Structure of BRS-Invariant Local Functionals

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Abstract

For a large class of gauge theories a nilpotent BRS-operator s is constructed and its cohomology in the space of local functionals of the off-shell fields is shown to be isomorphic to the cohomology of $\tilde{s} = s + d$ on functions $f(\tilde{C}, \mathcal{T})$ of tensor fields $\mathcal{T}$ and of variables $\tilde{C}$ which are constructed of the ghosts and the connection forms. The result allows general statements about the structure of invariant classical actions and anomaly candidates whose BRS-variation vanishes off-shell. The assumptions under which the result holds are thoroughly discussed.

1 Introduction

The BRS-formalism [2, 1] is an elegant and powerful tool for dealing with gauge symmetries in quantum field theory. It is available for a large class of gauge theories whose classical symmetries have an algebra which closes on the off-shell fields. In particular the BRS-formalism allows to characterize classical actions and anomalies as BRS-invariant local functionals of the fields. Namely an action is a BRS-invariant functional with ghost number 0 and anomalies correspond to BRS-invariant functionals with ghost number 1. The latter can occur in the quantized theory if there is no regularization preserving all symmetries of the classical theory. The BRS-invariance of the classical action follows from its gauge invariance (and vice versa) and BRS-trivial contributions to the action can be used to construct elegantly a gauge fixing and the corresponding Faddeev-Popov ghost contributions to the action [1]. The BRS-invariance of anomalies comprises consistency conditions which anomalies have to satisfy as a consequence of the algebra of the symmetries of the classical theory and which have been first derived for Yang–Mills theories [14]. In summary, actions and anomalies are solutions of the so-called consistency equation

$$s\mathcal{W}^G = 0, \quad \mathrm{gh}(\mathcal{W}^G) = G \tag{1.1}$$

where s denotes the BRS-operator and $\mathcal{W}^G$ is a local functional with ghost number (gh) G. In this paper I restrict the investigation to the case that (1.1) holds identically in the fields (i.e. for off-shell fields). I remark however that according to recent results [13, 8] there might be also anomalies corresponding to on-shell solutions of (1.1). More precisely these functionals satisfy (1.1) only weakly in the sense that they are BRS-invariant only up to functionals which vanish for solutions of the classical field equations.

The above-mentioned locality of $\mathcal{W}^G$ is in the case of an invariant action $\mathcal{W}^0$ a physical requirement (input) and follows in the case of an anomaly $\mathcal{W}^1$ from renormalization theory. Here a functional is called local if its integrand is a formal (not necessarily finite) power series in the undifferentiated elementary fields and a polynomial in their partial derivatives. In addition the integrand may depend explicitly on the coordinates (e.g. via background fields). The nilpotency of the BRS-operator (see below) implies that (1.1) represents a cohomological problem, i.e. in order to solve (1.1) one has to determine the cohomology of s in the space of local functionals with ghost number G. In particular each functional of the form sX^{G-1} solves (1.1) and is therefore called a trivial solution provided X^{G-1} is a local functional. Two solutions of (1.1) are called equivalent if they differ by a trivial solution:

$$\mathcal{W}^G \cong \mathcal{W}'^G \quad \Leftrightarrow \quad \mathcal{W}^G - \mathcal{W}'^G = sX^{G-1}. \tag{1.2}$$

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A local functional of the fields is an integrated local volume form ω_D^G of the fields (the superscript of a form denotes its ghost number, the subscript its form-degree, D denotes the space-time dimension)

$$\mathcal{W}^G = \int \omega_D^G$$

and BRS-invariance of $\mathcal{W}^G$ requires its integrand to satisfy

$$s\,\omega_D^G + d\,\omega_{D-1}^{G+1} = 0 \tag{1.3}$$

where ω_{D-1}^{G+1} is a local $(D-1)$-form with ghost number $G+1$ and d denotes the exterior derivative

$$d = dx^m \partial_m. \tag{1.4}$$

The BRS-operator is by construction a nilpotent antiderivation which anticommutes with d:

$$s^2 = \{s, d\} = d^2 = 0. \tag{1.5}$$

Therefore each volume form which can be written as $s\,\eta_D^{G-1} + d\,\eta_{D-1}^G$ solves (1.3). Such solutions of (1.3) are called trivial provided η_D^{G-1} and η_{D-1}^G depend locally on the fields. Analogously to (1.2) two solutions of (1.3) are called equivalent if they differ by a trivial solution,

$$\omega_D^G \cong \omega'^G_D \quad \Leftrightarrow \quad \omega'^G_D - \omega_D^G = s\,\eta_D^{G-1} + d\,\eta_{D-1}^G. \tag{1.6}$$

(1.1) resp. (1.3) have been investigated by many authors for various theories. In particular (1.3) has been solved completely for Yang–Mills theories [5] where the solutions can be divided into two classes: 'Lagrangian solutions' constructed of tensor fields and undifferentiated ghosts and 'chiral solutions' which can be written completely in terms of 0–forms given by the ghosts, connection 1–forms constructed of the gauge fields and the corresponding curvature 2–forms. Examples for 'Lagrangian solutions' are invariant Lagrangians (times the volume element d^Dx), examples of 'chiral solutions' are Chern-Simons forms and the integrands of chiral anomalies (see e.g. [9, 15]). Analogous results hold for Einstein–Yang–Mills theories in an Poincaré-invariant background [6]. The present paper shows that the results for Yang–Mills and Einstein–Yang–Mills theories hold in a generalized form for a large class of gauge theories. We shall see however that in the general case it does not make much sense to distinguish 'Lagrangian' and 'chiral solutions'—both arise from the same universal structure of solutions. The derivation of this structure is the main result of this paper. It will become clear that it originates in the very definition of the gauge theories investigated here and is intimately related with a generalized tensor calculus. Therefore this definition will be thoroughly discussed. It will be given in an approach which treats all gauge symmetries on an equal footing and in particular takes into consideration diffeomorphisms 'automatically'. As a consequence the latter always contribute to the BRS-transformations. Therefore ordinary Yang–Mills theories are strictly speaking not covered by the investigation of this paper, unlike Einstein–Yang–Mills theories. Other prominent theories to which the results of this paper apply are supergravity theories [7].

The paper is organized as follows. Section 2 defines the class of gauge theories for which the results of this paper hold, including the construction of the BRS–operator. Section 3 contains the result of the investigation, section 4 sketches its proof. Sect. 5 comments on the importance of some underlying assumptions for the validity of the result and in particular discusses modifications of the result if these assumptions are not fulfilled. The paper is completed by a summary and an appendix which contains the proof of two lemmas used in section 4.

2 Assumptions

2.1 Algebra on tensor fields

The gauge theories treated in this paper can be characterized starting from an algebra of operators Δ_M which is realized on tensor fields. I postpone a precise definition of tensor fields to subsection 2.3

but I remark that a tensor field is a function $\mathcal{T}(x,\varphi,\partial\varphi,\partial\partial\varphi,\dots)$ depending on the elementary fields and their partial derivatives (and explicitly on the coordinates via the background vielbein, see below) such that the BRS–transformation of $\mathcal{T}$ has a special simple form. The operators Δ_M whose number is not necessarily finite are assumed to be independent and to have a closed algebra on tensor fields which is of the form

$$[\Delta_M,\Delta_N\} := \Delta_M\Delta_N - (-)^{|M||N|}\Delta_N\Delta_M = \mathcal{F}_{MN}{}^P\Delta_P, \quad |M|,|N|\in\{0,1\} \tag{2.1}$$

where $|M|$ denotes the grading of Δ_M (i.e. we allow for bosonic and fermionic Δ's) and the structure functions $\mathcal{F}_{MN}{}^P$ are generally field dependent, i.e. they are tensor fields themselves. According to (2.1) they are symmetric or antisymmetric in their lower indices and even or odd graded depending on the grading of the corresponding generators

$$\mathcal{F}_{MN}{}^P = -(-)^{|M||N|}\mathcal{F}_{NM}{}^P, \quad |\mathcal{F}_{MN}{}^P| = |M|+|N|+|P| \mod 2. \tag{2.2}$$

By assumption they satisfy the Jacobi- resp. Bianchi-identities

$$\sum_{MNP}(-)^{|M||P|}\left(\Delta_M\mathcal{F}_{NP}{}^Q - \mathcal{F}_{MN}{}^R\mathcal{F}_{RP}{}^Q\right) = 0 \tag{2.3}$$

which guarantee the consistency of the algebra (2.1). The sum in (2.3) is the cyclic sum, i.e.

$$\sum_{MNP}(-)^{|M||P|}X_{MNP} = (-)^{|M||P|}X_{MNP} + (-)^{|N||M|}X_{NPM} + (-)^{|P||N|}X_{PMN}. \tag{2.4}$$

Notice that by assumption the operators Δ_M are covariant in the sense that they map tensor fields to tensor fields. It is not assumed that they can be defined on all fields in a way that (2.1) holds. In particular they are generally not defined on the gauge fields $\mathcal{A}_m{}^N$ and the ghost fields $\hat{C}^N$ which can be introduced elegantly by requiring that the exterior derivative d and the BRS–operator s act on tensor fields $\mathcal{T}$ according to

$$d\mathcal{T} = \mathcal{A}^N\Delta_N\mathcal{T}, \tag{2.5}$$
$$s\mathcal{T} = \hat{C}^N\Delta_N\mathcal{T} \tag{2.6}$$

where $\mathcal{A}^N$ denotes the gauge field 1–form

$$\mathcal{A}^N = dx^m\mathcal{A}_m{}^N. \tag{2.7}$$

Inserting $d = dx^m\partial_m$ and (2.7) into (2.5) one obtains

$$\partial_m\mathcal{T} = \mathcal{A}_m{}^M\Delta_M\mathcal{T}. \tag{2.8}$$

The grading of the ghosts, gauge fields and differentials is fixed by the requirement that s and d are odd graded operators while ∂_m is even graded. This gives

$$|\hat{C}^N| = |N|+1 \mod 2, \quad |\mathcal{A}_m{}^N| = |N|, \quad |dx^m| = 1. \tag{2.9}$$

According to (2.5) and (2.6) both d and s can be written on tensor fields as linear combinations of the operators Δ_N where the coefficients are the gauge field 1–forms $\mathcal{A}^N$ in one case and the ghosts $\hat{C}^N$ in the other case. This similarity of s and d is one of the decisive reasons behind the result of this paper and therefore it is commented now. (2.6) of course is the definition of the BRS–operator on tensor fields and at the same time serves as the defining property of tensor fields (see subsection 2.3). (2.5) can be viewed as the definition of the covariant derivatives. Namely the latter form by assumption a subset of $\{\Delta_M\}$. In order to distinguish the covariant derivatives from the remaining Δ's we label the former by latin indices from the beginning of the alphabet and the latter by greek indices from the middle of the alphabet. Furthermore the more customary notation $\mathcal{D}_a$ is introduced for the covariant derivatives:

$$\{\Delta_M\} = \{\mathcal{D}_a, \Delta_\mu\}, \quad \mathcal{D}_a \equiv \Delta_a, \quad a = 1, \dots, D, \quad \mu = D+1, D+2, \dots$$

The corresponding gauge and ghost fields are denoted by $\mathcal{A}_m{}^a$, $\mathcal{A}_m{}^\mu$, $\hat{C}^a$ and $\hat{C}^\mu$. The covariant derivatives of course are by assumption bosonic (even graded) operators:

$$|a| = 0. \tag{2.10}$$

The special role played by the covariant derivatives originates in the important assumption that the gauge fields $\mathcal{A}_m{}^a$ form (at each point) an invertible $D \times D$-matrix which is called the vielbein. We shall therefore also use the more customary notation $e_m{}^a$ instead of $\mathcal{A}_m{}^a$. The entries of the inverse vielbein are denoted by $E_a{}^m$:

$$\mathcal{A}_m{}^a \equiv e_m{}^a, \quad e_m{}^a E_a{}^n = \delta_m^n, \quad E_a{}^m e_m{}^b = \delta_a^b. \tag{2.11}$$

The invertibility of the vielbein allows to solve (2.8) for $\mathcal{D}_a \mathcal{T}$ and thus (2.5) indeed can be regarded as the definition of the covariant derivatives according to

$$\mathcal{D}_a \mathcal{T} = E_a{}^m(\partial_m - \mathcal{A}_m{}^\mu \Delta_\mu)\mathcal{T} \tag{2.12}$$

which is familiar from the Einstein–Yang–Mills theory where the Δ_μ are the elements of a Liealgebra. Notice however that this restrictive assumption has been dropped and replaced by the more general requirement that the Δ_μ constitute together with the $\mathcal{D}_a$ a set of independent (generally graded) operators which have a closed algebra (2.1) on tensor fields. In addition the Δ_M are assumed to be local operators in the sense that each of them maps local tensor fields to local tensor fields.

2.2 BRS–operator and field strengths

So far three kinds of fields have been introduced, namely tensor, gauge and ghost fields. In order to be able to construct a gauge fixed BRS-invariant action one introduces in addition an antighost ζ^N and a 'Lagrange multiplier field' b^N for each gauge field with gradings given by

$$|\zeta^N| = |N| + 1 \mod 2, \quad |b^N| = |N|.$$

On tensor fields the BRS–operator has been defined already in (2.6). In addition we define the BRS–transformations of the antighosts, explicit coordinates and differentials according to[1]

$$s\,\zeta^N = b^N, \quad s\,x^m = 0, \quad s\,dx^m = 0 \tag{2.13}$$

and further require that

$$(s+d)^2 = 0 \quad \Leftrightarrow \quad s^2 = [s, \partial_m] = [\partial_m, \partial_n] = 0. \tag{2.14}$$

This implies in particular

$$s\,\partial_{m_1} \dots \partial_{m_k} \zeta^N = \partial_{m_1} \dots \partial_{m_k} b^N, \quad s\,\partial_{m_1} \dots \partial_{m_k} b^N = 0. \tag{2.15}$$

The BRS–transformations of the ghosts and gauge fields follow from requiring (2.14) to hold on tensor fields. Notice that (2.5) and (2.6) imply

$$(s+d)\,\mathcal{T} = \hat{C}^N \Delta_N \mathcal{T} \tag{2.16}$$

[1]The fields and their partial derivatives are not regarded as functions of variables x but as fundamental variables themselves, see subsection 2.3.

where

$$\tilde{C}^N = \hat{C}^N + A^N. \tag{2.17}$$

$(s+d)^2\mathcal{T}$ can be evaluated using (2.1) and requiring it to vanish one obtains

$$(s+d)\,\tilde{C}^P = \tfrac{1}{2}(-)^{|N|}\tilde{C}^N\tilde{C}^M\mathcal{F}_{MN}{}^P \tag{2.18}$$

which reads more explicitly

$$\begin{aligned} s\,\hat{C}^P &= \tfrac{1}{2}(-)^{|N|}\hat{C}^N\hat{C}^M\mathcal{F}_{MN}{}^P, && (2.19)\\ s\,A_m{}^P &= \partial_m\hat{C}^P + \hat{C}^M A_m{}^N\mathcal{F}_{NM}{}^P, && (2.20)\\ 0 &= \partial_m A_n{}^P - \partial_n A_m{}^P - A_m{}^M A_n{}^N\mathcal{F}_{NM}{}^P. && (2.21) \end{aligned}$$

Notice that (2.18) does not only define the BRS–transformation of the ghosts and gauge fields (2.19), (2.20) but also contains the identity (2.21) which guarantees the consistency of (2.8) and $[\partial_m, \partial_n]\mathcal{T} = 0$. (2.21) is not a differential equation restricting the gauge fields and the structure functions but determines the structure functions $\mathcal{F}_{ab}{}^N$ in terms of the gauge fields, their partial derivatives and the remaining structure functions $\mathcal{F}_{\mu\nu}{}^N$ and $\mathcal{F}_{\mu a}{}^N$. Namely (2.21) can be solved for $\mathcal{F}_{ab}{}^N$ due to the invertibility of the vielbein:

$$\mathcal{F}_{ab}{}^N = -E_a{}^m E_b{}^n(\partial_m A_n{}^N - \partial_n A_m{}^N + e_n{}^c A_m{}^\mu\mathcal{F}_{\mu c}{}^N - e_m{}^c A_n{}^\mu\mathcal{F}_{\mu c}{}^N + A_n{}^\nu A_m{}^\mu\mathcal{F}_{\mu\nu}{}^N). \tag{2.22}$$

(2.22) justifies to call the $\mathcal{F}_{ab}{}^N$ the field strengths corresponding to $A_m{}^N$. Notice that they generally do not only depend on the gauge fields and their derivatives but also on additional fields which contribute via the structure functions $\mathcal{F}_{\mu\nu}{}^N$ and $\mathcal{F}_{\mu a}{}^N$ to the r.h.s. of (2.22). In Einstein–Yang–Mills theory all these structure functions are constant and the field strengths therefore depend only on the gauge fields but, for instance, in supergravity theories this is not the case and (2.22) yields in this case the correct extension of the Yang–Mills field strengths and the Riemann tensor and defines the field strengths of the gravitino such that these fields transform covariantly under general coordinate, Yang–Mills, Lorentz and supersymmetry transformations.

Using (2.16) and (2.18) one can check that (2.14) holds also for the ghosts and the gauge fields by virtue of the Jacobi- and Bianchi-identities (2.3).

It will turn out that the solutions of (1.3) can be more easily written in terms of new ghost variables C^m, C^μ than in terms of the ghosts $\hat{C}^M$. The different ghost basis' are related by

$$\hat{C}^a = e_m{}^a C^m, \quad \hat{C}^\mu = C^\mu + C^m A_m{}^\mu. \tag{2.23}$$

The new ghost basis is chosen such that the BRS–transformation of a tensor field now takes the form

$$s\,\mathcal{T} = (C^m\partial_m + C^\mu\Delta_\mu)\,\mathcal{T} \tag{2.24}$$

as can be easily verified by inserting (2.12) into (2.6). In terms of the new ghost variables the BRS–transformations of the ghosts and gauge fields read

$$\begin{aligned} s\,e_m{}^a &= C^n\partial_n e_m{}^a + (\partial_m C^n)\,e_n{}^a + C^\mu A_m{}^N\mathcal{F}_{N\mu}{}^a, && (2.25)\\ s\,A_m{}^\mu &= C^n\partial_n A_m{}^\mu + (\partial_m C^n)\,A_n{}^\mu + \partial_m C^\mu + C^\nu A_m{}^N\mathcal{F}_{N\nu}{}^\mu, && (2.26)\\ s\,C^m &= C^n\partial_n C^m + \tfrac{1}{2}(-)^{|\mu|}C^\mu C^\nu\mathcal{F}_{\nu\mu}{}^a E_a{}^m, && (2.27)\\ s\,C^\mu &= C^n\partial_n C^\mu + \tfrac{1}{2}(-)^{|\nu|}C^\nu C^\rho(\mathcal{F}_{\rho\nu}{}^\mu - \mathcal{F}_{\rho\nu}{}^a E_a{}^m A_m{}^\mu). && (2.28) \end{aligned}$$

Of course this version of the BRS–algebra is completely equivalent to the version using the $\hat{C}^M$ due to the invertibility of the vielbein.

2.3 Field content, variables and tensor fields

We restrict the investigation to theories whose field content can be characterized by means of the fields introduced so far. Thus the 'classical' field content consists of the components $e_m{}^a$ of the vielbein, the set of gauge fields $A_m{}^\mu$ and a set of further fields Ψ^i which by assumption are tensor fields. The Ψ^i are therefore called elementary tensor fields. The field content is completed by the ghosts C^N, the antighosts ζ^N and the Lagrange multiplier fields b^N. By this assumption we exclude for instance theories which contain gauge potentials $B = dx^{m_1}\dots dx^{m_p}B_{m_1\dots m_p}$ but it is expected that the presence of such fields (and corresponding ghosts and ghosts of ghosts) leads only to small modifications of the result, at least in cases of interest (cf. the discussion of new minimal supergravity given in [7]).

However some additional remarks are in order. First it is stressed that not the fields $e_m{}^a$ themselves are regarded as elementary fields but their deviations $h_m{}^a$ from the entries $\hat{e}_m{}^a(x)$ of some background vielbein which is also assumed to be invertible (but else arbitrary)

$$e_m{}^a = \hat{e}_m{}^a(x) + h_m{}^a. \tag{2.29}$$

We allow that some (or all) gauge fields $A_m{}^\mu$ are not elementary fields but local functions of the fields $h_m{}^a$, Ψ^i and the remaining gauge fields. Gauge fields $A_m{}^\mu$ which are not elementary fields are called *composite* gauge fields in the following[2]. I stress however that compositeness of gauge fields is required to be consistent with eqs. (2.19)–(2.21) resp. (2.25)–(2.28) which must hold identically in the elementary fields and their partial derivatives. In particular these equations must not impose differential equations for the ghosts. Namely it is assumed that $h_m{}^a$, Ψ^i, C^m ,C^μ, ζ^N, b^N and a subset of $\{A_m{}^\mu\}$ form a complete set of *elementary fields* $\{\varphi^\alpha\}$ which means that there are no algebraic identities relating the variables

$$\partial_{m_1}\dots\partial_{m_k}\varphi^\alpha, \quad m_{i+1}\geq m_i, \quad k\geq 0 \tag{2.30}$$

apart from those which follow from their grading according to (2.31). Thus we regard (2.30) as a set of infinitely many independent variables on which the 'partial derivatives' ∂_m act algebraically (for instance ∂_m maps the variable φ^α to the variable $\partial_m\varphi^\alpha$, and $\partial_m\partial_n\varphi^\alpha$ and $\partial_n\partial_m\varphi^\alpha$ are regarded as the same variable)[3]. Notice that here it is essential that all fields are off-shell fields. Explicit coordinates x^m and the differentials dx^m are treated as additional independent variables on which the ∂_m act according to

$$\partial_m\, x^n = \delta_m^n, \quad \partial_m\, dx^n = 0.$$

Notice that these assumptions exclude cases where constant ghosts (corresponding e.g. to global symmetries) are contained in s though these cases may be treated on a completely equal footing with the case where constant ghosts are absent. Nevertheless the presence of constant ghosts complicates the investigation as we shall see in section 5.

Each variable has a definite grading which determines its statistics according to

$$\begin{aligned} &z^A z^B = (-)^{|z^A||z^B|} z^B z^A, \quad z^A\in\{\partial_{m_1}\dots\partial_{m_k}\varphi^\alpha,\ x^m,\ dx^m\},\\ &|\partial_{m_1}\dots\partial_{m_k}\varphi^\alpha| = |\varphi^\alpha|, \quad |x^m| = 0, \quad |dx^m| = 1. \end{aligned} \tag{2.31}$$

I conclude this section with some remarks on tensor fields. In subsection 2.1 tensor fields have been characterized by the property that the operators Δ_M are realized on them according to the algebra (2.1). In particular this allowed to define the BRS–transformation of tensor fields according to (2.6). One may ask whether it is possible to characterize tensor fields more concretely. The answer is yes and in fact surprisingly simple if we now use (2.6) as a defining property for a function of the fields to be a tensor field. More precisely we require that a tensor field is a local function $\mathcal{T}$ of x^m, $h_m{}^a$, Ψ^i, $A_m{}^\mu$

[2]An example for a composite gauge field is the spin–connection in gravity or supergravity theories with vanishing structure function (torsion) $\mathcal{F}_{ab}{}^c$.

[3]This approach can be formalized using the jet bundle theory, see e.g. [12].

such that sT does not contain partial derivatives of the ghosts. Using this definition one can *prove* that each tensor field depends on the fields and their partial derivatives only via the elements $\mathcal{T}^r$ of

$$\{\mathcal{T}^r\} = \{\mathcal{D}_{a_1} \ldots \mathcal{D}_{a_k} \Psi^i,\ \mathcal{D}_{a_1} \ldots \mathcal{D}_{a_k} \mathcal{F}_{ab}{}^N :\ k \geq 0\} \tag{2.32}$$

where $\mathcal{F}_{ab}{}^N$ is given by (2.22). In order to prove this statement one uses the fact that according to appendix A each function of x^m, $h_m{}^a$, $A_m{}^\mu$, Ψ^i and their partial derivatives can be written also in terms of the variables x^m, $\mathcal{T}^r$ and the elements of

$$\{U_l\} = \{h_m{}^a,\ A_m{}^\mu,\ \partial_{(m_k} \ldots \partial_{m_1} e_{m_0)}{}^a,\ \partial_{(m_k} \ldots \partial_{m_1} A_{m_0)}{}^\mu :\ k \geq 1\}$$

which is a subset of $\{u_l\}$ defined in appendix A. The requirement that the BRS–transformation of a local function $f(x, \mathcal{T}, U)$ does not depend on derivatives of the $\tilde{C}^M$ is then easily seen to require that f does not depend on the U_l at all[4].

Thus each tensor field can be written as a function of the $\mathcal{T}^r$ which of course are tensor fields themselves, i.e. we know all tensor fields constructable of the fields and their partial derivatives. Notice that the $\mathcal{T}^r$ are 'created' by acting with $\mathcal{D}_a$ on Ψ^i and $\mathcal{F}_{ab}{}^N$. Therefore the $\mathcal{D}_a$ are in a sense not only a subset of $\{\Delta_M\}$ but also generate the space on which the Δ_M are realized, similarly as the ∂_m generate the variables (2.30). Notice that the invertibility of the vielbein is responsible for this special part played by the $\mathcal{D}_a$ compared to the remaining Δ's since it relates the set $\{\mathcal{D}_a, \Delta_\mu\}$ of operators to the set $\{\partial_m, \Delta_\mu\}$ (on tensor fields).

Finally I remark that the definition of tensor fields given above requires that each tensor field transforms scalarly under general coordinate transformations since otherwise its BRS–transformation would contain partial derivatives of the ghosts C^m of diffeomorphisms. This definition represents no loss of generality since 'world indices' $m, n, \ldots$ which indicate a nonscalar behaviour under general coordinate transformations can always be converted to 'flat indices' $a, b, \ldots$ by means of the vielbein and its inverse (here of course the invertibility of the vielbein again is essential). Notice however that this in particular means that neither the vielbein (or a metric g_{mn} built from it) is considered as a tensor field nor, for instance, the quantities $\mathcal{F}_{mn}{}^N := e_m{}^a e_n{}^b \mathcal{F}_{ab}{}^N$.

3 Result

In order to formulate the result of this paper I define the cohomology of

$$\tilde{s} = s + d \tag{3.1}$$

on local functions

$$f^g(\tilde{C}, \mathcal{T}) = \tilde{C}^{N_1} \ldots \tilde{C}^{N_g} f_{N_1 \ldots N_g}(\mathcal{T}). \tag{3.2}$$

This cohomology is denoted by $H^g(\tilde{s})$. It is well-defined since $\tilde{s}$ leaves the space of functions $f(\tilde{C}, \mathcal{T})$ invariant, i.e. the $\tilde{s}$-variation of a function $f(\tilde{C}, \mathcal{T})$ can be always written completely in terms of the variables $\tilde{C}^N$, $\mathcal{T}^r$ due to (2.16) and (2.18). Recalling basic properties of cohomological problems I remark that the general solution of

$$\tilde{s} f^g(\tilde{C}, \mathcal{T}) = 0 \tag{3.3}$$

is a linear combination of solutions f_i^g which, loosely speaking, form a basis for the solutions of (3.3). More precisely $\{f_i^g\}$ is a set of solutions of (3.3) which represent the cohomology classes of $H^g(\tilde{s})$:

$$f^g(\tilde{C}, \mathcal{T}) \cong \sum_i a_i f_i^g(\tilde{C}, \mathcal{T}), \quad \tilde{s} f_i^g(\tilde{C}, \mathcal{T}) = 0,$$

$$\sum_i a_i f_i^g(\tilde{C}, \mathcal{T}) \cong 0 \quad \Leftrightarrow \quad a_i = 0 \quad \forall i$$

[4] I stress that this holds due to the assumption that the ghosts are elementary fields.

where a_i are constant coefficients (c-numbers) and two solutions of (3.3) are called equivalent if they differ only by a trivial solution or a constant piece:

$$f^g(\tilde{C},\mathcal{T}) \cong f'^g(\tilde{C},\mathcal{T}) \quad \Leftrightarrow \quad f'^g(\tilde{C},\mathcal{T}) - f^g(\tilde{C},\mathcal{T}) = \tilde{s}\, h^{g-1}(\tilde{C},\mathcal{T}) + const.. \tag{3.4}$$

I remark that the inclusion of the constant in (3.4) in our case serves to treat elegantly the fact that c-numbers are solutions of (3.3) with $g=0$ (and in fact they are the only solutions with $g=0$ according to the corollary given below). Notice that a constant can contribute to $f^g(\tilde{C},\mathcal{T})$ only for $g=0$ as a consequence of our assumption that constant ghosts are absent. On the other hand there are no contributions $\tilde{s}\,h^{-1}$ in the case $g=0$ since there are no functions (3.2) for $g<0$. Thus $f^g \cong f'^g$ in fact holds in the case $g=0$ iff $f^0 - f'^0 = const.$ and in the case $g>0$ iff $f'^g - f^g = \tilde{s}\, h^{g-1}$. I stress however that in presence of constant ghosts the inclusion of constants is less trivial.

Furthermore we define $H^{n,p}(s|d)$ as the cohomology of s modulo d on local p–forms with ghost number n. This cohomology is defined by the problem

$$s\omega_p^n + d\omega_{p-1}^{n+1} = 0, \quad \omega_p^n \cong \omega'^n_p \;\Leftrightarrow\; \omega_p^n - \omega'^n_p = s\eta_p^{n-1} + d\eta_{p-1}^n \tag{3.5}$$

where ω_p^n, ω_{p-1}^{n+1}, η_p^{n-1} and η_{p-1}^n are local forms whose subscript (superscript) denotes their form degree (ghost number). Again the general solution of (3.5) can be written as

$$\omega_p^n \cong \sum_i a_i\,(\omega_p^n)_i$$

where $\{(\omega_p^n)_i\}$ denotes a set of solutions of (3.5) which represent the cohomology classes of $H^{n,p}(s|d)$. In particular in order to solve (1.3) one has to determine $H^{G,D}(s|d)$, i.e. one has to find a set $\{(\omega_D^G)_i\}$. The main result of the present paper is the following:

Theorem: *In contractible manifolds the cohomologies $H^{G,D}(s|d)$ and $H^{G+D}(\tilde{s})$ are isomorphic, i.e. the cohomology classes of $H^{G,D}(s|d)$ correspond one-to-one to those of $H^{G+D}(\tilde{s})$ provided there are no constant ghosts. This correspondence is very explicit: If $f_i^{G+D}(\tilde{C}^a,\tilde{C}^\mu,\mathcal{T})$ is a solution of (3.3) representing a cohomology class of $H^{G+D}(\tilde{s})$ then the corresponding cohomology class of $H^{G,D}(s|d)$ is represented by a nontrivial solution $(\omega_D^G)_i$ of (1.3) given by*

$$(\omega_D^G)_i = \left[f_i^{G+D}(e^a, C^\mu + A^\mu, \mathcal{T})\right]_D, \quad e^a = dx^m e_m{}^a, \quad A^\mu = dx^m A_m{}^\mu \tag{3.6}$$

where $[f_i^{G+D}(e^a,C^\mu+A^\mu,\mathcal{T})]_D$ denotes the volume form contained in $f_i^{G+D}(e^a,C^\mu+A^\mu,\mathcal{T})$. Thus the solutions of (1.3) can be easily obtained from those of (3.3).

This theorem will now be spelled out in some more detail in order to comment it and give some insight into its origin. Each function $f^{G+D}(\tilde{C},\mathcal{T})$ decomposes according to

$$f^{G+D}(\tilde{C},\mathcal{T}) = \sum_{p=0}^{D} \omega_p^{G+D-p} \tag{3.7}$$

where ω_p^{G+D-p} is a p-form with ghost number $(G+D-p)$. (3.3) decomposes into the so-called descent equations

$$0 < p \le D: \quad s\omega_p^{G+D-p} + d\omega_{p-1}^{G+D-p+1} = 0, \quad s\omega_0^{G+D} = 0. \tag{3.8}$$

In particular (3.8) contains (1.3) i.e. the volume form part ω_D^G contained in $f^{G+D}(\tilde{C},\mathcal{T})$ solves (1.3). In order to realize that this part is given by (3.6) one uses

$$\tilde{C}^a = \tilde{C}^m e_m{}^a, \quad \tilde{C}^\mu = C^\mu + \tilde{C}^m A_m{}^\mu, \quad \tilde{C}^m = dx^m + C^m \tag{3.9}$$

which holds according to (2.23) and (2.17). Written in terms of the C^m and C^μ, $f^{G+D}(\tilde{C},\mathcal{T})$ therefore depends on the ghost C^m and the differential dx^m only via their sum $\tilde{C}^m$. Since the $\tilde{C}$'s anticommute

$f^{G+D}(\tilde{C},\mathcal{T})$ contains no piece of higher degree than D in the $\tilde{C}$'s. This implies that the volume form part contained in $f^{G+D}(\tilde{C},\mathcal{T})$ is given by

$$\left[f^{G+D}(\tilde{C}^a,\tilde{C}^\mu,\mathcal{T})\right]_D=\left[f^{G+D}(\epsilon^a,C^\mu+\mathcal{A}^\mu,\mathcal{T})\right]_D \tag{3.10}$$

and thus leads to (3.6). Notice that ω_D^G does not depend on the ghosts of diffeomorphisms C^m at all. This holds in particular for $G=1$ and thus may indicate that diffeomorphisms are not anomalous for the theories investigated here. I stress that this result holds due to our assumptions that the vielbein is invertible and that the manifold is contractible. In fact, in noncontractible manifolds which allow for closed but not exact 1–forms $\hat{\omega}_1=dx^m\omega_m(x)$ there are generally solutions of (1.3) with $G=1$ which depend on the C^m (and on the $\hat{\omega}_1$) as will be shown in section 5. Notice that (3.10) shows that ω_D^G is more conveniently written using the ghost basis $\{C^m,C^\mu\}$ than using the basis $\{\hat{C}^N\}$. In contrast, the 0–form ω_0^{G+D} is most conveniently written in terms of the ghosts $\hat{C}^N$ according to lemma 1 of the next section since it is BRS-invariant:

$$\omega_0^{G+D}=f^{G+D}(\hat{C},\mathcal{T}). \tag{3.11}$$

For all other forms occurring in the descent equations there is no preferred ghost basis in which they take a simple form since they depend both on the differentials (via the gauge field 1–forms $\mathcal{A}^N$) and on the ghosts C^m.

Remarks:

(i) Notice that (2.17) and (2.23) imply

$$f^{G+D}(\hat{C},\mathcal{T})=\omega(C^m,C^\mu,\mathcal{A}_m{}^N,\mathcal{T})\quad\Rightarrow\quad f^{G+D}(\tilde{C},\mathcal{T})=\omega(\tilde{C}^m,C^\mu,\mathcal{A}_m{}^N,\mathcal{T}). \tag{3.12}$$

Thus we can describe our result also in the following way: Modulo trivial contributions each solution $\{\omega_p^{G+D-p}\ :\ 0\leq p\leq D\}$ of the descent equations can be obtained from a BRS-invariant 0–form $\omega_0^{G+D}=f^{G+D}(\hat{C},\mathcal{T})$ by expressing the latter in terms of the ghosts C^m and C^μ and then replacing each ghost C^m by $\tilde{C}^m=dx^m+C^m$. The resulting function $\omega(\tilde{C}^m,C^\mu,\mathcal{A}_m{}^N,\mathcal{T})$ is the formal sum of all ω_p^{G+D-p}. Such a result has been derived already in [6] for the special case of Einstein–Yang–Mills theory. The derivation given in [6] however uses the assumption that the background vielbein is constant. The result of the present paper shows that this assumption is superfluous.

(ii) According to (3.6) each solution of (1.3) with negative ghost number is trivial since the r.h.s. of (3.6) does not depend on fields with negative ghost number. The isomorphism of $H^{G,D}(s|d)$ and $H^{G+D}(\tilde{s})$ implies then that (3.3) has nontrivial solutions only for $g\geq D$. These statements are in fact not restricted to contractible manifolds. Namely the vanishing of $H^{G,D}(s|d)$ for $G<0$ is a well-known consequence of the simple BRS-transformation $s\zeta^N=b^N$ of the antighosts and can be proved as the analogous statement in Yang–Mills theories (see e.g. [5]). The vanishing of $H^g(\tilde{s})$ for $g<D$ is proved as follows: A solution $f^g=\sum_p\omega_p^{g-p}$ of (3.3) with $g<D$ decomposes into the descent equations (3.8) with vanishing volume form ω_D^{g-D} (recall that a function (3.2) does not contain parts of negative ghost number). By means of arguments used at the end of section 4 one concludes from $\omega_D^{g-D}=0$ that in fact all forms ω_p^{g-p} are trivial and thus that f^g is trivial.

Corollary: *$H^g(\tilde{s})$ is zero for $g<D$ provided there are no constant ghosts.*

4 Proof of the result

In this section the result given in the previous section is proved by means of the following lemmas:

Lemma 1:

a) Nontrivial contributions to local BRS-invariant forms can be chosen to depend on the fields and *their derivatives only via the undifferentiated ghosts $\hat{C}^N$ and via the tensor fields T^r:*

$$\begin{aligned} & s\,\omega(dx,x,\varphi,\partial\varphi,\partial\partial\varphi,\ldots)=0 \\ \Rightarrow\quad & \omega=f(dx,x,\hat{C},T)+s\,\eta(dx,x,\varphi,\partial\varphi,\partial\partial\varphi,\ldots). \end{aligned} \tag{4.1}$$

b) A local function of the variables $dx^m, x^m, \hat{C}^N, T^r$ is BRS–trivial (in the space of local forms) if and only if it is the BRS–variation of a local function of the same variables:

$$\begin{aligned} \exists\,\eta:\quad & f(dx,x,\hat{C},T)=s\,\eta(dx,x,\varphi,\partial\varphi,\partial\partial\varphi,\ldots) \\ \Leftrightarrow\quad \exists\,g:\quad & f(dx,x,\hat{C},T)=s\,g(dx,x,\hat{C},T). \end{aligned} \tag{4.2}$$

Lemma 1 is proved in appendix A.

Lemma 2: *BRS-invariance of a function $\omega(\hat{C},T)$ implies $\tilde{s}$-invariance of the function $\omega(\tilde{C},T)$ which arises from $\omega(\hat{C},T)$ by replacing $\hat{C}^N$ with $\tilde{C}^N=\hat{C}^N+A^N$:*

$$s\,\omega(\hat{C},T)=0\quad\Rightarrow\quad \tilde{s}\,\omega(\tilde{C},T)=0. \tag{4.3}$$

Lemma 2 holds since $\tilde{s}$ acts on the variables $\tilde{C}^N$ and T^r exactly as s acts on the variables $\hat{C}^N$ and T^r, see (2.6), (2.19), (2.16) and (2.18).

Lemma 3: *Algebraic Poincaré Lemma in contractible manifolds and in absence of constant ghosts: In the space of local forms*

$$\omega_p=dx^{m_1}\ldots dx^{m_p}\omega_{m_1\ldots m_p}(x,\varphi,\partial\varphi,\partial\partial\varphi,\ldots)$$

closed forms are also exact unless they are volume forms or constant 0–forms. Volume forms are exact if and only if their Euler derivative

$$\frac{\hat{\partial}}{\hat{\partial}\varphi^\alpha}=\sum_{n\geq 0}\ \sum_{m_{r+1}\geq m_r}(-)^n\partial_{m_1}\ldots\partial_{m_n}\frac{\partial}{\partial(\partial_{m_1}\ldots\partial_{m_n}\varphi^\alpha)}$$

with respect to each elementary field φ^α vanishes:

$$\begin{aligned} p=0:\quad & d\omega_0=0 & \Leftrightarrow\quad & \omega_0=const., \\ 0<p<D:\quad & d\omega_p=0 & \Leftrightarrow\quad & \omega_p=d\omega_{p-1}, \\ p=D:\quad & \omega_D=d\omega_{D-1} & \Leftrightarrow\quad & \forall\varphi^\alpha:\ \hat{\partial}\omega_D/\hat{\partial}\varphi^\alpha=0. \end{aligned} \tag{4.4}$$

For forms which do not depend explicitly on the coordinates but only on the φ^α and their partial derivatives this lemma has been proved e.g. in [3, 5, 12]. In appendix B this result is used to prove a generalized version of lemma 3.

We now turn to the proof of the result stated in the previous section. In order to simplify the notation the ghost number of the forms is omitted in the following. Assume that ω_D solves (1.3). Applying s to (1.3) one concludes by means of (1.5) that $s\omega_{D-1}$ satisfies $d(s\omega_{D-1})=0$. Since $s\omega_{D-1}$ is not a volume form this implies according to (4.4) the existence of a local $(D-2)$–form such that $s\omega_{D-1}+d\omega_{D-2}=0$. Repeating the argument one deduces the existence of a set of local p–forms ω_p which satisfy

$$s\omega_D+d\omega_{D-1}=0,\quad s\omega_{D-1}+d\omega_{D-2}=0,\quad\ldots\quad,\quad s\omega_{L+1}+d\omega_L=0,\quad s\omega_L=0. \tag{4.5}$$

We shall see that in contractible manifolds one actually has $L=0$, i.e. in this case the descent equations (4.5) terminate always with a nontrivial 0–form unless ω_D solves (1.3) trivially (this does not hold for instance in pure Yang–Mills theory unless one includes diffeomorphisms in s). I first note that the set of forms ω_p is not unique since if the set $\{\omega_p\}$ satifies eqs. (4.5) then the set $\{\omega'_p\}$ with

$\omega'_p = \omega_p - s\eta_p - d\eta_{p-1}$ also satisfies these equations where $\{\eta_p\}$ denotes an arbitrary set of local forms with $\mathrm{gh}\,(\eta_p) = \mathrm{gh}\,(\omega_p) - 1$. The sets $\{\omega_p\}$ and $\{\omega'_p\}$ are called equivalent since their elements differ only by trivial contributions. In particular, trivial contributions $s\eta_L + d\eta_{L-1}$ to ω_L can be always absorbed by choosing an equivalent set. By means of (4.1) one therefore concludes that without loss of generality ω_L can be assumed to depend on the fields and their derivatives only via the $\hat{C}^N$ and T^r

$$\omega_L = \omega_L(dx, x, \hat{C}, T).$$

Evidently ω_L can be written in the form

$$\omega_L = \sum_\tau \omega^\tau(dx, x) f_\tau(\hat{C}, T) \tag{4.6}$$

where the ω^τ are linear independent L-forms which do not depend on the fields and the f_τ are linear independent functions of the variables $\hat{C}^N$, T^r. Without loss of generality one can further assume that no nontrivial linear combination of the f_τ combines to a BRS-trivial function:

$$\sum_\tau \lambda^\tau f_\tau = sB \quad \Leftrightarrow \quad \lambda^\tau = 0 \quad \forall \tau. \tag{4.7}$$

Namely otherwise an appropriate trivial piece $s\eta_L$ can be subtracted from ω_L such that (4.7) holds for $\omega'_L = \omega_L - s\eta_L$ (cf. eq. (3.10) of [6] and the arguments used there). $s\omega_L{=}0$ requires

$$sf_\tau = 0 \quad \forall \tau \tag{4.8}$$

due to the linear independence of the ω^τ. According to (4.3) each f_τ can be completed to a solution $\tilde{f}_\tau = f_\tau(\tilde{C}, T)$ of $\tilde{s}\tilde{f}_\tau = 0$. In particular this implies the existence of a 1-form h_τ whose BRS-transformation equals $-df_\tau$:

$$df_\tau + sh_\tau = 0, \quad h_\tau = A^N \frac{\partial}{\partial \hat{C}^N} f_\tau(\hat{C}, T). \tag{4.9}$$

Inserting (4.6) into the last but one eq. (4.5) one obtains (using (4.9))

$$\sum_\tau (d\omega^\tau) f_\tau + s(\omega_{L+1} - \sum_\tau \omega^\tau h_\tau) = 0. \tag{4.10}$$

If $d\omega^\tau$ does not vanish for all τ then (4.10) requires that a nontrivial linear combination of the f_τ is BRS-exact. This however contradicts (4.7) and therefore each of the forms ω^τ has to be closed,

$$d\omega^\tau = 0 \quad \forall \tau. \tag{4.11}$$

Furthermore the ω^τ can be assumed not to be exact,

$$\omega^\tau \neq d\eta^\tau \quad \forall \tau \tag{4.12}$$

since $\omega^\tau = d\eta^\tau$ implies that the contribution $\omega^\tau f_\tau$ to ω_L is trivial due to (4.9) and therefore can be subtracted from ω_L:

$$\omega^\tau = d\eta^\tau \quad \Rightarrow \quad \omega^\tau f_\tau = d(\eta^\tau f_\tau) + s(\eta^\tau h_\tau) \quad \text{(no summation over } \tau).$$

Since we assumed the manifold to be contractible we conclude from (4.11) and (4.12) that ω^τ is a constant 0-form, i.e. in fact ω_L is a 0-form depending only on the $\hat{C}^N$ and T^r:

$$\omega_L = \omega_0(\hat{C}, T). \tag{4.13}$$

Thus we have proved that to each nontrivial solution ω_D of (1.3) there corresponds a solution ω_0 of

$$s\,\omega_0(\hat{C},T)=0, \quad \omega_0\neq s\,\eta_0(\hat{C},T). \tag{4.14}$$

From (4.3) we also know that conversely each nontrivial solution ω_0 of (4.14) gives rise to a solution $\tilde{\omega}$ of (3.3) which in fact is nontrivial since $\tilde{\omega}=\tilde{s}\tilde{\eta}$ would imply the existence of a function η_0 such that $s\eta_0=\omega_0$ in contradiction to (4.14). Finally each nontrivial solution of (3.3) gives rise to a nontrivial solution ω_D of (1.3) according to the following argument: Suppose that ω_D is trivial, i.e. that there are local forms η_D and η_{D-1} such that $\omega_D=s\eta_D+d\eta_{D-1}$. Inserting this into (1.3) one obtains $d(-s\eta_{D-1}+\omega_{D-1})=0$ which implies according to (4.4) the existence of a local form η_{D-2} such that $\omega_{D-1}=s\eta_{D-1}+d\eta_{D-2}$. Thus the triviality of ω_D implies the triviality of ω_{D-1}. Repeating the arguments one concludes that all other forms ω_p are trivial as well and thus that $\tilde{\omega}=\tilde{s}\tilde{\eta}$ which contradicts the assumption that $\tilde{\omega}$ is nontrivial. Thus the nontrivial solutions of (1.3) and (3.3) correspond one-to-one which completes the proof.

5 Discussion of some assumptions

This section comments on three assumptions which have been made in order to derive the results presented in sect. 3. These assumptions are the contractibility of the manifold, the absence of constant ghosts, and the algebraic independence of the ghosts and their partial derivatives. For simplicity it is only discussed how the results may change if only one of these assumptions is not fulfilled respectively.

a) Noncontractible manifold:

If the manifold is not contractible there may be non-exact solutions of (4.11) apart from constant 0–forms and thus the descent equations (4.5) may terminate with a form ω_L whose degree L differs from 0 and is of the form

$$\omega_L=\sum_\tau \hat{\omega}_L^\tau(dx,x)f_\tau(\hat{C},T) \tag{5.1}$$

where $f_\tau(\hat{C},T)$ are BRS-invariant functions satisfying (4.7) and the $\hat{\omega}_L^\tau(dx,x)$ are closed but not exact L–forms which can be chosen such that no nontrivial linear combination of them combines to an exact form:

$$d\hat{\omega}_L^\tau(dx,x)=0 \quad \forall\tau, \quad \sum_\tau \lambda^\tau\hat{\omega}_L^\tau(dx,x)=d\eta(dx,x) \quad \Leftrightarrow \quad \lambda^\tau=0 \quad \forall\tau. \tag{5.2}$$

One easily completes (5.1) to an $\tilde{s}$-invariant function:

$$\tilde{\omega}=\sum_\tau \hat{\omega}_L^\tau(dx,x)f_\tau(\tilde{C},T). \tag{5.3}$$

$\tilde{\omega}$ decomposes into parts of definite form degree and ghost number which solve the descent equations (4.5) and in particular contains a solution of (1.3) which is given by

$$\omega_D^G=\sum_\tau \hat{\omega}_L^\tau(dx,x)\left[f_\tau(\tilde{C},T)\right]_{D-L} \tag{5.4}$$

where $\left[f_\tau(\tilde{C},T)\right]_{D-L}$ denotes the $(D-L)$–form contained in $f_\tau(\tilde{C},T)$. Notice that (5.4), unlike (3.6), generally depends on the ghosts of diffeomorphisms C^m even if one writes it in terms of the ghost basis $\{C^m,C^\mu\}$ since $\tilde{\omega}$ depends on C^m and dx^m not only via their sum $\tilde{C}^m=C^m+dx^m$ unless $L=0$. I remark however that the corollary given at the end of section 3 implies that without loss of generality functions f_τ contributing to (5.1) resp. (5.3) can be assumed to have degree $g\geq D$ in the $\hat{C}^N$ resp. $\tilde{C}^N$ (provided there are no constant ghosts). One easily verifies that this implies

$$L=D+G-g\leq G \tag{5.5}$$

i.e. only closed but not exact forms with degree $L \leq G$ can contribute to a solution ω_D^G. In particular, apart from constant 0–forms, closed but not exact forms can contribute to anomalies only if they are 1–forms.

b) Presence of constant ghosts

If the manifold is contractible the descent equations always terminate at form–degree 0 (unless ω_D is trivial) but in presence of constant ghosts their general form is not given by (3.8) but reads

$$0 < p \leq D: \ s\omega_p^{G+D-p} + d\omega_{p-1}^{G+D-p+1} = 0, \quad s\omega_0^{G+D} = \hat{\omega}_0^{G+D+1}(C_0),$$
$$\Leftrightarrow \ \tilde{s} f^{G+D} = \hat{\omega}_0^{G+D+1}(C_0), \quad f^{G+D} = \sum_{p=0}^{D} \omega_p^{G+D-p} \tag{5.6}$$

where the possible occurrence of a nontrivial function $\hat{\omega}_0^{G+D+1}$ of the constant ghosts, denoted collectively by C_0, complicates the situation. A simple example shows that such a function really may occur. We introduce the following set of fields: the vielbein fields $e_m{}^a$, a set of bosonic scalar fields Ψ^μ, a corresponding set of constant fermionic ghosts $C_0^\mu = C^\mu$, $\mu = D, \ldots, 2D$ and the ghosts C^m of diffeomorphisms. On these variables we define the BRS–operator according to

$$se_m{}^a = C^n\partial_n e_m{}^a + e_n{}^a\partial_m C^n, \quad s\Psi^\mu = C^m\partial_m\Psi^\mu + C^\mu, \quad sC^m = C^n\partial_n C^m, \quad sC^\mu = 0.$$

I remark that this example can be formulated in the framework given in sect. 2 according to

$$[\mathcal{D}_a, \mathcal{D}_b] = \mathcal{F}_{ab}{}^c\mathcal{D}_c, \quad [\mathcal{D}_a, \Delta_\mu] = [\Delta_\mu, \Delta_\nu] = 0, \quad \mathcal{F}_{ab}{}^c = E_a{}^m E_b{}^n(\partial_n e_m{}^c - \partial_m e_n{}^c),$$
$$\mathcal{D}_a\Psi^\mu = E_a{}^m\partial_m\Psi^\mu, \quad \Delta_\nu\Psi^\mu = \delta_\nu^\mu,$$

i.e. the set $\{\Delta_\mu\}$ consists in this case of abelian generators $\Delta_\mu = \partial/\partial\Psi^\mu$ which commute with the covariant derivatives and the corresponding gauge fields $\mathcal{A}_m{}^\mu$ vanish. One can easily verify that $\tilde{s} = s+d$ acts on the variables Ψ^μ and C^μ according to

$$\tilde{s}\Psi^\mu = F^\mu + C^\mu, \quad \tilde{s}C^\mu = \tilde{s}F^\mu = 0, \quad F^\mu := \tilde{C}^m\partial_m\Psi^\mu, \quad \tilde{C}^m = dx^m + C^m$$

(notice that one has $C^\mu = \hat{C}^\mu = \tilde{C}^\mu$ in this case). Furthermore one can check that a solution of (5.6) with $G = 0$ is given by

$$\begin{aligned} \hat{\omega}_0^{D+1} &= \varepsilon_{\mu_0\ldots\mu_D}C^{\mu_0}\ldots C^{\mu_D}, \\ f^D &= \varepsilon_{\mu_0\ldots\mu_D}\Psi^{\mu_0}(C^{\mu_1}\ldots C^{\mu_D} - F^{\mu_1}C^{\mu_2}\ldots C^{\mu_D} \\ &\quad +F^{\mu_1}F^{\mu_2}C^{\mu_3}\ldots C^{\mu_D} - \ldots + (-)^D F^{\mu_1}\ldots F^{\mu_D}) \end{aligned}$$

(all terms in $\tilde{s}f^D$ cancel apart from $\varepsilon_{\mu_0\ldots\mu_D}C^{\mu_0}\ldots C^{\mu_D}$ and $\varepsilon_{\mu_0\ldots\mu_D}F^{\mu_0}\ldots F^{\mu_D}$; the latter vanishes since it has degree $(D+1)$ in the D anticommuting variables $\tilde{C}^m$).

c) Identities relating the ghosts:

The third assumption which is commented here is the assumption that the ghosts are elementary fields (of course it does not matter whether one regards the $\hat{C}^N$ or the C^m, C^μ as the basic ghost fields). This assumption is needed for the proof of lemma 1 (see appendix A) and the following example shows that the results given in section 3 indeed generally do not hold if this assumption is not satisfied. We consider D–dimensional gravity with Weyl transformations, i.e. the set Δ_μ consists in this case of the generators $l_{ab} = -l_{ba}$ of Lorentz transformations and the generator δ of Weyl transformations. The algebra (2.1) is chosen torsionless, i.e. it is given by

$$[\mathcal{D}_a, \mathcal{D}_b] = -\tfrac{1}{2}R_{ab}{}^{cd}l_{cd} - F_{ab}\delta, \quad [\delta, \mathcal{D}_a] = -\mathcal{D}_a, \quad [l_{ab}, \mathcal{D}_c] = \eta_{bc}\mathcal{D}_a - \eta_{ac}\mathcal{D}_b,$$
$$[l_{ab}, l_{cd}] = 2\eta_{a[c}l_{d]b} - 2\eta_{b[c}l_{d]a}, \qquad [l_{ab}, \delta] = 0 \tag{5.7}$$

where $\eta_{ab} = diag(1, -1, \ldots, -1)$ denotes the D-dimensional Minkowski-metric, and $R_{ab}{}^{cd}$ and F_{ab} denote the components of the Riemann tensor and the field strength of Weyl-transformations. They are obtained from (2.22) which gives in this case (with $R_{mn}{}^{ab} = e_m{}^c e_n{}^d R_{cd}{}^{ab}$, $F_{mn} = e_m{}^a e_n{}^b F_{ab}$)

$$\begin{aligned} R_{mn}{}^{ab} &= \partial_m \omega_n{}^{ab} - \partial_n \omega_m{}^{ab} - \omega_{mc}{}^a \omega_n{}^{cb} + \omega_{nc}{}^a \omega_m{}^{cb}, && (5.8)\\ F_{mn} &= \partial_m A_n - \partial_n A_m, && (5.9)\\ 0 &= \partial_m e_n{}^a - \partial_n e_m{}^a - \omega_{mb}{}^a e_n{}^b + \omega_{nb}{}^a e_m{}^b + e_m{}^a A_n - e_n{}^a A_m && (5.10)\end{aligned}$$

where $\omega_m{}^{ab}$ are the components of the spin-connection (gauge field for Lorentz transformations) and A_m denote the components of the gauge field for Weyl transformations (indices a are raised and lowered by means of η_{ab}). (5.10) expresses the vanishing of the torsion in $[\mathcal{D}_a, \mathcal{D}_b]$. It can be solved for $\omega_m{}^{ab}$ and thus determines it in terms of the vielbein and A_m:

$$\begin{aligned} \omega_m{}^{ab} &= E^{an} E^{br} (\omega_{[mn]r} - \omega_{[nr]m} + \omega_{[rm]n}),\\ \omega_{[mn]r} &= \tfrac{1}{2} e_{ra} (\partial_m e_n{}^a + e_m{}^a A_n - \partial_n e_m{}^a - e_n{}^a A_m). && (5.11)\end{aligned}$$

The covariant derivatives (2.12) take the form

$$\mathcal{D}_a = E_a{}^m (\partial_m - \tfrac{1}{2} \omega_m{}^{ab} l_{ab} - A_m \delta) \tag{5.12}$$

and the BRS-transformations (2.25)-(2.28) of $e_m{}^a$, A_m and the ghosts read

$$\begin{aligned} s\, e_m{}^a &= C^n \partial_n e_m{}^a + e_n{}^a \partial_m C^n + C_b{}^a e_m{}^b + C e_m{}^a, && (5.13)\\ s\, A_m &= C^n \partial_n A_m + A_n \partial_m C^n + \partial_m C, && (5.14)\\ s\, C^m &= C^n \partial_n C^m, && (5.15)\\ s\, C &= C^n \partial_n C, && (5.16)\\ s\, C^{ab} &= C^n \partial_n C^{ab} + C_c{}^b C^{ac} && (5.17)\end{aligned}$$

where C^{ab} denote the Lorentz ghosts and C is the Weyl ghost. As long as $h_m{}^a$, A_m, C^m, C^{ab} and C are elementary fields the results given in section 3 are valid. However there is a particular choice of C in terms of C^m and $e_m{}^a$ for which (5.13)-(5.17) are satisfied without dropping the assumption that the remaining fields are elementary. This choice is

$$C = -\frac{1}{e\,D}\, \partial_m (C^m e), \qquad e = det(e_m{}^a) \tag{5.18}$$

One can check that the identification (5.18) does not contradict any of the assumptions given in section 2 apart from the assumption that all ghosts are elementary fields. In particular A_m is still an elementary field and not a function of $h_m{}^a$ and its derivatives, and the $h_m{}^a$ are not subject to some identity which restricts their independence (like, for instance, $e = const.$). As a consequence of (5.18) there are local solutions of (1.3) which are not obtainable by (3.6). They are simply given by $d^D x$ times an arbitrary function of e:

$$\omega_D^0 = d^D x\, f(e). \tag{5.19}$$

(5.19) indeed solves (1.3) due to

$$s\, e = 0 \tag{5.20}$$

which can be explicitly verified using (5.13) and (5.18). (5.19) is a nontrivial solution of (1.3) since it is not a total derivative (unless $f = const.$). The descent equations arising from (5.19) just read $s\omega_D^0 = 0$, i.e. they terminate at form-degree D, not at degree 0. In particular there is no element of $H^D(\bar{s})$ which corresponds to (5.19). Thus the results given in section 3 evidently do not hold in this case. This shows that the assumption that the ghosts are elementary fields is indeed decisive for the validity of the results given in section 3. I remark that (5.13)-(5.18) is the non-supersymmetric version of an analogous example discussed in [11] for D=4, N=1 supergravity.

6 Summary

It has been shown for the class of gauge theories described in section 2 that each off-shell solution of the consisteny equation (1.1) can be obtained from a nontrivial solution of (3.3). This follows from the fact that the descent equations (3.8) which emerge from the consisteny equation and which can be written compactly in the form

$$\tilde{s}\, f^{G+D} = 0, \quad f^{G+D} = \sum_{p=0}^{D} \omega_p^{G+D-p}, \quad \tilde{s} = s + d$$

have a solution of the form

$$f^{G+D} \cong \tilde{C}^{N_1} \ldots \tilde{C}^{N_{G+D}} f_{N_1 \ldots N_{G+D}}(\mathcal{T})$$

where $\cong$ denotes equality up to trivial solutions of the form $\tilde{s}g$. Thus the nontrivial part of f^{G+D} can be written as a function of the tensor fields $\mathcal{T}$ given in (2.32) and the variables $\tilde{C}$ given in (2.17) (resp. (3.9)). The latter should be considered as appropriate generalizations of the connection 1–forms $\mathcal{A}^N = dx^m \mathcal{A}_m{}^N$.

This particular form of f^{G+D} implies a remarkable statement about the dependence of the solutions ω_D^G of (1.3) on the various variables (fields and their partial derivatives, explicit coordinates and differentials). Namely according to (3.6) ω_D^G can be written, up to trivial contributions, entirely in terms of the undifferentiated ghosts C^μ, the 1–forms $e^a = dx^m e_m{}^a$ and $\mathcal{A}^\mu = dx^m \mathcal{A}_m{}^\mu$ and the tensor fields. In particular the differentials and the gauge fields (including the vielbein) contribute only in the combination of the 1–forms $\mathcal{A}^N \in \{e^a, \mathcal{A}^\mu\}$ apart from the dependence of the tensor fields on the gauge fields. Since ω_D^G is a volume form with ghost number G it is of the form

$$\omega_D^G \cong \mathcal{A}^{N_1} \ldots \mathcal{A}^{N_D} C^{\mu_1} \ldots C^{\mu_G} \Gamma_{N_1 \ldots N_D \mu_1 \ldots \mu_G}(\mathcal{T}). \tag{6.1}$$

If one converts the 'world index' m of $\mathcal{A}_m{}^\mu$ into a 'flat index' a according to

$$\mathcal{A}_a{}^\mu = E_a{}^m \mathcal{A}_m{}^\mu \quad \Rightarrow \quad \mathcal{A}^\mu = e^a \mathcal{A}_a{}^\mu$$

then the differentials contribute in (6.1) only via the 1–forms e^a. Using

$$e^{a_1} \ldots e^{a_D} = d^D x\, e\, \varepsilon^{a_1 \ldots a_D}, \quad d^D x = dx^1 \ldots dx^D, \quad e = det(e_m{}^a), \quad \varepsilon^{1 \ldots D} = 1$$

one therefore can write ω_D^G as the volume element $d^D x\, e$ times a function of the C^μ, $\mathcal{A}_a{}^\mu$ and $\mathcal{T}^r$ whose degree in the C^μ is fixed by its ghost number and which is a polynomial of degree D in the $\mathcal{A}_a{}^\mu$:

$$\omega_D^G \cong d^D x\, e \sum_{n=0}^{D} C^{\mu_1} \ldots C^{\mu_G} \mathcal{A}_{a_1}{}^{\nu_1} \ldots \mathcal{A}_{a_n}{}^{\nu_n} \omega^{a_1 \ldots a_n}_{\nu_1 \ldots \nu_n \mu_1 \ldots \mu_G}(\mathcal{T}). \tag{6.2}$$

I stress that these results hold for arbitrary background metric $\hat{e}_m{}^a(x)$ (provided it is invertible). In particular, though we allowed ω_D^G to depend arbitrarily (but locally) on $h_m{}^a$ and explicit coordinates, it has been shown that the solutions in fact depend on these variables only via $e_m{}^a = \hat{e}_m{}^a(x) + h_m{}^a$ and its derivatives up to trivial contributions which of course can depend arbitrarily on $h_m{}^a$ and x^m. Furthermore the fact that ω_D^G does not depend on the ghosts of diffeomorphisms may indicate that diffeomorphisms are not anomalous (provided the vielbein is invertible). I stress again that the results may change if the assumptions are weakend, see section 5. In particular nontrivial solutions ω_D^G of (1.3) may depend on the ghosts of diffeomorphisms if the manifold is not contractible and possesses closed but not exact p–forms with $p \leq G$. These forms then generally contribute also themselves to ω_D^G according to (5.4). This does not affect the results for classical actions since they have ghost number 0, but there may be anomaly candidates which depend linearly on the C^m and on closed but not exact 1–forms if the manifold allows for such 1–forms. I remark that this holds also for Yang-Mills theories in non-contractible manifolds as follows from the results of [10] (recall that pure Yang–Mills theories are strictly speaking not covered by the investigation of this paper unless one extends them to Einstein–Yang–Mills theories).

One can check our result for known examples of solutions of (1.3). E.g. in $D = 4$ Einstein-Yang–Mills theory the solution of (1.3) corresponding to the nonabelian chiral anomaly can be written in the following ways (see e.g. [6]):

$$\omega^1_{4,chir} = Tr\left\{Cd(AdA + \tfrac{1}{2}A^3)\right\} \tag{6.3}$$

$$= Tr\left\{C(F^2 - \tfrac{1}{2}A^2F - \tfrac{1}{2}AFA - \tfrac{1}{2}FA^2 + \tfrac{1}{2}A^4)\right\} \tag{6.4}$$

$$= d^4x\, e\, \varepsilon^{abcd} Tr\{C\,(F_{ab}F_{cd} - \tfrac{1}{2}\mathcal{A}_a\mathcal{A}_bF_{cd} - \tfrac{1}{2}\mathcal{A}_aF_{bc}\mathcal{A}_d - \tfrac{1}{2}F_{ab}\mathcal{A}_c\mathcal{A}_d + \tfrac{1}{2}\mathcal{A}_a\mathcal{A}_b\mathcal{A}_c\mathcal{A}_d)\} \tag{6.5}$$

with

$$C = C^iT_i, \quad A = dx^m \mathcal{A}_m{}^iT_i, \quad \mathcal{A}_a = \mathcal{A}_a{}^iT_i, \quad F = dA + A^2 = \tfrac{1}{2}\, e^a e^b F_{ab}$$

where $f_{ij}{}^k$ are the structure constants of the Liealgebra $\mathcal{G}$ of the Yang–Mills gauge group, $\{T_i\}$ is a set of constant matrices representing $\mathcal{G}$ according to $[T_i, T_j] = f_{ij}{}^kT_k$ and C^i and $\mathcal{A}_m{}^i$ denote the Yang–Mills ghost and gauge fields whose BRS–transformations (2.28) resp. (2.26) read

$$sC^i = C^m\partial_mC^i - \tfrac{1}{2}C^jC^kf_{jk}{}^i, \quad s\mathcal{A}_m{}^i = C^n\partial_n\mathcal{A}_m{}^i + \mathcal{A}_n{}^i\partial_mC^n + \partial_mC^i - C^j\mathcal{A}_m{}^kf_{jk}{}^i.$$

(6.4) evidently is of the form (6.1) and (6.5) is of the form (6.2) since the entries of F_{ab} are tensor fields. Furthermore one can verify that $\omega^1_{4,chir}$ arises via (3.6) from

$$f^5_{chir}(\tilde{C}, \mathcal{T}) = Tr(\mathcal{C}\mathcal{F}^2 - \tfrac{1}{2}\mathcal{C}^3\mathcal{F} + \tfrac{1}{10}\mathcal{C}^5), \quad \mathcal{C} = \tilde{C}^iT_i, \quad \mathcal{F} = \tfrac{1}{2}\tilde{C}^a\tilde{C}^bF_{ab}$$

which solves (3.3) as one can verify easily by means of $\tilde{s}\,\mathcal{C} = -\mathcal{C}^2 + \mathcal{F}$, $\tilde{s}\,\mathcal{F} = \mathcal{F}\mathcal{C} - \mathcal{C}\mathcal{F}$. Namely one obtains $\tilde{s}f^5_{chir}(\tilde{C}, \mathcal{T}) = Tr(\mathcal{F}^3)$ which is a 6-form in the anticommuting $\tilde{C}^a$ and therefore vanishes in four dimensions.

Appendix

A Proof of Lemma 1

In order to prove Lemma 1 of section 4 we introduce variables which are better suited to this problem than the original set of variables containing the elementary fields and their partial derivatives. The new set of variables is given by

$$\{dx^m, x^m, \hat{C}^N, \mathcal{T}^r, u_l, v_l\},$$

$$\{u_l\} = \{h_m{}^a,\ \mathcal{A}_m{}^\mu,\ \zeta^N,\ \partial_{(m_k}\ldots\partial_{m_1}\mathcal{A}_{m_0)}{}^N,\ \partial_{(m_k}\ldots\partial_{m_1)}\zeta^N :\ k \geq 1\}, \quad v_l = s\,u_l \tag{A.1}$$

where total symmetrization is defined according to

$$T_{(m_1\ldots m_n)} = \frac{1}{n!}\sum_\pi T_{m_{\pi(1)}\ldots m_{\pi(n)}}$$

(π runs over all permutations in the symmetric group S_n). We claim that each local function of the variables $dx^m, x^m, \varphi^\alpha, \partial_m\varphi^\alpha, \partial_m\partial_n\varphi^\alpha, \ldots$ can be also expressed as a function of the variables (A.1), i.e.

$$\omega(dx, x, \varphi, \partial\varphi, \partial\partial\varphi, \ldots) = \Omega(dx, x, \hat{C}, \mathcal{T}, u, v). \tag{A.2}$$

In order to prove (A.2) it is sufficient to show that each of the variables $\partial_{m_k} \ldots \partial_{m_1} \varphi^\alpha$ can be expressed as a local function of the variables (A.1). This can be shown by induction on the order k of partial derivatives of the φ^α. For $k = 0$ the statement is obviously true. Let us denote an arbitrary kth order partial derivative of one of the φ's by $\partial^{(k)}\varphi$ and assume that the statement has been proved up to order k, i.e. $\partial^{(k)}\varphi = \Omega^{(k)}(x, \hat{C}, \mathcal{T}, u, v)$. In order to prove it for order $k+1$ we consider

$$\partial_m \partial^{(k)}\varphi = \partial_m \Omega^{(k)}(x, C, \mathcal{T}, u, v) = \left\{ \frac{\partial}{\partial x^m} + (\partial_m \hat{C}^N) \frac{\partial}{\partial \hat{C}^N} + (\partial_m \mathcal{T}^r) \frac{\partial}{\partial \mathcal{T}^r} \right.$$
$$\left. + (\partial_m u_l) \frac{\partial}{\partial u_l} + (\partial_m v_l) \frac{\partial}{\partial v_l} \right\} \Omega^{(k)}(x, C, \mathcal{T}, u, v). \tag{A.3}$$

Now one considers the various terms appearing on the r.h.s. of (A.3). According to (2.20) $\partial_m \hat{C}^N$ can be written as $s\mathcal{A}_m{}^N - \hat{C}^M \mathcal{A}_m{}^P \mathcal{F}_{PM}{}^N$ which for all values of N is a function of the $x^m, \hat{C}^N, \mathcal{T}^r, u_l, v_l$ since by assumption $\mathcal{F}_{PM}{}^N = \mathcal{F}_{PM}{}^N(\mathcal{T})$ and since $se_m{}^a = sh_m{}^a$. According to (2.8) $\partial_m \mathcal{T}^r$ can be written as $\mathcal{A}_m{}^N \Delta_N \mathcal{T}^r$ which is evidently a function of the variables $x^m, \mathcal{T}^r, u_l$ since by assumption $\Delta_N \mathcal{T}^r$ is a function $R_N{}^r(\mathcal{T})$. Therefore the first three contributions to the r.h.s. of (A.3) can indeed be written in terms of the variables (A.1) but we still have to show that $\partial_m u_l$ and $\partial_m v_l$ can also written in terms of these variables. This is trivial if u_l or v_l denote derivatives of ζ^N or b^N and holds for the remaining u's and v's if it holds for all derivatives of the $\mathcal{A}$'s and $\hat{C}$'s. To prove it for all $\partial^{(k+1)}\mathcal{A}$ we use

$$\partial_{m_{k+1}} \ldots \partial_{m_1} \mathcal{A}_{m_0}{}^N = \partial_{(m_{k+1}} \ldots \partial_{m_1} \mathcal{A}_{m_0)}{}^N + \tfrac{k+1}{k+2} \partial_{(m_1} \ldots \partial_{m_k} X_{m_{k+1})m_0}{}^N \tag{A.4}$$

where

$$X_{mn}{}^N = \partial_m \mathcal{A}_n{}^N - \partial_n \mathcal{A}_m{}^N.$$

According to (2.21) the second term on the r.h.s. of (A.4) is proportional to

$$\partial_{(m_1} \ldots \partial_{m_k} (\mathcal{A}_{m_{k+1})}{}^M \mathcal{A}_{m_0}{}^P \mathcal{F}_{PM}{}^N)$$

and can be expressed as a polynomial in the $\partial^{(n)}\mathcal{A}$, $n \leq k$ with coefficients which are functions of the $\mathcal{T}^r$ (recall that $\mathcal{F}_{MN}{}^P = \mathcal{F}_{MN}{}^P(\mathcal{T})$ and $\partial_m \mathcal{T}^r = \mathcal{A}_m{}^N R_N{}^r(\mathcal{T})$). Thus each partial derivative of $\mathcal{A}_m{}^N$ of order $k+1$ can be expressed in terms of the variables (A.1) provided this holds also for all lower orders and since it holds for $k = 0$ it is true for all k. In order to prove that $\partial^{(k+1)}\hat{C}$ can be expressed in terms of the variables (A.1) we use

$$\partial_{(m_1} \ldots \partial_{m_{k+1})} \hat{C}^N = s\, \partial_{(m_1} \ldots \partial_{m_k} \mathcal{A}_{m_{k+1})}{}^N - \partial_{(m_1} \ldots \partial_{m_k} (\hat{C}^N \mathcal{A}_{m_{k+1})}{}^P \mathcal{F}_{PM}{}^N). \tag{A.5}$$

Since the second term on the r.h.s. depends only on those $\partial^{(n)}\hat{C}$ with $n \leq k$ and on derivatives of $\mathcal{A}$'s and $\mathcal{T}$'s the induction works also for the derivatives of the $\hat{C}^N$. This completes the proof of (A.2).

One now can apply standard techniques (for instance the Basic Lemma [4]): One introduces the 'contracting' operator r defined by

$$r = \sum_l u_l \frac{\partial}{\partial v_l}$$

whose anticommutator with s is the counting operator $\mathcal{N}$ for the variables u_l and v_l,

$$\{r, s\} = \sum_l (v_l \frac{\partial}{\partial v_l} + u_l \frac{\partial}{\partial u_l}) =: \mathcal{N},$$

decomposes a solution Ω of $s\Omega = 0$ according to $\Omega = \sum_n \Omega_n(dx, x, \hat{C}, \mathcal{T}, u, v)$ into eigenfunctions Ω_n of $\mathcal{N}$ with eigenvalue n and concludes that each part Ω_n, $n \neq 0$ is BRS-exact. The locality of Ω implies that Ω_0 does not depend on the u_l and v_l and this proves lemma 1.

I stress that this proof takes advantage of the fact that there are no algebraic identities between the u's and v's since otherwise neither $\mathcal{N}$ nor r are well-defined. The decisive assumption which guarantees the absence of such identities is the assumption that the ghosts are independent elementary fields. Namely on the one hand this guarantees the absence of algebraic identities between the v's and on the other hand it requires also the absence of algebraic identities containing the variables u's since the BRS-variation of each identity between the u's would imply an identity involving the v's.

B Proof of an extension of the algebraic Poincaré lemma

We prove a lemma which reduces to lemma 3 of section 4 if the manifold is contractible and if there are no constant ghosts. The latter are denoted by C_0 as in section 5. The non-constant ghosts are contained in $\{\varphi^\alpha\}$.

Lemma 4: *Closed but not exact contributions to local forms*

$$\omega_p = dx^{m_1}\dots dx^{m_p}\omega_{m_1\dots m_p}(x, C_0, \varphi, \partial\varphi, \partial\partial\varphi, \dots)$$

do not depend on the φ^α and their partial derivatives unless ω_p is a volume form with nonvanishing Euler derivative $\hat\partial/\hat\partial\varphi^\alpha$ with respect to at least one φ^α:

$$\begin{array}{rl} p=0: & d\omega_0 = 0 \quad\Leftrightarrow\quad \omega_0 = \hat\omega_0(C_0),\\ 0<p<D: & d\omega_p = 0 \quad\Leftrightarrow\quad \omega_p = d\omega_{p-1} + \hat\omega_p(dx, x, C_0),\\ p=D: & \omega_D = d\omega_{D-1}(dx, x, C_0, [\varphi]) + d^D x\mathcal{L}(x, C_0, [\varphi]) + \hat\omega_D(dx, x, C_0),\\ & \exists\,\alpha: \quad \hat\partial\mathcal{L}/\hat\partial\varphi^\alpha \neq 0. \end{array} \tag{B.1}$$

The closed but not exact parts $\hat\omega_p(dx, x, C_0)$ are linear combinations of closed forms $\hat\omega_p^\tau(dx, x)$ with linear independent coefficients $f_\tau(C_0)$,

$$\hat\omega_p(dx, x, C_0) = \sum_\tau f_\tau(C_0)\hat\omega_p^\tau(dx, x), \quad d\hat\omega_p^\tau(dx, x) = 0 \tag{B.2}$$

which can be chosen such that

$$\sum_\tau \lambda^\tau\hat\omega_p^\tau(dx, x) = d\eta(dx, x) \quad\Leftrightarrow\quad \lambda^\tau = 0 \quad \forall\tau. \tag{B.3}$$

Lemma 4 is an extension of an analogous result which holds for forms depending locally on the elementary fields φ^α and their partial derivatives but not explicitly on the coordinates. Using the notation $[\varphi]$ for the variables (2.30) this result reads

Algebraic Poincaré lemma for local forms $\omega_p(dx, [\varphi])$ [3, 12, 5]:

$$\begin{array}{rll} p=0: & d\omega_0([\varphi]) = 0 & \Leftrightarrow\quad \omega_0 = const.,\\ 0<p<D: & d\omega_p(dx,[\varphi]) = 0 & \Leftrightarrow\quad \omega_p = d\omega_{p-1}(dx,[\varphi]) + \hat\omega_p(dx),\\ p=D: & \omega_D(dx,[\varphi]) = d\omega_{D-1}(dx,[\varphi]) & \Leftrightarrow\quad \hat\partial\omega_D/\hat\partial\varphi^\alpha = 0 \quad \forall\varphi^\alpha. \end{array} \tag{B.4}$$

Of course the lemma holds analogously in presence of constant ghosts which then just have to be added to the arguments of all forms and functions appearing in (B.4) in order to get the correct version of the lemma for local forms $\omega_p(dx, C_0, [\varphi])$. Using (B.4) we now prove (B.1) by an inspection of the dependence of a closed form $\omega_p(dx, x, C_0, [\varphi])$ on the partial derivatives of the φ^α. To this end we introduce the counting operator N_∂ of the total number of partial derivatives acting on the φ^α. It acts as follows:

$$N_\partial\, \partial^{(r)}\varphi^\alpha = r\,\partial^{(r)}\varphi^\alpha, \quad N_\partial\,(\partial^{(r)}\varphi^\alpha\partial^{(s)}\varphi^\beta) = (r+s)\,\partial^{(r)}\varphi^\alpha\partial^{(s)}\varphi^\beta, \quad \dots$$

where $\partial^{(r)}\varphi^\alpha$ denotes an arbitrary partial derivative of φ^α of rth order, i.e. $\partial^{(r)}\varphi^\alpha \in \{\partial_{m_1}\ldots\partial_{m_r}\varphi^\alpha\}$. Each local form $\omega_p(dx,x,C_0,[\varphi])$ can be decomposed uniquely into N_∂–eigenfunctions $\omega_p^{(n)}$ where n denotes the N_∂–eigenvalue. The locality of ω_p guarantees that only nonnegative integers n appear in this decomposition and that there is a largest eigenvalue which is denoted by $\bar n$:

$$\omega_p(dx,x,C_0,[\varphi]) = \sum_{n=0}^{\bar n}\omega_p^{(n)}(dx,x,C_0,[\varphi]),\quad N_\partial\,\omega_p^{(n)} = n\,\omega_p^{(n)}. \tag{B.5}$$

d is decomposed into a part d^x which acts nontrivially only on the variables x^m and a part d^φ acting nontrivially only on the variables $[\varphi]$:

$$\begin{array}{lll} d^x\,dx^m = d^x\,C_0 = 0, & d^x\,x^m = dx^m, & d^x\,\partial^{(r)}\varphi^\alpha = 0,\\ d^\varphi\,dx^m = d^\varphi\,C_0 = 0, & d^\varphi\,x^m = 0, & d^\varphi\,\partial^{(r)}\varphi^\alpha = dx^m\partial_m\partial^{(r)}\varphi^\alpha. \end{array} \tag{B.6}$$

Evidently d^x and d^φ satisfy $[N_\partial, d^x] = 0$, $[N_\partial, d^\varphi] = d^\varphi$. Therefore $d\omega_p = 0$ decomposes into the set of equations

$$d^x\omega_p^{(0)} = 0,\quad 0 \le n < \bar n:\quad d^\varphi\omega_p^{(n)} + d^x\omega_p^{(n+1)} = 0,\quad d^\varphi\omega_p^{(\bar n)} = 0. \tag{B.7}$$

Keeping in mind that d^φ treats the variables x and C_0 as constants one concludes by means of (B.4) from the last of the equations (B.7) that $\omega_p^{(\bar n)}$ (i) is d^φ–exact or (ii) is a volume form with nonvanishing Euler derivative with respect to at least one φ^α or (iii) does not depend on the $[\varphi]$ at all. (i) can hold only if $\bar n \neq 0$ and $p \neq 0$, (iii) only if $\bar n = 0$. Thus one has one of the following mutually excluding cases:

$$\begin{array}{lll} \text{(i)} & \bar n \neq 0,\ p \neq 0: & \omega_p^{(\bar n)} = d^\varphi\omega_{p-1}^{(\bar n-1)}(dx,x,C_0,[\varphi]),\\ \text{(ii)} & p = D: & \exists\,\alpha:\ \hat\partial\omega_D^{(\bar n)}/\hat\partial\varphi^\alpha \neq 0,\\ \text{(iii)} & \bar n = 0: & \omega_p = \omega_p(dx,x,C_0). \end{array} \tag{B.8}$$

In the case (i) one considers the form ω'_p defined by

$$\text{(i)}:\quad \omega'_p(dx,x,C_0,[\varphi]) := \omega_p(dx,x,C_0,[\varphi]) - d\omega_{p-1}^{(\bar n-1)}(dx,x,C_0,[\varphi]) \tag{B.9}$$

where $\omega_{p-1}^{(\bar n-1)}$ is the form appearing in (B.8(i)). (B.9) states that ω_p is exact up to a closed form ω'_p whose decomposition into N_∂–eigenfunctions ${\omega'}_p^{(n)}$ contains only parts with eigenvalues $n < \bar n$ as can be seen by inserting (B.5) into (B.9) taking into account (B.8(i)). In the case (ii) one analogously considers the form

$$\text{(ii)}:\quad \omega'_D := \omega_D - \omega_D^{(\bar n)} \tag{B.10}$$

which also contains only N_∂–eigenfunctions with eigenvalues smaller than $\bar n$. By induction on n starting from the highest eigenvalue $\bar n$ one therefore can prove

$$\begin{array}{rl} p = 0: & \omega_0 = \omega_0(x,C_0),\\ 0 < p < D: & \omega_p = d\omega_{p-1}(dx,x,C_0,[\varphi]) + \hat\omega_p(dx,x,C_0),\\ p = D: & \omega_D = d\omega_{D-1}(dx,x,C_0,[\varphi]) + d^Dx\,\mathcal{L}(x,C_0,[\varphi]) + \hat\omega_D(dx,x,C_0),\\ & \exists\,\alpha:\quad \hat\partial\mathcal{L}/\hat\partial\varphi^\alpha \neq 0 \end{array} \tag{B.11}$$

where ω_{p-1}, ω_{D-1} and $\mathcal{L}$ are local since they are finite linear combinations of forms whose N_∂–eigenvalues do not exceed $\bar n$ as follows from the proof. Finally one easily verifies that $d\omega_0 = 0$ requires $\omega_0 = \omega_0(C_0)$ and that the forms $\hat\omega_p(dx,x,C_0)$ appearing in (B.11) for $p > 0$ can be chosen to satisfy (B.3) by absorbing exact contributions into ω_{p-1}. This completes the proof of (B.1) but I add a remark about the global existence of ω_{p-1}. It is assumed that $\omega_p(dx,x,C_0,[\varphi])$ exists globally (on the whole manifold) with regard to its dependence on the x^m. Together with $d\omega_p = 0$ this requires:

a) each $\omega_p^{(n)}$ exists globally (in the same sense) since the $\omega_p^{(n)}$ are independent due to the assumption of the independence of the variables (2.39) (cases where this assumption is not justified are clearly beyond the scope of this paper),
b) due to a) each $d^{\varphi}\omega_p^{(n)}$ also exists globally since d^{φ} does not change the x–dependence,
c) (B.7) and b) shows that $d^x\omega_p^{(n)}$ exists globally.
One conludes:
d) (B.8(i)) shows that $\omega_{p-1}^{(\bar{n}-1)}$ and $d^x\omega_{p-1}^{(\bar{n}-1)}$ exist globally since this holds for $\omega_p^{(\bar{n})}$ and $d^x\omega_p^{(\bar{n})}$ according to a) and c).
e) Due to a) and d), ω'_p given in (B.9) also exists globally. The same holds of course for ω'_D in (B.10). Since ω'_p and ω'_D contain only parts with N_∂–eigenvalues smaller than $\bar{n}$ one proves by iteration of the arguments that all forms occurring in (B.11) exist globally.

The version (4.4) of (B.1) holding in contractible manifolds follows from the fact that in this case each closed form $\hat{\omega}_p(C_0, dx, x)$ is exact apart from constant 0–forms $\hat{\omega}_0(C_0)$.

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