ON THE KINETIC THEORY OF A FULLY IONIZED GAS

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1. Introduction

The standard method used to determine successive approximations to the transport coefficients from the Boltzmann equation of a single gas and ionized gases is that of Chapman-Enskog (see for example [1]). The deviation from equilibrium of the distribution function in this method is found as a solution of an integral equation that follows from the Boltzmann equation.

Other method used is that of moments of Grad [2], where the deviation from equilibrium of the distribution function is written in terms of the moments of the distribution function. The transport coefficients are then obtained from the field equations of the moments through an iterate scheme. For a single gas, successive approximations to the transport coefficients were obtained in Refs. [3] and [4] from a $(13+9N)$-field theory, and the proof of the equivalence between the two methods can be found in [4].

Recently [5] an alternative method for kinetic theory was proposed that combines the features of the Chapman-Enskog and Grad methods, but neither a solution of the integral equation is needed nor the field equations of the moments are used. Like in the method of Grad, the deviation from equilibrium of the distribution function is written in terms of the moments of the distribution function, but the constitutive equations follow direct from the Boltzmann equation through the Chapman-Enskog method. The aim of this work is to apply the alternative method to ionized gases.

2. The alternative method

We denote by $\mathbf{x}$ and $\mathbf{c}_\alpha$ the position and the velocity of a particle of constituent α in a mixture of ions $\alpha = I$ and electrons $\alpha = E$ of a fully ionized gas. A state of this mixture will be characterized by the set of one-particle distribution functions

$$f_\alpha \equiv f(\mathbf{x}, \mathbf{c}_\alpha, t) \qquad (\alpha = I, E) \tag{1}$$

such that $f_\alpha(\mathbf{x}, \mathbf{c}_\alpha, t) d\mathbf{x} d\mathbf{c}_\alpha$ gives, at time t, the number of α particles in the volume element $d\mathbf{x} d\mathbf{c}_\alpha$ around $\mathbf{x}$ and $\mathbf{c}_\alpha$. The one-particle distribution function f_α obeys the Boltzmann equation

$$\frac{\partial f_\alpha}{\partial t} + c_i^\alpha \frac{\partial f_\alpha}{\partial x_i} + \frac{e_\alpha}{m_\alpha}\left[E_i - (\mathbf{c}_\alpha \times \mathbf{B})_i\right] \frac{\partial f_\alpha}{\partial c_i^\alpha} = \sum_{\beta=I}^{E} \int \left(f'_\alpha f'_\beta - f_\alpha f_\beta\right) g^{\beta\alpha} b\, db\, d\varepsilon\, d\mathbf{c}_\beta. \tag{2}$$

In Eq. (2) m_α and e_α denote the mass and electric charge of a particle of constituent α, E_i the electric field and B_i the magnetic flux intensity. Also, $\mathbf{g}^{\beta\alpha} = \mathbf{c}^\beta - \mathbf{c}^\alpha$ is the relative velocity of two particles before collision, b and ε are impact parameters, and the primes refer to after collision velocities. Besides, f'_β denotes $f(\mathbf{x}, \mathbf{c}'_\beta, t)$ and so forth.

We base on the method of Grad and characterize a macroscopic state of the ionized gas by the moments

$$\varrho_\alpha = \int m_\alpha f_\alpha d\mathbf{c}_\alpha. \qquad \text{with} \qquad \varrho = \sum_{\alpha=1}^{E} \varrho_\alpha. \tag{3}$$

$$v_i = \frac{1}{\varrho}\sum_{\alpha=1}^{E} \varrho_\alpha v_i^\alpha, \qquad \text{with} \qquad v_i^\alpha = \frac{1}{\varrho_\alpha}\int m_\alpha c_i^\alpha f_\alpha d\mathbf{c}_\alpha. \tag{4}$$

$$T = \frac{2m_\alpha}{3k\varrho_\alpha}\int \frac{1}{2} m_\alpha C_\alpha^2 f_\alpha d\mathbf{c}_\alpha. \qquad p^\alpha_{\langle ij\rangle} = \int m_\alpha C^\alpha_{\langle i} C^\alpha_{j\rangle} f_\alpha d\mathbf{c}_\alpha. \tag{5}$$

$$J_i^\alpha = \int m_\alpha \xi_i^\alpha f_\alpha d\mathbf{c}_\alpha. \qquad q_i^\alpha = \int \frac{1}{2} m_\alpha C_\alpha^2 C_i^\alpha f_\alpha d\mathbf{c}_\alpha. \tag{6}$$

In the set of moments (3)-(6) ϱ_α denotes the mass density of constituent α, v_i the velocity of the mixture, T the temperature of the mixture. $p^\alpha_{\langle ij\rangle} = p^\alpha_{ij} - \frac{1}{3}p^\alpha_{rr}\delta_{ij}$ the pressure deviator. J_i^α the diffusion flux, and q_i^α the heat flux of constituent α in the mixture. Besides, ϱ is the mass density of the mixture, v_i^α the velocity of constituent α, k the Boltzmann constant, $C_i^\alpha = c_i^\alpha - v_i^\alpha$ and $\xi_i^\alpha = c_i^\alpha - v_i$ partial peculiar velocities. We have supposed that the constituents in the mixture are at the same temperature T.

According to the method of Grad we expand the distribution function in polynomials of the peculiar velocity ξ_i^α about a locally Maxwellian function $f_\alpha^{(0)}$, with the coefficients related to the moments (3)-(6), e.g.,

$$f_\alpha = f_\alpha^{(0)}\left\{1 + \frac{\beta_\alpha}{\varrho_\alpha}\xi_i^\alpha J_i^\alpha + \frac{\beta_\alpha^2}{2\varrho_\alpha}\left[\xi_i^\alpha \xi_j^\alpha p^\alpha_{\langle ij\rangle} + \frac{4}{5}\xi_i^\alpha q_i^\alpha\left(\frac{\beta_\alpha \xi_\alpha^2}{2} - \frac{5}{2}\right)\right]\right\}. \tag{7}$$

where

$$f_\alpha^{(0)} = \frac{\varrho_\alpha}{m_\alpha}\left(\frac{\beta_\alpha}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{\beta_\alpha \xi_\alpha^2}{2}}, \qquad \beta_\alpha = \frac{m_\alpha}{kT}. \tag{8}$$

Now we apply the method of Chapman-Enskog, insert Eq. (7) into the Boltzmann equation (2) neglect all non-linear terms and all derivatives of $p^\alpha_{\langle ij\rangle}$, J_i^α, and q_i^α. Therefore, it follows by eliminating the time derivatives of ϱ_α, v_i, and T

$$f_\alpha^{(0)}\left\{\beta_\alpha \xi_k^\alpha\left[\frac{1}{\varrho_\alpha}\frac{\partial p_\alpha}{\partial x_k} - \frac{1}{\varrho}\frac{\partial p}{\partial x_k} + \left(\sum_{\beta=1}^{E}\frac{\varrho_\beta e_\beta}{\varrho m_\beta} - \frac{e_\alpha}{m_\alpha}\right)\mathcal{E}_k\right] + \frac{1}{T}\frac{\partial T}{\partial x_k}\xi_k^\alpha\left(\frac{\beta_\alpha \xi_\alpha^2}{2} - \frac{5}{2}\right)\right.$$

$$+\beta_\alpha \xi_k^\alpha \xi_l^\alpha \frac{\partial v_{\langle k}}{\partial x_{l\rangle}}\Bigg\} + \frac{\beta_\alpha}{\varrho_\alpha}\xi_k^\alpha\left[\frac{\varrho_\alpha}{\varrho}\sum_{\beta=1}^{E}\frac{e_\beta}{m_\beta}(\mathbf{J}^\beta \times \mathbf{B})_k - \frac{e_\alpha}{m_\alpha}(\mathbf{J}^\alpha \times \mathbf{B})_k\right] + \frac{e_\alpha}{m_\alpha}(\xi^\alpha \times \mathbf{B})_k \frac{\beta_\alpha^2}{\varrho_\alpha}\left[\xi_l^\alpha p^\alpha_{\langle kl\rangle}\right.$$

$$\left.\left. + \frac{2}{5}q_k^\alpha\left(\frac{\beta_\alpha \xi_\alpha^2}{2} - \frac{5}{2}\right)\right]\right\} = \sum_{\beta=1}^{E}\left\{I_{\alpha\beta}\left[\xi_k^\alpha \xi_l^\alpha\right]\frac{\beta_\alpha^2}{2\varrho_\alpha}p^\alpha_{\langle kl\rangle} + I_{\alpha\beta}\left[\xi_k^\beta \xi_l^\beta\right]\frac{\beta_\beta^2}{2\varrho_\beta}p^\beta_{\langle kl\rangle} + I_{\alpha\beta}\left[\xi_k^\alpha\right]\frac{\beta_\alpha}{\varrho_\alpha}J_k^\alpha\right.$$

$$\left. + I_{\alpha\beta}\left[\xi_k^\beta\right]\frac{\beta_\beta}{\varrho_\beta}J_k^\beta + \frac{2}{5}I_{\alpha\beta}\left[\left(\frac{\beta_\alpha \xi_\alpha^2}{2} - \frac{5}{2}\right)\xi_k^\alpha\right]\frac{\beta_\alpha^2}{\varrho_\alpha}q_k^\alpha + \frac{2}{5}I_{\alpha\beta}\left[\left(\frac{\beta_\beta \xi_\beta^2}{2} - \frac{5}{2}\right)\xi_k^\beta\right]\frac{\beta_\beta^2}{\varrho_\beta}q_k^\beta\right\}. \tag{9}$$

where $\mathcal{E}_i$ and $I_{\alpha\beta}[\phi_\alpha]$ are given by

$$\mathcal{E}_i = E_i + (\mathbf{v} \times \mathbf{B})_i, \qquad I_{\alpha\beta}[o_\alpha] = \int f_\alpha^{(0)} f_\beta^{(0)} (o'_\alpha - o_\alpha)\, g^{\beta\alpha} b\, db\, d\varepsilon\, d\mathbf{c}_\beta, \tag{10}$$

and p is the pressure of the mixture

$$p = \sum_{\alpha=1}^{E} p_\alpha = \sum_{\alpha=1}^{E} \varrho_\alpha \frac{k}{m_\alpha} T \tag{11}$$

The multiplication of Eq. (9) by $\frac{\beta_\alpha}{\varrho_\alpha}\xi_i$ and integration over all values of ξ_α leads to

$$-\left[\frac{1}{\varrho_\alpha}\frac{\partial p_\alpha}{\partial x_k} - \frac{1}{\varrho}\frac{\partial p}{\partial x_k} + \left(\sum_{\beta=1}^{E}\frac{\varrho_\beta e_\beta}{\varrho m_\beta} - \frac{e_\alpha}{m_\alpha}\right)\mathcal{E}_k\right] - \left[\frac{1}{\varrho}\sum_{\beta=1}^{E}\frac{e_\beta}{m_\beta}(\mathbf{J}^\beta \times \mathbf{B})_i - \frac{e_\alpha}{\varrho_\alpha m_\alpha}(\mathbf{J}^\alpha \times \mathbf{B})_i\right]$$

$$= \sum_{\beta=1}^{E} A_{\alpha\beta} J_i^\beta - \sum_{\beta=1}^{E} \beta_\beta F_{\alpha\beta} q_i^\beta. \tag{12}$$

Also, the multiplication of Eq. (9) by $\frac{\beta_\alpha^2}{\varrho_\alpha}\left(\frac{\beta_\alpha \xi_\alpha^2}{2} - \frac{5}{2}\right)\xi_i$ and integration over all values of ξ_α leads to

$$\frac{5}{2T}\frac{\partial T}{\partial x_i} - \frac{e_\alpha \beta_\alpha^2}{m_\alpha \varrho_\alpha}(\mathbf{q}^\alpha \times \mathbf{B})_i = \frac{5}{2}\sum_{\beta=1}^{E} \beta_\alpha F_{\alpha\beta} J_i^\beta - \sum_{\beta=1}^{E} H_{\alpha\beta} q_i^\beta. \tag{13}$$

In Eqs. (12) and (13) $A_{\alpha\beta}$, $F_{\alpha\beta}$, and $H_{\alpha\beta}$ are collision integrals and their expressions are omitted here. If we know these integrals the constitutive equations for the heat flux q_i^α and for the diffusion flux J_i^α of constituent α in the mixture can be obtained from the system of equations (12) and (13). On the other hand, the constitutive equation for the heat flux q_i and for the total electric current J_i of the mixture in a linearized theory follow from

$$q_i = \sum_{\alpha=1}^{E}\left(q_i^\alpha + \frac{5}{2}\frac{k}{m_\alpha} T J_i^\alpha\right), \qquad J_i = \sum_{\alpha=1}^{E} \frac{e_\alpha}{m_\alpha} J_i^\alpha. \tag{14}$$

Hence, it follows from Eqs. (12)-(14)

$$q_i = -K_{ij}\frac{\partial T}{\partial x_j} - Q_{ij}^q\left[\frac{m_E m_l}{e_E m_l - e_l m_E} T \frac{\partial}{\partial x_j}\left(\frac{\mu_E - \mu_l}{T}\right) - \mathcal{E}_j\right]. \tag{15}$$

$$J_i = -\sigma_{ij}\left[\frac{m_E m_l}{e_E m_l - e_l m_E} T \frac{\partial}{\partial x_j}\left(\frac{\mu_E - \mu_l}{T}\right) - \mathcal{E}_j\right] - Q_{ij}^J \frac{\partial T}{\partial x_j}. \tag{16}$$

In the above equations $\mu_\alpha = T\left(\frac{5}{2}\frac{k}{m_\alpha} - \eta_\alpha\right)$ is the chemical potential, and η_α the specific entropy of constituent α. Equation (15) represents the law of Fourier, while (16) the law of Ohm. K_{ij} is the thermal conductivity tensor, σ_{ij} the electric conductivity tensor, and Q_{ij}^q, Q_{ij}^J denote coefficients of cross effects. These coefficients depend on the integrals $A_{\alpha\beta}$, $F_{\alpha\beta}$, and $H_{\alpha\beta}$.

Likewise the multiplication of Eq. (9) by $\beta_\alpha \xi_{\langle i}^\alpha \xi_{j\rangle}^\alpha$ and integration over all values of ξ_α leads to

$$-2\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} - \frac{c_\alpha}{\varrho_\alpha kT}(\varepsilon_{riq}B_q p^\alpha_{\langle rj\rangle} + \varepsilon_{rjq}B_q p^\alpha_{\langle ri\rangle}) = \sum_{\beta=1}^{E} L_{\alpha\beta}p^\beta_{\langle ij\rangle}, \tag{17}$$

where $L_{\alpha\beta}$ denote collision integrals. Once the integrals $L_{\alpha\beta}$ are known, one can obtain from Eq. (17) the constitutive equation for the pressure deviator $p^\alpha_{\langle ij\rangle}$ of constituent α. Hence, it follows the constitutive equation for the pressure deviator of the mixture $p_{\langle ij\rangle}$ since in a linearized theory it is given by

$$p_{\langle ij\rangle} = \sum_{\alpha=1}^{E} p^\alpha_{\langle ij\rangle}. \tag{18}$$

The final expression is

$$p_{\langle ij\rangle} = -2\eta_{\langle ij\rangle\langle kl\rangle}\frac{\partial v_{\langle k}}{\partial x_{l\rangle}} \tag{19}$$

Equation (19) is the mathematical expression of the Navier-Stokes law with $\eta_{\langle ij\rangle\langle kl\rangle}$ denoting the shear viscosity tensor. The shear viscosity tensor is a function of the collision integrals $L_{\alpha\beta}$.

Acknowledgements

One of the authors (G. M. K.) gratefully acknowledges the support of the Conselho Nacional de Desenvolvimento Científico e Tecnológico of Brazil.

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