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HIGH ENERGY ASYMPTOTICS OF PERTURBATIVE MULTI-COLOR QCD *

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Abstract

It is shown that the Bethe-Salpeter equations for compound states of many reggeized gluons are conformally invariant in the two-dimensional impact parameter space. Their solutions can be written in holomorphically factorized form and there is a differential operator commuting with the holomorphic part of the corresponding Hamiltonian.

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The structure functions of deep-inelastic scattering at small-x satisfy two different equations in the leading logarithmic approximation (LLA). The first one – the GLAP equation, describes the Q^2-evolution of partonic distributions $h_i(x)$[1]. The second one–the BFKL, equation determines the x-dependence of parton densities $h_i(x, k_\perp)$[2]. Analogous equations for matrix elements of higher twist operators were constructed in Refs. [3] and [4]. Here we discuss the possibility of finding an exact solution for multi-gluon compound states[4] in LLA for the color group SU(N), in the limit $N \to \infty$. The contributions of diagrams with many reggeized gluons are important for the unitarization of the perturbative Pomeron in QCD[5,6]. In the t-channel angular momentum representation, the partial waves $f_\omega(\vec{\rho}_1, \vec{\rho}_2, \cdots \vec{\rho}_n; \vec{\rho}_o)$ describe colorless compound states with the transverse coordinate $\vec{\rho}_o$ and the gluon impact parameters $\vec{\rho}_i (i = 1, 2 \cdots n)$ and satisfy the following Bethe-Salpeter equation[4,5].

$$\omega f_\omega = \sum_{\substack{i,k = 1,2\cdots n \\ i<k}} K_{ik} \, f_\omega \tag{1}$$

The eigenvalues ω of this equation give the positions $j = 1 + \omega$ of the Regge singularities of the t-channel scattering amplitudes and the eigenfunctions f_ω are proportional to the Reggeon-particle-particle couplings.

The color structure of the pair kernels K_{ik} is factorized

$$K_{ik} = T_i^a \, T_k^a \, \frac{g^2}{8\pi^2} \, \mathcal{H}_{ik}, \tag{2}$$

where g is the QCD coupling constant, T_i^a are the gauge group generators acting upon the color indices of the gluon i. $\mathcal{H}_{ik}$ is the pair hamiltonian, depending on complex coordinates ρ_i, ρ_i^*, ρ_k, ρ_k^* of two-dimensional impact parameters $\vec{\rho}_i$, $\vec{\rho}_k$ of gluons and on their momenta $P = i\frac{\partial}{\partial \rho}$. The hamiltonian has the property of holomorphic separability[5].

$$\mathcal{H}_{ik} = H_{ik} + H_{ik}^* \tag{3}$$

Its holomorphic part H_{ik}

$$H_{ik} = P_i^{-1} \ell n(\rho_{ik}) \, P_i + P_k \ell n(\rho_{ik}) P_k^{-1} + \ell n(P_i P_k) - 2\psi(1), \tag{4}$$

and its antiholomorphic part H_{ik}^* act on the coordinates ρ_i, ρ_k and ρ_i^*, ρ_k^* correspondingly [in Eq. (4), $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$, $\rho_{ik} = \rho_i - \rho_k$].

Due to property (3) the solution of Eq. (1) is significantly simplified for the case of the three-particle compound state (Odderon) and in the general case of n-particle compound states for multicolor QCD ($N \to \infty$). Indeed, here one can use the following ansatz:

$$f_\omega(\vec{\rho}_1,\ \vec{\rho}_2 \cdots \vec{\rho}_n,\ \vec{\rho}_0) = \sum_r\ C_r f^r(\rho_1,\ \rho_2,\ \cdots \rho_n;\ \rho_0)\tilde{f}^r(\rho_1^*, \rho_2^*, \cdots; \rho_0^*), \tag{5}$$

corresponding to the holomorphic factorization of Green functions in two-dimensional conformal field theories[5]. The holomorphic and antiholomorphic functions $f^r(\rho_i)$, $\tilde{f}^r(\rho_i^*)$ satisfy the following equations

$$\epsilon f^r(\rho_i) = H f^r(\rho_i), \quad \tilde{\epsilon}\tilde{f}^r(\rho_i) = H^* \tilde{f}^r(\rho_i^*), \tag{6}$$

where

$$H = \sum_{i=1}^{n}\ H_{i,i+1} \tag{7}$$

and the indices $j = n+1$ and $j = 1$ are equivalent. The eigenvalue ω of Eq. (1) is expressed through the sum of "energies" ϵ and $\tilde{\epsilon}$ by the formulae:

$$\omega = -\frac{g^2}{16\pi^2} N(\epsilon + \tilde{\epsilon}). \tag{8}$$

For the case of multi-color QCD the validity of the ansatz(5) is a consequence of the fact, that Eq. (1) is diagonalized in color space for $N \to \infty$ by the product of color structures.

$$Tr\,(\lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_k}) \tag{9}$$

where the λ_{i_r} are the Gell-Mann matrices for the gluons i_r. One can neglect, in this limit, interaction terms H_{ij} for the gluons $i,\ j$ belonging to different factors in (9) and leave only terms with the neighboring labels in the product of λ-matrices. Thus, eigenfunctions of Eq. (1) turn out to be products of the eigenfunctions for each color structure (9). In the irreducible case of one structure we can enumerate the particles in accordance with the order of multiplying the λ_i in Eq. (9) and obtain Eqs. (6) and (7) from Eq. (1) after the substitution (see Eq. (2))

$$T_i^a T_k^a \longrightarrow -\frac{N}{2}\delta_{k,i+1}. \tag{10}$$

This simplification corresponds to the well-known result of 't Hooft, for glue dynamics, that if one substitutes, in color space, each gluon line by two quark lines with an opposite arrow on them, then only the quark diagrams with a cylinder topology will give the maximal number of quark loops, needed to obtain the maximal power of N.

For the Hamiltonian (7) describing a gluon chain, in which only neighboring particles interact, one can obtain the following representations:

$$H = \sum_i \left[P_i^{-1} \ell n \left(\rho_{i-1,i} \cdot rho_{i,i+1} \right) P_i + 2\ell n(P_i) - 2\psi(1) \right] =$$

$$= \sum_i \left[\rho_{i-1,i} \ell n(P_i) \rho_{i-1,i}^{-1} + \rho_{i,i+1} \ell n(P_i) \rho_{i,i+1}^{-1} + \ell n \left(\rho_{i-1,i} \cdot \rho_{i,i+1} \right) - 2\psi(1) \right] = \tag{11}$$

$$= \sum_i \left[\ell n \left(\rho_{i-1,i}^2 P_i \right) + \ell n \left(\rho_{i,i+1}^2 P_i \right) - \ell n \left(\rho_{i-1,i} \rho_{i,i+1} \right) - 2\psi(1) \right].$$

From the last representation the invariance of H under the Möbious group of conformal transformations is obvious. According to the first and second representations, H is not a symmetric operator ($H^T \neq H$) but H^T can be obtained from H with two different similarity transformations:

$$H^T = A_1 H A_1^{-1} = A_2 H A_2^{-1}, \qquad A = \Pi_{i=1}^n P_i \ , \ \ A_2 = \left(\Pi_{i=1}^n \rho_{i,i+1} \right)^{-1}. \tag{12}$$

This corresponds to two possible definitions of scalar products (χ, Φ), compatible with Eq. (5): $(\chi, \Phi)_{1,2} = \int \Pi_i d\rho_i \ \chi A_{1,2} \phi$.

From Eq. (12) we conclude that the operator

$$A = A_2^{-1} A_1 = \rho_{12} \rho_{23} \cdots \rho_{n1} P_1 P_2 \cdots P_n \tag{13}$$

commutes with H:

$$[A, H] = O. \tag{14}$$

Therefore, the eigenfunctions for the Schrödinger equation (6) are also the eigenfunctions of the simple differential equation

$$A f^{\tau} = \lambda f^{\tau} \tag{15}$$

There is another representation of the operator A given by (13):

$$A \sim Tr \ M(\rho_1) M(\rho_2) \cdots M(\rho_n), \tag{16}$$

where

$$M(\rho) = \begin{pmatrix} M_0(\rho) & M_+(\rho) \\ -M_-(\rho) & -M_0(\rho) \end{pmatrix}, \quad M_0(\rho) = \rho \partial, \ M_+ = \partial, \ M_- = \rho^2 \partial \tag{17}$$

and $M_{0,\pm}$ are the generators of the conformal group. According to Eq. (15), A can be interpreted as a transfer matrix for a two-dimensional lattice with the weights $M(\rho_i)$ in each vertex, multiplied as 2×2 matrices in the "space" directions and as differential operators in the "time" direction.

For the simplest nontrivial case $n = 3$ (Odderon), due to the conformal invariance of H (11), one can search for the solution in the form [5,7]:

$$f(\rho_1, \rho_2, \rho_3; \rho_0) = \left(\frac{\rho_{12}\rho_{13}\rho_{23}}{\rho_{10}^2\rho_{20}^2\rho_{30}^2}\right)^{\frac{m}{3}} \psi^m(x) \tag{18}$$

where $x = \frac{\rho_{12}\rho_{30}}{\rho_{10}\rho_{32}}$ is the anharmonic ratio and m is the conformal weight of the Odderon ($m = \frac{1}{2} + i\nu + \frac{n}{2}$ for real ν and integer n). In the x representation H equals:

$$H = \ell n\left(\frac{1-x}{x}k_1\right) + \ell n\left(\frac{x}{1-x}k_1\right) + \ell n\left(\frac{x}{1-x}k_2\right) + \ell n\left(\frac{1}{x(1-x)}k_2\right) + \ell n\left(\frac{1-x}{x}k_3\right)$$
$$+ \ell n\left(\frac{1}{x(1-x)}k_3\right) - 6\psi(1) \tag{19}$$

where the "momenta" k_r are given by

$$k_r = i\nabla_r, \quad \nabla_1 = \frac{m}{3}(1-2x)+x(1-x)\partial, \quad \nabla_2 = \frac{m}{3}(1+x)+x(1-x)\partial, \quad \nabla_3 = -\frac{m}{3}(2-x)+x(1-x)\partial. \tag{20}$$

The equation (15) is simplified because in the Odderon case expression (13) is an ordinary differential operator of the third order.

$$A(x) = k_1\frac{1}{x(1-x)}k_2k_3 = k_1\frac{1}{x(1-x)}k_3k_2 = \cdots = \frac{1}{x}k_3\frac{x}{1-x}k_2\frac{1}{x}k_1. \tag{21}$$

Note that H (19) is some function of A (21) in accordance with Eq. (14). This function can be found for $x \to 0$, when $A \to \infty$:

$$\frac{1}{2}H(A)_{A\to\infty} = \ell nA = -\ell nx + \frac{1}{2}\psi(\frac{m}{3} + x\partial) + \frac{1}{2}\psi(1 + \frac{m}{3} + x\partial) + \frac{1}{2}\psi(1 - 2\frac{m}{3} + x\partial)$$
$$+ \frac{1}{2}\psi(1 - \frac{m}{3} - x\partial) + \frac{1}{2}\psi(-\frac{m}{3} + x\partial) + \frac{1}{2}\psi(2\frac{m}{3} - x\partial) - 3\psi(1). \tag{22}$$

In particular, from the above asymptotic of H, one can find the solution of Eqs.(6) at small x:

$$\psi^{m,m}(x,x^*)_{x\to 0} = |x|^{\frac{4m}{3}} + |x|^{2-\frac{2}{3}m}(a_1 + a_2 \ell n|x|). \tag{23}$$

This small x behavior of $\psi^{m,m}$ was used in Ref. [7] to search for a good ansatz for the Odderon wave function in the variational approach to the Odderon problem in QCD. It will be interesting if this function could be found in an explicit way from the differential Eq. (15).

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References

1. V. N. Gribov and L. N. Lipatov, Sov. J. Nucl. Phys. **15**, (1972) 675; L. N. Lipatov, Sov. J. Nucl. Phys. **20** (1975) 94; G. Altarelli, G. Parisi, Nucl. Phys. **B126** (1977) 298.
2. E. A. Kuraev, L. N. Lipatov, V. S. Fadin, Sov. Phys. JETP **45** (1977) 199; Ya. Ya. Balitsky, L. N. Lipatov, Sov. J. Nucl. Phys. **28** (1978) 822.
3. A. P. Bukhvostov, G. V. Frolov, L. N. Lipatov, Nucl. Phys. **B258** (1985) 601.
4. J. Bartels, Nucl. Phys. **B175** (1980) 365; J. Kwiecinski, M. Praszalowicz, Phys. Lett. **B94** (1980) 413.
5. L. N. Lipatov, Phys. Lett. **B251** (1990) 284.
6. L. N. Lipatov, Nucl. Phys. **B365** (1991) 641.
7. P. Gauron, L. Lipatov. B. Nicolescu, Phys. Lett. **B260** (1991) 407; **B304** (1993) 334.