pulse, whose simulation by the pile-up of small pulses we fear, rises will not be the true base-line referred to above but will itself be determined by the mean value of such pile-up, namely:

$$B = X \int_0^\infty g(c_{\mathrm{max}})\,\overline{c}(c_{\mathrm{max}})\,dc_{\mathrm{max}} \ , \tag{1}$$

where X is the average number of small pulses arriving within a time interval equal to the length in time of an individual pulse (taken here as "short" in the nomenclature of Parts I and II and so defining the unit of time) and where:

$$\overline{c}(c_{\mathrm{max}}) = c_{\mathrm{max}} \int_0^\infty c(t)\,dt \ , \tag{2}$$

the form $c(t)$ having $c_{\mathrm{max}} = 1$. In this case we must concern ourselves with pile-up of the small pulses to $D = P + B$ to mimic a genuine pulse of height P. However, in the case of bipolar pulses, if the area of the back-kick is equal to the area of the fore-kick (equal-area bipolar pulses), we have $B{=}0$ and so we are concerned with pile-up to a height $D = P$ to simulate a genuine pulse of height P since now the true and effective base-lines are the same. Now if we call the pile-up probability for unipolar pulses $\Omega(D)_{\mathrm{UP}}$ and that for equal-area bipolar pulses whose fore-kick is identical with the unipolar pulses $\Omega(D)_{\mathrm{BP}}$, where D is in both cases referred to the true base-line, then certainly $\Omega(D)_{\mathrm{BP}} < \Omega(D)_{\mathrm{UP}}$. However, as just remarked, it may be more proper to compare $\Omega(D)_{\mathrm{BP}}$ with $\Omega(D+B)_{\mathrm{UP}}$, and since $\Omega(D)_{\mathrm{UP}}$ is a rapidly falling function of D (see many illustrations in Parts I and II) we may well expect that $\Omega(D+B)_{\mathrm{UP}} < \Omega(D)_{\mathrm{BP}}$, i.e. that the effective pile-up will be smaller for unipolar than for bipolar pulses. This somewhat counter-intuitive result is based upon a simple model of the electronic circuitry that considers only a pulse-shaping network involving only short time constants and a single very much longer time constant responsible for base-line restoration; practical schemes may give different results as will, of course, pulse-shaping networks that deliver pulses of different positive-going and negative-going areas. In this paper, for purposes of our illustrations, we consider only equal-area bipolar pulses.

3. The pile-up probability

We have immediately:

$$\Omega(D)_{\rm BP} = \int_0^\infty P(\ell)\,\Omega(D+\ell)_{\rm UP}\,d\ell \ , \tag{3}$$

where $P(\ell)d\ell$ is the probability, at the arbitrary instant in time, that the back-kicks due to pulses originating longer ago than one unit of time (the length of the fore-kick) will themselves add up to between ℓ and $\ell+d\ell$ (in the negative sense). (Everywhere in this paper we compare unipolar pulses with bipolar pulses whose fore-kick is identical with the unipolar pulse.)

We now notice that owing to the fact that $\Omega(D)_{\rm UP}$ falls very rapidly with D quite a small value of ℓ will have a large effect on $\Omega(D)_{\rm BP}$ so that although we are, as in parts I and II, concerned only with $\Omega(D) << 1$ we are not concerned with small values of $P(\ell)$ and so we cannot use ruin theory, which is perfectly adequate for $\Omega(D)_{\rm UP}$ as we saw in Parts I and II, to estimate it but must use other methods. We return to quantify this remark later but first give some illustrations from cases for which exact solutions are available for $P(\ell)$.

4. Illustrations: symmetrical pulses

We first of all consider symmetrical bipolar pulses *viz.* those for which the back-kick is just the reverse of the fore-kick in form and height and of equal length in time. Exact solutions are available for:

1) square waves all of the same height;

2) square waves with exponential height distribution;

3) sawteeth all of the same height.

In all cases we present the results in the form of $\Omega(D)_{\mathrm{BP}}/\Omega(D^*)_{\mathrm{UP}}$ for the two cases that: (i) $D^* = D$; (ii) $D^* = D+B$ for the reason indicated in sec. 2, where B is given by eqs. (1) and (2) and where $\Omega(D)_{\mathrm{UP}}$ covers the range 10^{-8} to 10^{-3} as was considered extensively in Parts I and II.

4.1 Square pulses all of the same height

We must here replace the integral of eq. (3) by summation over the discrete pulse heights, the expression for $\Omega(D)_{\mathrm{BP}}$ being elementary. Fig. 1 gives the results. It is seen that, as expected, $\Omega(D)_{\mathrm{BP}}/\Omega(D)_{\mathrm{UP}}$ is less than unity but that, as was conjectured might be the case, $\Omega(D)_{\mathrm{BP}}/\Omega(D+B)_{\mathrm{UP}}$ is greater than unity, *viz.* that in the terms of our model, bipolar pulses pile up more severely than unipolar pulses. We also see that the pile-up probability ratios are rather insensitive to the pile-up probability. As we shall see, these are both general results.

4.2 Square waves with exponential height distribution

The exact solution for $\Omega(D)_{\mathrm{UP}}$ has been given in eq. (11) of Part I; for $P(\ell)$ we have:

$$P(\ell) = \mathrm{e}^{-X+\ell} \sum_{N=1}^{\infty} \frac{X^N}{N!} \frac{\ell^{N-1}}{(N-1)!} \,. \tag{4}$$

The results are given in fig. 2 which resembles fig. 1 but with much smaller ranges for the pile-up probability ratios.

4.3 Sawteeth all of the same height

The exact solutions for $\Omega(D)$ and $P(\ell)$ have been given in eqs. (15), (16) and (17) of Part I. The results are given in fig. 3 which also resembles figs. 1 and 2, being closer to fig. 1 which is also for pulses all of equal height.

5. Illustrations: asymmetrical pulses

It is of interest for a number of reasons to consider pulses whose back-kick is not symmetrical with their fore-kick although of the same area. The points of interest will be brought out as the illustrations develop.

5.1 Sawteeth with square-wave back-kick and exponential height distribution

In this illustration the square-wave back-kick has the same length in time as the positive-going sawtooth (and therefore half the height). The positive-going sawtooth is here treated by ruin theory (using eq. (19) of Part I) and the back-kick exactly. The results are given in fig. 4. The point of interest is the resemblance between fig. 4 and fig. 2, both of which relate to exponential height distributions and both of which have substantially smaller pile-up probability ratios than for the equal-height distributions of figs. 1 and 3. It is probably a general result that the effect of bipolar pulses is greater for narrow than for broad height distributions $g(c_{max})$.

5.2 Back-kick of different duration from fore-kick

We have so far considered bipolar pulses whose positive-going and negative-going portions are of equal length in time. It is of interest to enquire into the effect of different lengths in time for the two portions while preserving equality of form and equality of area. We should expect that as the length of the back-kick is reduced to zero, while keeping the fore-kick the same in all respects, $\Omega(D)_{BP}$ will tend to $\Omega(D)_{UP}$ and that as the length of the back-kick is increased towards infinity $\Omega(D)_{BP}$ will tend to $\Omega(D+B)_{UP}$.

Figs. 5 and 6 show this expected asymptotic behaviour for square waves with exponential height distribution and for sawteeth all of the same height, respectively. S is the duration of the back-kick in units of the duration of the fore-kick.

We see from figs. 5 and 6 that the lengthening of the back-kick can be advantageous in reducing pile-up although we must bear in mind that the time constant effective for ultimate base-line restoration is assumed long in relation to the longest time constant involved in pulse-shaping.

6. Comparison of back-kicks of differing form

It is of interest to enquire into the degree to which the pile-up of bipolar pulses is affected by changes in the form of the back-kick for a constant fore-kick. As before we consider equal-area bipolar pulses.

In fig. 7 we compare sawteeth all of the same height having symmetrical sawteeth back-kicks (giving $\Omega(D)_{\rm ST}$) with sawteeth having square-wave back-kicks of duration unity (giving $\Omega(D)_{\rm SQ}$). It is seen that over the whole range studied pile-up is the same for the two forms of back-kick to better than a factor of 2.

Fig. 8 makes a similar comparison for sawteeth with an exponential height distribution (again using ruin theory for $\Omega(D)_{\rm UP}$): (i) having square-wave back-kicks of duration unity; (ii) having back-kicks of exponential form with a time constant of unity. (i) gives $\Omega(D)_{\rm SQ}$ and (ii) gives $\Omega(D)_{\rm E}$. The exact treatment of the back-kicks of exponential form to generate their $P(\ell)$ for use in eq. (3) is given by appropriate differentiation of eq. (22) of Part I:

$$P(\ell) = (e^{-\ell}\,\ell^{X-1})/\Gamma(X)\ . \tag{5}$$

Again, only feeble dependence of pile-up on the form of the back-kick is seen.

This result, that bipolar pile-up is not strongly sensitive to the form of the back-kick, is in sharp contrast to the case for $\Omega(D)_{\rm UP}$ itself. Thus we see from fig. 8 that for X=5 there is only a 30-35% difference between the pile-up of exponentially distributed sawteeth with square-wave back-kicks and with back-kicks of exponential form over the whole range of $\Omega(D)_{\rm UP}$ from 10^{-8} to 10^{-3}. However, the $\Omega(D)_{\rm UP}$ of exponentially distributed square waves and exponentials are themselves very different: for X=5 and D=15, $\Omega(D)$ for square waves is greater than that for exponentials (itself about 10^{-3}) by an order of magnitude; for X=5 and D=30 the difference is more than two orders of magnitude ($\Omega(D)$ for exponentials being there about 10^{-8}) see figs. 10 and 14 of Part I for square waves and exponentials, respectively.

This observation of comparative insensitivity to the form of the back-kick will be important when we come to consider the case of arbitrary pulse form in sec. 8.

7. Build-up of $\Omega(D)_{\rm BP}$ with ℓ

We have noted qualitatively in sec. 3 that we cannot use ruin theory for generating the $P(\ell)$ of eq. (3). This we now illustrate quantitatively with two examples.

The first example is for symmetrical bipolar sawteeth all of the same height. Consider the contribution to $\Omega(D)_{\rm BP}$ that is due to the superposition of back-kicks in the range $0 < \ell \leq L$:

$$\Omega(D,L)_{\rm BP} = \int_{\epsilon}^{L} P(\ell)\,\Omega(D+\ell)_{\rm UP}\,d\ell\,, \tag{6}$$

where the lower limit of integration ϵ is infinitesimally small but excludes zero itself. Now define the fraction of $\Omega(D)_{\rm BP}$ that is associated with back-kicks of magnitude L or less as $\Omega(D,L)_{\rm BP}/\Omega(D,\infty)_{\rm BP}$. (Note that $\Omega(D,\infty)_{\rm BP}$ differs from $\Omega(D)_{\rm BP}$ by the exclusion of $\ell \equiv 0$, *viz.* the case that no pulse was initiated in the interval of one to two units of time prior to the arbitrary instant in question; such a case is of no interest in our present context of studying the build-up of $\Omega(D)_{\rm BP}$ with ℓ.)

Fig. 9 shows the fraction of $\Omega(D)_{\rm BP}$ as defined in the previous paragraph as a function of L for $X=1$ with D = 3, 5 and 7 (for which $\Omega(D)_{\rm UP}$ has the approximate values 2×10^{-3}, 4×10^{-6} and 4×10^{-9}, respectively) and for $X=5$ with D = 7, 10 and 13 (for which $\Omega(D)_{\rm UP}$ has the approximate values 2×10^{-3}, 9×10^{-6} and 1×10^{-8}, respectively). We see that indeed the build-up of $\Omega(D)_{\rm BP}$ is very rapid and this quantifies the uselessness of ruin theory for $P(\ell)$: not only has ruin theory become very inaccurate long before the ℓ-values in major question have been reached but it has become meaningless – the $\Omega(D)_{\rm UP}$ of ruin theory has no solution in this case for $\frac{D}{X} < \frac{1}{2}$ where, as may be seen from fig. 9, almost all the interest resides.

Fig. 10 presents the similar analysis for the case of symmetrical bipolar square waves with exponential height distribution for $X=1$ with $D = 7$ and 23 (for which $\Omega(D)_{UP}$ has the approximate values 7×10^{-3} and 2×10^{-8}, respectively) and for $X=5$ with $D = 19$ and 39 (for which $\Omega(D)_{UP}$ has the approximate values 1×10^{-3} and 4×10^{-9}, respectively). The conclusion is the same as in the previous case, namely that ruin theory is useless for $P(\ell)$: in this case it has no solution for $\frac{D}{X} < 1$.

8. Arbitrary pulse form and pulse-height distribution

We have seen in sec. 7 that ruin theory cannot be used for generating the $P(\ell)$ of eq. (3), although it is perfectly satisfactory for the $\Omega(D+\ell)_{UP}$ of that equation for which it may be derived for arbitrary pulse form $c(t)$ and arbitrary pulse-height distribution $g(c_{max})$ using the methods presented in detail in Part I. In the present context it is essential that $P(\ell)$ be available exactly. However, we have also seen, in sec. 6, that $\Omega(D)_{BP}$ is not very sensitive to the form of the back-kick, so that it will be acceptable to use for $P(\ell)$ the exact solution for a back-kick probability distribution that is itself only a (sufficiently close) approximation to the exact distribution.

Specifically:

(i) Generate the "exact" back-kick $P(\ell)_1$, where the subscript 1 refers to a single pulse, by using eq. (9) of Part I with integration over $g(c_{max})$. (Note that in our present case, in contrast to that stressed in Part I, it is adequate, when analytical folding of the pulse form and pulse-height distribution is not possible, to effect a numerical convolution since there is now no great sensitivity to the tail of the $P(\ell)$ distribution in contrast to the case in generating the $\Omega(D)_{UP}$.)

(ii) Replace the "exact" $P(\ell)_1$ by a step-function approximation to it at equal increments $\Delta\ell$ of ℓ with successive step-heights $p_1, p_2 \ldots p_n$ such that $\sum_{x=1}^{n} p_x = 1$ and such that the probability p_x refers to a value of ℓ in the range $(x-1)\Delta\ell$ to $x\Delta\ell$, say

$\ell = (x - \frac{1}{2})\Delta\ell$

(iii) We now have:

$$P(\ell) = \sum_{N=1}^{\infty} \frac{X^N}{N!} e^{-X} P(\ell)_N \ , \tag{7}$$

where ℓ varies in steps of $\Delta\ell$ and so where the integral in eq. (3) is replaced by the appropriate discrete summation. (As in sec. 7 the case $\ell \equiv 0 \quad (N = 0)$ is excluded from eq. (7) and is to be added separately into eq. (3)). In eq. (7) $P(\ell)_N$ is the probability that the random superposition of N back-kicks, each of which has a probability p_x of being of magnitude $(x-\frac{1}{2})\Delta\ell$ at the arbitrary instant, will add up to ℓ where ℓ ranges from $\frac{1}{2}N\,\Delta\ell$ to $N(n-\frac{1}{2})\Delta\ell$. This is a standard combinational problem the solution to which is that $P([m-\frac{1}{2}N]\Delta\ell)_N$ is the coefficient of y^m in $A(n,N)$, where:

$$A(n,N) = (p_1 y + p_2 y^2 + \ldots p_n y^n)^N \ . \tag{8}$$

In view of the demonstrated lack of sensitivity of $\Omega(D)_{BP}$ to the form of the back-kick it is likely that quite small values of n, say 5 or 6, may be adequate for many practical purposes. On the other hand, N may have to range up to quite high values for X-large, say up to $N{=}10$ or so for $X{=}5$. This makes the evaluation of $A(n,N)$ by hand completely impracticable; its evaluation by a computer algebra program, however, is trivial even for quite large values of n, N.

We now illustrate the utility of this procedure with reference to the pile-up of bipolar symmetrical sawteeth all of the same height in which we synthesize the $P(\ell)_1$ of the back-kick (in this case all of the same height corresponding to $p_x = \frac{1}{n}$) so that $n{=}1$ corresponds to a square-wave back-kick. Fig. 11 shows the result: the rapid convergence with increasing n towards the exact result is seen.

We may finally note that if tailed back-kicks (in the sense of Parts I and II) are in question it is adequate to truncate them at, say, 0.1 of their maximum value, renormalize their area and treat as above (where we have tacitly assumed that we are dealing with short back-kicks in the sense of Parts I and II – and where, of course, X refers to the back-kicks in the case that back-kick and fore-kick are not of equal duration).

References

[1] D.H. Wilkinson, Nucl. Instr. and Meth. A297 (1990) 230.

[2] D.H. Wilkinson, Nucl. Instr. and Meth. A297 (1990) 244.

[3] D.H. Wilkinson, preceding paper.

Figure captions

1. The pile-up of symmetrical bipolar square waves all of the same height (unity). $\Omega(D)_{\mathrm{BP}}$ is the probability that, at an arbitrary instant, the pile-up exceeds D. $\Omega(D^{*})_{\mathrm{UP}}$ is the similar probability, for exceeding D^{*}, of unipolar pulses identical with the fore-kick of the bipolar pulses. For the lower set of curves $D^{*} = D$; for the upper set of curves $D^{*} = D + B$ where B is the mean level of pile-up for unipolar pulses. ($B = X$ in this case.) The abscissa is the pile-up probability for the unipolar pulses. X-values given on the curves.
2. As for fig. 1 but for square waves with exponential distribution of pulse-height of mean height unity.
3. As for fig. 1 but for sawteeth all of the same height (unity). ($B = \frac{X}{2}$ in this case.)
4. As for fig. 1 but for pulses with sawtooth fore-kick and square-wave back-kick of the same area and duration as the fore-kick; exponential distribution of pulse-height of mean height unity. ($B = \frac{X}{2}$).
5. Dependence of pile-up for equal-area bipolar square waves of exponential height distribution on the duration S of the back-kick in units of that of the fore-kick. The arrows to the left indicate the respective values of $\Omega(D)_{\mathrm{UP}}$; those to the right show $\Omega(D+B)_{\mathrm{UP}}$. For X=2. D-values given on the curves.
6. As for fig. 5 but for bipolar sawteeth all of the same height (unity).
7. Pile-up for symmetrical bipolar sawteeth all of the same height (unity), $\Omega(D)_{\mathrm{ST}}$, compared with that for pulses of identical sawtooth fore-kick but with the back-kick replaced by a square wave of the same area and duration, $\Omega(D)_{\mathrm{SQ}}$. X-values given on the curves.

8. Pile-up for pulses of exponential height distribution with a sawtooth fore-kick and a square-wave back-kick of the same area and duration as the forekick, $\Omega(D)_{SQ}$, compared with that for pulses of identical sawtooth fore-kick but with the back-kick replaced by one of exponential form of time constant equal to the duration of the fore-kick and of equal area, $\Omega(D)_{E}$. X-values given on the curves.

9. Pile-up of symmetrical bipolar sawteeth all of the same height (unity). The ordinate is the fraction of the pile-up that is associated with superposition of back-kicks totalling less than L. D-values given on the curves.

10. As for fig. 9 but for symmetrical bipolar square pulses of exponential height distribution with mean height unity. D-values given on the curves.

11. Pile-up of symmetrical bipolar sawteeth all of the same height, $\Omega(D)_{E}$, compared with that for the same pulses with the back-kicks approximated by n steps, $\Omega(D)_{A}$. n-values given on the curves.

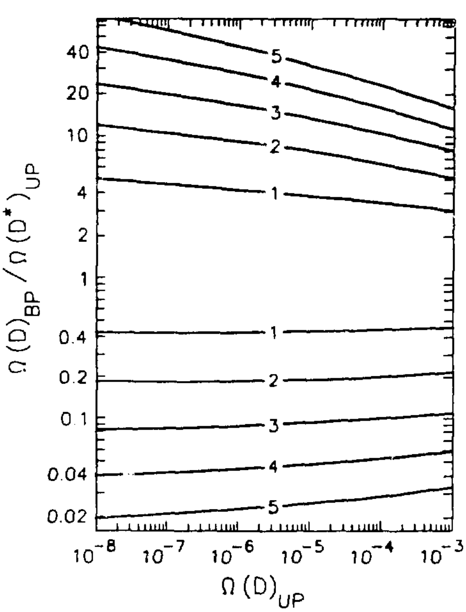

Fig. 1

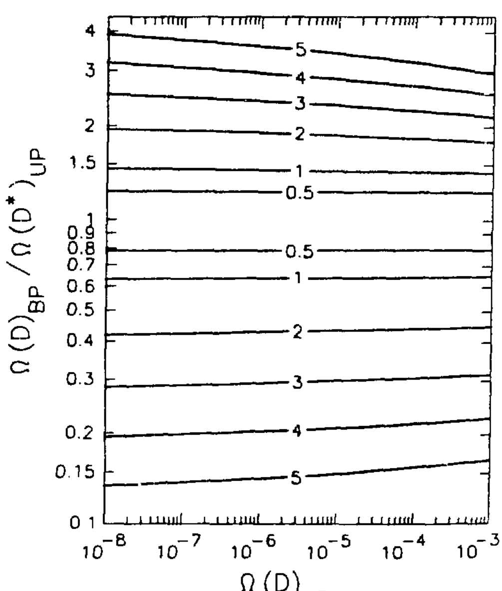

Fig. 2

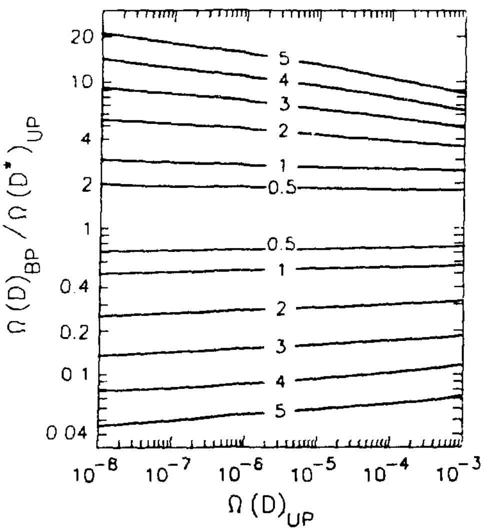

Fig. 3

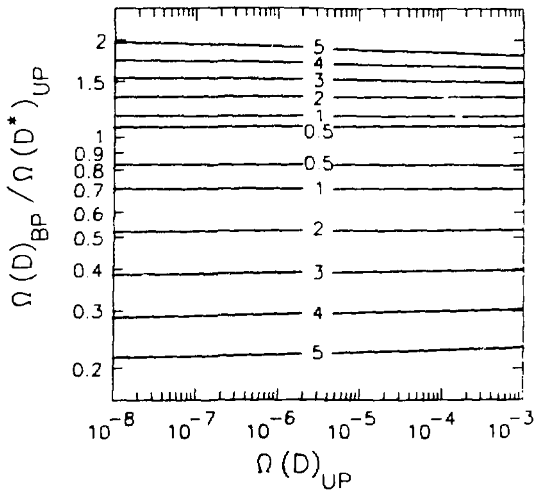

Fig. 4

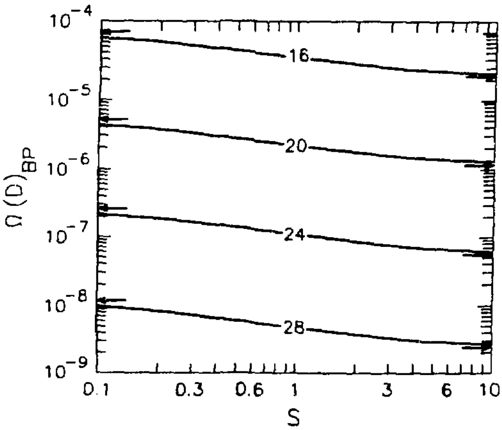

Fig. 5

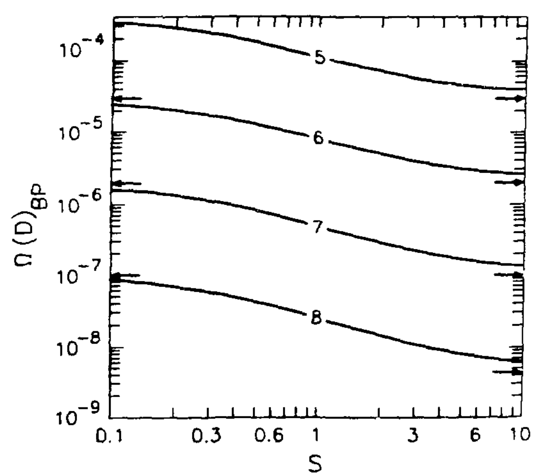

Fig. 6

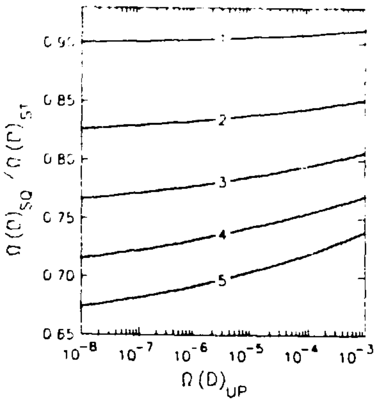

Fig. 7

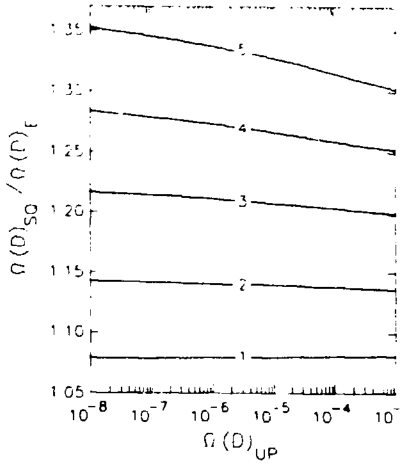

Fig. 8

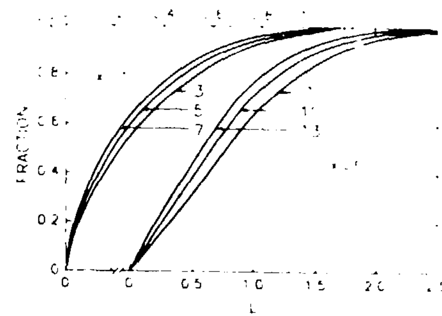

Fig. 9

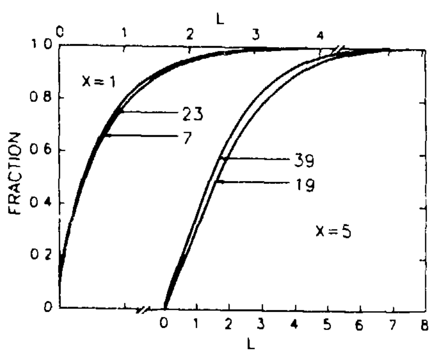

Fig. 10

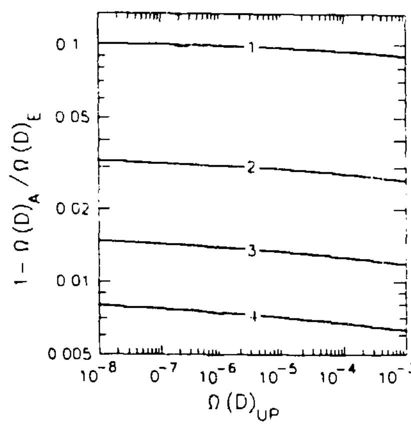

Fig. 11