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PRIMITIVITY AND WEAK DISTRIBUTIVITY IN NEAR RINGS AND MATRIX NEAR RINGS

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ABSTRACT

This paper shows the structure of matrix near ring constructed over a weakly distributive and primative near ring. It is proved that a weakly distributive primitive near ring is a ring and the matrix near rings constructed over it is also a bag.

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1. Prerequisites

(R,+,.) is a right near ring if (R,+) is a, not necessarily abelian, group, (R,.) is a semigroup and $(x+y)z = xz + yz \ \forall \ x, y, z \in R$. Throughout this paper, R is a right near ring, not necessarily with identity. Let R and T be two near rings. A mapping θ from R to T is called a near ring homomorphism if $\theta(x+y) = \theta x + \theta y$ and $\theta(xy) = (\theta x)(\theta y) \ \forall \ x, y \in R$. If there is a near ring homomorphism $\theta : R \rightarrow M(G)$, the set of all functions from G to itself, then G is called a (left) R-module. G is faithful if $\ker\theta = \{0\}$. A subgroup H of G is called an R-submodule of G if $RH \subseteq H$. If there exists g in G such that $Rg = G$ then G is called a monogenic R-module which is strongly monogenic if $\forall \ g \in G$ either $Rg = G$ or $Rg = \{0\}$. In Van Der Walt [13], an R-module G is called connected if for any $g_1, g_2 \in G \ \exists \ g \in G, x, y \in R$ such that $g_1 = xg$ and $g_2 = yg$. A monogenic R-module G is an R-module of type 0 if G is simple, of type 1 if G is simple and strongly monogenic, and of type 2 if G is non trivial and has no non trivial proper R-submodule. R is called ν-primitive on G for $\nu = 0, 1, 2$, if G is faithful and of type ν. A subset I of R is called an ideal of R if $(I, +)$ is a normal subgroup of $(R, +)$, $IR \subseteq I$ and $x(a+y) - xy \in I \ \forall x, y \in R, a \in I$. Let I, J and P be ideals of R. P is called a prime ideal if whenever $IJ \subseteq P$ then $I \subseteq P$ or $J \subseteq P$. We call R a prime ring if $\{0\}$ is a prime ideal. An element d of R is called a distributive element if $d(x+y) = dx + dy \ \forall \ x, y \in R$. R_d, the set of all distributive elements of R, forms a semigroup under multiplication. If $R = R_d$, then R is distributive. Let $(S, .) \subseteq (R_d, .)$. R is called distributively generated (d.g. in short) if (R,+) is generated as a group by (S,.). For the rest of this section, R is a d.g. near ring. We define distributor of x over y and z as $(x; y, z) = x(y+z) - xz - xy \ \forall x, y, z \in R$. If X, Y, Z are subsets of R, then

$$(X; Y, Z) = Gp < \{(x; y, z) : x \in X, y \in Y, z \in Z\} >$$

where $Gp < X >$ is group generated by X. Let I be an ideal of R. The distributor series of ideals of R is defined as

$$D^0(I) = I, D^{m+1}(I) = Gp < (R; D^m(R), D^m(R)) >^R$$

where $Gp < X >^R$ is a normal subgroup of R generated by X. A d.g. near ring R is called weakly distributive (w.d. in short) if $D^m(R) = \{0\}$ for some integer m.

For further details on near rings, we refer Meldrum [8].

Let n be a natural number and R^n denotes the direct sum of n copies of (R,+). Elementary matrices over R are defined as functions from R^n to itself as

$$f_{ij}^r = \iota_i f^r \pi_j$$

for $r \in R, 1 \leq i, j \leq n$ where $f^r : R \rightarrow R$ such that $f^r(x) = rx \ \forall \ x \in R$, ι_j and π_j are j-th co-ordinate injection and projection functions respectively.

The near ring of n x n matrices over R, denoted by $M_n(R)$, is defined as a subnear ring of $M(R^n)$, generated by $\{f_{ij}^r : r \in R, 1 \leq i, j \leq n\}$.

More details on matrix near rings are available in [1]-[6] and [9]-[14]. There is a rich collection of work on this area in Abbasi [1] and Meyer [11].

2. On Primitive and W.d. Near Rings

Our goal in this section is to show the action of primitivity on w.d. near rings. For our main results, we need the following lemmas of Fröhlich [7].

LEMMA 2.1: *An abelian d.g. near ring is a ring.*

LEMMA 2.2: *If R is d.g. with $R^2 = R$, then R is w.d. if and only if $(R, +)$ is soluble.*

LEMMA 2.3: *If R is d.g. with $(R, +)$ soluble then $\delta_1(R)$, the commutator of R, is multiplicatively nilpotent.*

Another useful result available in Meldrum[8, 6.31] is

LEMMA 2.4: *Any ν-primitive ideal is a prime ideal for $\nu = 0, 1, 2$.*

We are now able to present one of the major results of this section.

THEOREM 2.5: *Let R be a prime d.g. near ring. If $(R, +)$ is soluble, then R is a ring.*

Proof: The solubility of $(R, +)$ forces $\delta_1(R)$ to be multiplicatively nilpotent. Now since R is a prime near ring therefore $\delta_1(R) = \{0\}$ and so $(R, +)$ is abelian. Rest of the proof follows from lemma 2.1.

This theorem together with lemma 2.2 gives us

COROLLARY 2.6: *If R is a prime w.d. near ring with $R^2 = R$, then R is a ring.*

COROLLARY 2.7: *Let $\nu \in \{0, 1, 2\}$. If R is ν-primitive w.d. near ring with $R^2 = R$, then R is a ring.*

Proof: Follows from above result and lemma 2.4.

An immediate consequence of this result is

COROLLARY 2.8 *If $\nu \in \{0, 1, 2\}$ and R is a w.d. ν-primitive near ring on (R,+), then R is a ring.*

Our last result of this section is also an immediate consequence of lemma 2.2 and will be used in the next section.

COROLLARY 2.9: *Let R be a connected R module. Then R is w.d. if and only if $(R, +)$ is soluble.*

3. on Primitive and W.d. Matrix Near Rings

For $\nu \in \{0, 2\}$, it is shewed in Van Der Walt [13] that if R is ν-primitive on G, then $M_n(R)$ is ν-primitive on G^n, since the author only considers near rings with identity for which 1 and 2 primitivity coincide. As we are interested in near rings not necessarily with identity , our following result extends the work for $\nu \in \{0, 1, 2\}$.

LEMMA 3.1: *If G is an strongly monogenic R module, then G^n is an strongly monogenic $M_n(R)$ module.*

Proof: If G is strongly monogenic by g, then it can be checked easily that G^n is strongly monogenic by $< g, 0, ..., 0 >$.

THEOREM 3.2: *Let $\nu \in \{0,1,2\}$. If R is ν-primitive on G, then $M_n(R)$ is ν-primitive on G^n.*

To extend our earlier work on w.d. near rings presented in Abbasi, Meldrum and Meyer [3], we need these results of Van Der Walt [13] and Abbasi and Meldrum [2].

LEMMA 3.3: *If G is a connected R module, then G^n is a connected $M_n(R)$ module.*

LEMMA 3.4: *Let $(R,+)$ be a connected R-module. Then $(R,+) \in V$, a variety of additive groups, if and only if $(M_n(R),+) \in V$.*

THEOREM 3.5: *Let R be a connected R module. Then R is w.d. if and only if $M_n(R)$ is w.d.*

Proof: The solubility of $(R,+)$, followed from weak distributivity of R, gives us the solubility of $(R^n,+)$. Now since $(R^n,+)$ is a connectec $M_n(R)$ module, therefore corollary 2.9 forces $M_n(R)$ to be a w.d. near ring.
The proof of converse follows from corollary 2.9, lemmas 3.3, 3.4 and 2.2.

Our next result follows immediately from the above theorem.

COROLLARY 3.6: *Let $\nu \in \{0,1,2\}$ and R be a ν-primitive near ring on $(R,+)$. Then R is w.d. if and only if $M_n(R)$ is w.d.*

We conclude this paper by looking at the structure of matrix near rings constructed over ν-primitive w.d. near rings.

THEOREM 3.7: *Let R be a ν-primitive w.d. near ring for $\nu = 0,1,2$. Then $M_n(R)$ is a ring.*

Proof: Theorem 3.2 together with corollaries 2.8 and 3.6 gives us what we want.

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