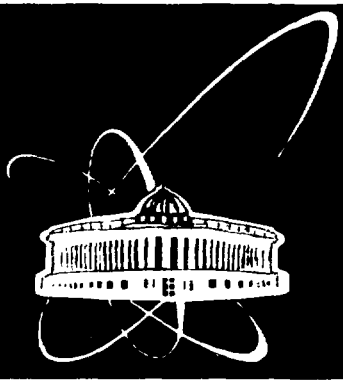

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ANHARMONIC POTENTIAL IN THE OSCILLATOR REPRESENTATION

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1 Introduction

One of the basic problems of nonrelativistic quantum mechanics is to find the energy spectrum and eigenfunctions of a microsystem described by the Schrödinger equation with an appropriate potential. Exact solutions of this equation have been found ([1]-[4]) for a quite limited class of potentials like the harmonic oscillator, the Coulomb potential and some others. However, most quantum systems are described by potentials for which the Schrödinger equation cannot be solved analytically. Thus, the solution of the Schrödinger equation with a sufficiently arbitrary potential represents the main mathematical task. For this aim, many approximate analytical and numerical methods were worked out. The great progress in the development of computer technique and effective algorithms of numerical solutions of differential equations permit one to obtain numerical solutions for the energy spectrum and wave functions with a quite high accuracy although practical calculations are usually very laborious and require powerful computers.

Approximate analytical methods imply a perturbative procedure when the Hamiltonian is divided into two parts $H = H_0 + H_I$, the solution of the zeroth approximation $H_0\Psi^{(0)} = E^{(0)}\Psi^{(0)}$ is supposed to be obtained and perturbation corrections to the zeroth approximation $E^{(0)}$ and $\Psi^{(0)}$ can be calculated. The physical and mathematical top point is that the Hamiltonian H_0 in an appropriate representation of the Schrödinger equation should be chosen in such a way as to catch the main dynamic properties of a quantum system and to give a possibility to calculate analytically all physical characteristics of the system under consideration. The interaction Hamiltonian H_I should give small corrections to the zeroth approximation and these corrections can be calculated.

Here we mention the standard perturbation Reley-Schrödinger theory ([1]-[4]), the quasiclassical or WKB method ([1]-[4]), $1/N$ -expansion([5], [6]). We will not go into details of these methods and refer readers to the numerous literature (see, for example, [1]-[6]).

In this paper the oscillator representation method ([7],[8]) will be applied to Quantum Mechanics problems. The most remarkable difference between Quantum Field Theory and Quantum Mechanics is that quantized fields in QFT are set of oscillators and any interactions of fields do not change the oscillator nature of these quantized fields. At the same time, in Quantum Mechanics most of the potentials and therefore their corresponding wave functions are quite far from the oscillator behaviour. The application of the oscillator repesentation method implies that a wave function, being a bound ground state of a quantum system with an attractive potential, is expanded over the oscillator basis in the representation in which canonical variables (coordinate and momenta) are expressed through the creation and annihilation operators a^+ and a. However in most cases the asymptotic behaviour of a true wave function (for example, the Couloumb wave functions) for large distances does not coincide with the Gaussian asymptotic behaviour of oscillator wave functions. This means that the expansion of these wave functions over the oscillator basis, although being mathematically correct, leads to series converging not sufficiently fast for practical purposes.

Therefore, before applying the oscillator representation method we have to modify the variables in the starting Schrödinger equation to get a modified equation having solutions with the Gaussian asymptotic behaviour. In the Coulomb systems such a modification

is performed by going over to the four-dimensional space where the wave function of the Coulomb system becomes an oscillator one. In an early paper [9], Schrödinger has noted the existence of such a transformation which transforms the three-dimensional Coulomb system into an oscillator one in the four-dimensional space. Kustaanheimo and Stiefel [10] gave the explicit form of this transformation and used it to solve the classical Kepler problem.

It should be taken into account that these transformations are not the canonical ones. It means that a quantum system after a la Kustaanheimo-Stiefel transformation becomes another quantum system with another set of quantum numbers and corresponding wave functions. However this new set contains a subset of wave functions which are wave functions of the initial system at the same time, and we should be able to pick out necessary quantum numbers and wave functions. Therefore, these transformations should be considered as a successful mathematical technical method.

So we get a modified Schrödinger equation the eigenfunctions of which have the oscillator Gaussian asymptotical behaviour. The next step is to write the Hamiltonian in terms of normal products over the creation and annihilation operators a^+ and a. Now one faces the question what is the optimal way to determine the frequency of this oscillator. In the language of the Hamilton formalism the problem of calculation of the ground-state energy can be formulated in the following manner. Let the Hamiltonian of a system be given. Let us pick out the pure oscillator part with some unknown frequency ω and write this Hamiltonian in the form $H_0 = \omega a^+ a$. The rest of the Hamiltonian should be represented in terms of normal products over the operators a^+ and a. The requirement that this interaction Hamiltonian does not contain quadratic in the canonical variables terms leads to the equation which determines the oscillator frequency ω. This requirement is called the *oscillator representation condition* (ORC). As a result, the total Hamiltonian is written in the representation where the main quantum contributions to the ground state or vacuum of the system are taken into account.

The conception of normal products introduced into nonrelativistic quantum mechanics is actually not new (see,for example, [11]); however, the question is on what principles the realization of this idea should be based. All approaches which have used the formalism of creation and annihilation operators imply that the ground state wave function belongs to the oscillator basis although the true wave function can have completely different asymptotic behaviour. Besides, the "free" Hamiltonians H_0, for which the exact oscillator solutions exist, are usually chosen in the form which destroys completely the canonical quantum structure of the Hamiltonian (see, for example,[5]-[11]).

The paper is organized in the following way: In section 2, basic formulas of the oscillator representation method are given. In section 3, we consider the one- and three-dimensional anharmonic potentials and power-low as well as logarithmic potentials. The energy levels for the ground and orbitial as well as radial excited states are calculated. The results of our calculations in the zeroth approximation agree with the exact values very well. In section 4, the relativized Schrödinger equation is considered.

2 The Oscillator Representation

The method is based on the ideas and methods of quantum theory of scalar fields and consists in the following. In nonrelativistic quantum mechanics any ground state of a bound system can be approximated by a wave function of an oscillator. The question arises how to choose this oscillator in the optimal form. For this aim we take advantage of the ideas of quantum field theory developed in the papers [7],[8] .

Our starting point is the radial Schrödinger equation in 3 dimensions:

$$\Big[-\frac{1}{2r}\Big(\frac{d}{dr}\Big)^2 r+\frac{l(l+1)}{2r^2}+V(r)\Big]\psi_{nl}(r)=E_{nl}\psi_{nl}(r)\ . \tag{2.1}$$

We shall consider the potentials of the Coulomb or Yukawa type, i.e. the potentials decreasing for large distances

$$V(r)\to 0 \qquad \text{for} \qquad r\to\infty, \tag{2.2}$$

and the anharmonic potentials of the confinement type for which

$$V(r)\to r^{2\sigma}, \qquad (\sigma>0) \quad \text{for} \quad r\to\infty. \tag{2.3}$$

These potentials can have a repulsive region at short distances.

Our aim is to calculate the energy spectrum E_{nl} and to find the wave functions $\psi_{nl}(r)$ by using the oscillator representation method. This means that the wave functions $\psi_{nl}(r)$ should be expanded over the oscillator basis. This expansion can be done, but it will not be effective for the simple reason that the asymptotic behaviours of the true and oscillator wave functions disagree for large $r\to\infty$ and short $r\to 0$ distances. Thus we *cannot* apply the oscillator representation method directly, but we have to transform the Schrödinger equation (2.1) in such a way that the true wave function should become the necessary asymptotic behaviour for large and small distances. For this aim, we will use the well-known technique of changing the independent coordinate (see, for example, [12], [13]) which was applied to show the equivalence between solutions for different power-low potentials in the spaces of different dimensions. For example, there exists equivalence between the Coulomb potential in 3 dimensions and the oscillator potential in 4 dimensions (see [10], [13]).

Our idea consists in the following. We want to change the variable $r=r(s)$ and identify the transformed equation with a Schrödinger equation in the space with another dimension. The transition in the radial Schrödinger equation to the highest dimensions from the general point of view has been considered earlier (see, for example, [14]). Thus, the calculation of the function $\psi_{0l}(r)$ would be equivalent to the calculation of the ground state wave function of a modified Hamiltonian in another dimension. Moreover, the wave functions in this auxiliary space should have the oscillator Gaussian asymptotic behaviour. The radial excitation wave functions $\psi_{nl}(r)=|n_r\rangle$ will be equivalent to the highest oscillator states.

The Schrödinger equation (2.1) can be written in the form:

$$\int d^3\mathbf{r}\Psi(\mathbf{r})\Big[-\frac{1}{2}\Delta+V(r)-E\Big]\Psi(\mathbf{r})=0\ .$$

If $\Psi(\mathbf{r}) = \psi_{nl}(r)Y_{nl}(\theta,\phi)$, then this equation for the wave function of the l-th orbital excitation looks as

$$\int\limits_0^\infty dr\Big(r\psi_{nl}(r)\Big)\Big[-\frac{1}{2}\Big(\frac{d}{dr}\Big)^2+\frac{l(l+1)}{2r^2}+\Big(V(r)-E\Big)\Big]\Big(r\psi_{nl}(r)\Big)=0\,.$$

The wave function $\psi_{nl}(r)$ depends on one radial variable r only. After the substitutions

$$r=s^{2\rho}\qquad\text{and}\qquad r\psi_{nl}(r)=s^a\Phi(s), \tag{2.4}$$

where ρ and a are parameters, this equation becomes after some transformations

$$\int\limits_0^\infty dss^{D-1}\Phi(s)\Big[-\frac{1}{2}\Big(\Big(\frac{d}{ds}\Big)^2+\frac{D-1}{s}\cdot\frac{d}{ds}\Big)+W_I(s^2,E)\Big]\Phi(s)=0\,, \tag{2.5}$$

with

$$W_I(s^2,E)=\bar{W}(s^2;l,\rho,D;E)=-\frac{K(l,\rho,D)}{2s^2}+2\rho^2s^{4\rho-2}\Big(V(s^{2\rho})-E\Big)\,,$$
$$D=2a-2\rho+2\,,\qquad K(l,\rho,D)=\frac{1}{4}\Big((D-2)^2-4\rho^2(2l+1)^2\Big)\,.$$

One can see that in the case when the function $\Phi(\mathbf{s}) = \Phi(s)$ depends on s^2, only this equation can be identified with the equation in the space R^D with

$$D=2a-2\rho+2$$

on a wave function $\Phi(s)$ depending on the radius s only. The equation (2.5) can be rewritten

$$\int d^D\mathbf{s}\Phi(s)\Big[-\frac{1}{2}\Delta_D+W_I(s^2,E)-\varepsilon(E)\Big]\Phi(s)=0,$$

where the function

$$\varepsilon(E)=\varepsilon(l,\rho,D;E)$$

should be considered as an eigenvalue of the Schrödinger equation in D-dimensions

$$\Big[-\frac{1}{2}\Delta_D+W_I(s^2,E)\Big]\Phi(s)=\varepsilon(E)\Phi(s). \tag{2.6}$$

The desired energy E is defined by the equation

$$\varepsilon(E)=\varepsilon(l,\rho,D;E)=0. \tag{2.7}$$

Formulation of the problem. We would like to stress that the energy E enters into the Schrödinger equation (2.7) as a parameter. Thus, our problem is formulated in the following way. We have the Hamiltonian H in D dimensions

$$H(E)=-\frac{1}{2}\Delta_D+W_I(s^2,E)=\frac{p^2}{2}+W_I(s^2,E), \tag{2.8}$$

and we have to solve the Schrödinger equation

$$H(E)\Phi = \left[\frac{p^2}{2} + W_I(s^2, E)\right]\Phi = \varepsilon(E)\Phi \tag{2.9}$$

for radial excitations only. The desired energy spectrum $E_{n\ell}$ of the initial problem in (2.1) is contained in the radial excitation spectrum $\varepsilon^{[n]}$ of the Hamiltonian (2.8):

$$H(E)\Phi^{[n]}(s) = \varepsilon^{[n]}(E)\Phi^{[n]}(s)\,, \qquad (n = 0, 1, 2, ...), \tag{2.10}$$

and it is defined by the equation

$$\varepsilon^{[n]}(E) = 0. \tag{2.11}$$

We shall solve equation (2.10) by the oscillator representation method. The Hamiltonian (2.8) can be represented in the form (for details see [8])

$$H = H_0(E) + H_I(E) + \varepsilon_0(E) = H_0 + H_I + \varepsilon_0\,, \tag{2.12}$$

with

$$\begin{aligned} H_0 &= \omega(a^+a)\,, \\ H_I &= \int \left(\frac{dk}{2\pi}\right)^D \widetilde{W}_I(k^2, E) c^{-k^2/(4\omega)} : c_2^{ikq} :\,, \\ \varepsilon_0(E;\omega, D) &= \frac{D\omega}{4} + \int_0^\infty \frac{du u^{D/2-1} e^{-u}}{\Gamma(D/2)} \cdot W_I\left(\frac{u}{\omega}, E\right), \end{aligned}$$

where $: * :$ is the symbol of the normal ordering and $c_2^z = e^z - 1 - z - \frac{1}{2}z^2$.

The condition of the oscillator representation can be written as

$$\frac{\partial}{\partial\omega}\varepsilon_0(E;\omega, D) = 0\,. \tag{2.13}$$

This equation determines the parameter

$$\omega = \omega(E, D)\,,$$

as a function of the energy E, D and other parameters defining the potentail $V(r)$ in (2.1).

The ground state energy $\varepsilon(E, D)$ of the Hamiltonian H in (2.12) will be calculated by perturbation method over the interaction Hamiltonian H_I and in the N-th approximation order has the form:

$$\varepsilon_{(N)}(E, D) = \varepsilon_0(E, D) + \varepsilon_2(E, D) + ... + \varepsilon_N(E, D)\,.$$

According to (2.11), the ground state energy E of the initial problem in the N-th perturbation order of the oscillator representation method is defined by the equations (2.13) and

$$\varepsilon_{(N)}(E, D) = \varepsilon_0(E, D) + \varepsilon_2(E, D) + ... + \varepsilon_N(E, D) = 0\,. \tag{2.14}$$

This equation determines the energy $E_{(N)}(D)$ in the N-th perturbation order as a function of D and other parameters defining the potential. The parameter D can be defined by the condition

$$E_{(N)} = \min_{\{D\}} E_{(N)}(D) \ . \tag{2.15}$$

The ground state energy in the zeroth and second approximations. Really, we shall use the oscillator representation method in the zeroth and second approximations only. Here we give formulas simplifying these calculations for the ground state energy. The function $\varepsilon(E;\omega,D)$ in (2.12) depends on two parameters D and ω. We shall consider problems for solution of which a greater number of auxiliary parameters can be introduced. Let us denote these parameters by $\{\alpha_j\}$ and the auxiliary ground state energy

$$\varepsilon_{(2)}(E;\omega,\alpha_j) = \varepsilon_0(E;\omega,\alpha_j) + \varepsilon_2(E;\omega,\alpha_j) \ .$$

According to (2.12) the function $\varepsilon_0(E;\omega,\alpha_j)$ has the form

$$\varepsilon_0(E;\omega,\alpha_j) = A(\omega,\alpha_j) - EB(\omega,\alpha_j), \tag{2.16}$$

where $A(\omega,\alpha_j)$ and $B(\omega,\alpha_j)$ are known functions.

In the zeroth approximation, the equation defining the oscillator representation (2.13) and the equation (2.14)

$$\varepsilon_0(E_0;\omega,\alpha_j) = A(\omega,\alpha_j) - E_0B(\omega,\alpha_j) = 0$$

give

$$\frac{\partial}{\partial\omega}\left(\frac{A(\omega,\alpha_j)}{B(\omega,\alpha_j)}\right) = 0 \ .$$

The equations

$$\frac{\partial}{\partial\alpha_k}E_0(\omega,\alpha_j) = \frac{\partial}{\partial\alpha_k}\left(\frac{A(\omega,\alpha_j)}{B(\omega,\alpha_j)}\right) = 0 \qquad \text{for all } \ k$$

define the parameters $\{\alpha_j\}$ as functions of E_0

$$\alpha_j = \alpha_j(E). \tag{2.17}$$

As a result, in the zeroth approximation the energy E_0 is determined by the minimum

$$E_0 = \min_{\{\omega,\alpha_j\}} \frac{A(\omega,\alpha_j)}{B(\omega,\alpha_j)} = \frac{A(\omega_0,\alpha_j^0)}{B(\omega_0,\alpha_j^0)} \tag{2.18}$$

where the parameters ω_0 and α_j^0 define the minimum.

In the second approximation, we have to solve equation (2.13) and

$$\varepsilon_{(2)}(E;\omega,\alpha_j) = \varepsilon_0(E;\omega,\alpha_j) + \varepsilon_2(E;\omega,\alpha_j) = 0 \ . \tag{2.19}$$

We expect that the second correction is small so that in the second approximation the energy $E_{(2)} = E_0 + E_2$ and

$$\varepsilon_0(E_{(2)};\omega_0,\alpha_j) = A(\omega_0,\alpha_j) - E_{(2)}B(\omega_0,\alpha_j) = -E_2B(\alpha_j) + O(E_2^2)$$

where $\omega_0 = \omega(E_0)$. Thus the second correction is

$$E_2 = \frac{\varepsilon_2(E_0;\omega_0,\alpha_j)}{B(\omega_0,\alpha_j)} + O(E_2^2).$$

Finally we get

$$\begin{aligned} E_{(2)} &= E_0 + E_2 = \\ &= \min_{\{\omega,\alpha_j\}} \frac{A(\omega,\alpha_j)}{B(\omega,\alpha_j)} + \frac{\varepsilon_2(E_0;\omega_0,\alpha_j(E_0))}{B(\omega_0,\alpha_j(E_0))} + E_0O\left(\left|\frac{E_2}{E_0}\right|^2\right) \\ &= \frac{A(\omega_0,\alpha_j^0) + \varepsilon_2(E_0;\omega_0,\alpha_j^0)}{B(\omega_0,\alpha_j^0)} + E_0O\left(\left|\frac{E_2}{E_0}\right|^2\right) . \end{aligned} \tag{2.20}$$

Using this formula, calculations become simpler in comparison with (2.15) and (2.17).

The accuracy of the oscillator representation can be evaluated as

$$\delta \sim \left|\frac{\varepsilon_2}{\varepsilon_0}\right| .$$

The radial excitations. The radial excitations in the oscillator representation are defined in the form

$$|n_r\rangle = C_{n_r}(a_j^+a_j^+)^{n_r}|0\rangle , \qquad C_{n_r}^{-2} = 2^{2n_r}n_r!\frac{\Gamma(d/2+n_r)}{\Gamma(d/2)}. \tag{2.21}$$

We shall apply the oscillator representation to the Hamiltonian in the form (2.12), and then we get the Schrödinger equation (2.10) . The desired energies E_n $(n = 0, 1, ...)$ of the initial equation (2.1) for the ground and radial excited states are defined by equation (2.11), and therefore, we should find the functions $\varepsilon^{[n]}(E)$ for the ground and radial excited states. For the state $|n\rangle$ $(n = 0, 1, ...)$ the matrix element

$$\langle n|H_I|n\rangle = A^{[n]}(\omega,\alpha_j) - E_0B^{[n]}(\omega,\alpha_j) \neq 0 .$$

The energy $\varepsilon^{[n]}$ in the lowest approximation looks like

$$\begin{aligned} \varepsilon_1^{[n]}(E) &= \langle n|H|n\rangle = \varepsilon_0(E) + 2n\omega + \langle n|H_I|n\rangle \\ &= A_1^{(n)}(\omega,\alpha_j) - EB_1^{(n)}(\omega,\alpha_j) , \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} A_1^{(n)}(\omega,\alpha_j) &= A(\omega,\alpha_j) + 2n\omega + A^{[n]}(\omega,\alpha_j) , \\ B_1^{(n)}(\omega,\alpha_j) &= B(\omega,\alpha_j) + B^{[n]}(\omega,\alpha_j) . \end{aligned}$$

Two equations

$$\frac{\partial}{\partial\omega}A(\omega,\alpha_j) - E\frac{\partial}{\partial\omega}B(\omega,\alpha_j) = 0\ , \qquad (2.23)$$
$$A_1^{(n)}(\omega,\alpha_j) - EB_1^{(n)}(\omega,\alpha_j) = 0\ ,$$

determine the functions $\omega(\alpha_j)$ and $E(\alpha_j)$. The energy of the n-th excited state in the first approximation of the oscillator representation is determined as

$$E_1^{[n]} = \min_{\{\alpha_j\}} \frac{A_1^{[n]}(\omega(\alpha_j),\alpha_j)}{B_1^{[n]}(\omega(\alpha_j),\alpha_j)}.$$

In the second approximation the energy is defined as

$$\varepsilon_{(2)}^{[n]}(E) = \varepsilon_0(E) + 2n\omega(E) + \langle n|H_I|n\rangle \qquad (2.24)$$
$$-\langle n|\big(H_I - \langle n|H_I|n\rangle\big)\cdot\frac{1}{H_0 - 2n\omega(E)}\cdot\big(H_I - \langle n|H_I|n\rangle\big)|n\rangle\ .$$

The wave function in the second approximation is

$$\Phi_2^{[n]} = \{1 - \frac{1}{H_0 - 2n\omega(E)}\cdot\big(H_I - \langle n|H_I|n\rangle\big)\}|n\rangle\ .$$

The parameter D and the oscillator basis. Here we want to make a remark concerning the connection of the space R^D, for which D can be a noninteger, and the algebra of the creation and annihilation operators implying the number D to be an integer.

The parameters ρ and D are arbitrary and can be chosen in an appropriate way. They can be considered as additional variational parameters which can be found, for example, by the minimization of the energy in the zeroth approximation:

$$\varepsilon_0(E) = \min_{\{\rho,D\}} \varepsilon(l,\rho,D;E).$$

On the other hand, the parameter ρ can be connected with the behaviour of the wave function $\psi_{nl}(r)$, at large distances to get the Gaussian asymptotics. For example, for potentials (2.3) we can choose the parameter $\rho = 1/(1+\sigma)$ so that one can get

$$\Psi(r) \sim \exp(-r^{1+\sigma}) \sim \exp(-r^{1/\rho}) \sim \exp(-s^2) \sim \Phi(s).$$

The parameter a or D can be connected with the behaviour of the wave function at short distancies. If the potential $V(r)$ has no repulsive character for $r \to 0$, then we choose $K(l,\rho,D) = 0$ and from (2.6) we get

$$D = 2 + 2\rho(2l+1).$$

If the potential $V(r)$ has a repulsive character for $r \to 0$, then D is a parameter which should be chosen to dump the repulsive behaviour of the potential at short distances. For example, it can be found by the minimization of the background energy in the zeroth

approximation. It implies that D can be any positive number. In other words, the dimension D of the space R^D can be considered as an additional parameter which can be chosen to improve the zeroth approximation.

One can see that the radial quantum number n does not enter into the Schrödinger equation (2.9) in the explicit form. The orbital quantum number l enters into (2.9), but it is absorbed by the "dimension" parameter D. From the point of view of the space R^D the functions

$$\Phi_n(s) = s^{2\rho-a}\psi_{nl}(s^{2\rho}) \qquad \text{or} \qquad \psi_{nl}(r) = r^{\frac{D-2\rho-2}{4\rho}}\Phi_n(r^{1/2\rho})$$

for any n and for a fixed l are eigenfunctions of the basic series of radial excitations in the space R^D *with the radial quantum number* n *and the zeroth orbital momenta.*

Thus, the solution of the equation in 3-dimensions for the l-th orbital excitation is equivalent to the solution of the Schrödinger equation in the space R^D for states with zeroth angular momenta.

As a result, the initial Schrödinger equation is represented in the form (2.9) in which the wave function of the ground state $\Phi(s)$ has

- the Gaussian asymptotics for large distances $\Phi(s) \sim \exp(-s^2)$,
- maximum at the point $s = 0$.

The oscillator representation method consists in that we look for the solution of the Schrödinger equation (2.9) in the form

$$\Phi_n(s) = \exp(-\frac{\omega}{2}s^2)\sum_m c_{nm}P_m^{(D)}(s^2\omega)$$

where $\{P_m^{(D)}(t)\}$ is the class of orthogonal polynomials which are orthogonal on the interval $0 < t < \infty$ with the weight function

$$\rho_D(t) = t^{\frac{D}{2}-1}\exp(-t),$$

i.e.,

$$\int_0^\infty dt\; t^{\frac{D}{2}-1}e^{-t}P_n^{(D)}(t)P_m^{(D)}(t) = \delta_{nm}\ .$$

These orthogonal polynomials can be constructed by using the formalism of creation and annihilation operators a_j and a_j^+ in the space R^D (see [8]). We have

$$|0\rangle \sim \exp\Big(-\frac{\omega}{2}s^2\Big).$$

All radial excitations can be written in the form

$$\Phi_n \sim (a^+a^+)^n|0\rangle \sim P_n^{(D)}(\omega s^2)e^{-\frac{\omega}{2}s^2} \sim P_n^{(D)}(\omega r^{1/\rho})\exp(-\frac{\omega}{2}r^{1/\rho}), \qquad (2.25)$$

where $P_n^{(D)}(t)$ is a polynomial of the n-th order. The parameter D in this representation can be considered as any positive number. These polynomials satisfy the orthogonal condition:

$$\left(\Psi_n, \Psi_m\right) \sim \langle 0|(aa)^n(a^+a^+)^m|0\rangle \sim \int_0^\infty ds\, s^{D-1} \exp(-s^2) P_n^{(D)}(s^2) P_m^{(D)}(s^2)$$

$$\sim \int_0^\infty dt\, t^{D/2-1} \exp(-t) P_n^{(D)}(t) P_m^{(D)}(t) \sim \delta_{nm} .$$

This condition can be imposed on any positive D. Thus the algebra of creation and annihilation operators is nothing else than the mathematical method to perform any calculations connected with orthonormal polynomials.

3 Anharmonic Potentials

Anharmonic oscillator models have played an important role in the evolution of many branches of quantum physics. In spite of seeming simplicity it is not easy problem to find spectrum and eigenfunctions of an anharmonic interaction. There is a voluminous literature where different analytical and numerical methods are worked out to solve this problem (see, for example,[15]-[21]). On the other hand, the anharmonic potential is a good touchstone to test any new method.

Bender and Wu [16] have made a valuable contribution to investigation of the anharmonic oscillator, which is of particular interest to field theoreticians because it can be regarded as a field theory in one dimension. The main hope is that the unusual and unexpected properties of this nonlinear model may give some indication of the analytical structure of more realistic field theory. Nevertheless the developed technique turned out to be quite complicated even for this simple case.

The standard way of attacking this problem is to invoke perturbation theory. Perturbation series for any physical characteristics are asymptotical ones, i.e., they have the zeroth radius of covergence. Summation methods should be applied to calculate high order corrections. As a result, we have quite a cumbersome process. A thorough discussion of these difficulties has been done by Stevenson [17].

The quasiclassical approximation was applied to the three-dimensional anharmonic oscillator [18]. The problem of calculation of the energy levels is reduced to the solution of a very cumbersome transcendental equation invoking the complete elliptic integrals. However, its accuracy drastically worsens for the low lying energy levels and moderate anharmonicity.

Another known approach for treating systems with strong interaction is a modified perturbation theory [19]. The accuracy of the modified perturbation theory with the principle of minimal sensitivity has been carefully analyzed [20] for the anharmonic oscillator.

The $1/N$ expansion for the anharmonic oscillator was used in Refs.[21]. In [6] the $1/N$ expansion was applied to calculate the spectrum of the anharmonic oscillator.

In this section we would like to contribute to these numerous investigations. We shall demonstrate the oscillator representation method to calculate the bound state energies of anharmonic oscillators.

3.1 Anharmonic potentials in R^1

Here we demonstrate the oscillator representation method on calculation of the bound state energy of one-dimension anharmonic oscillator. The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{m\nu^2}{2}q^2 + \lambda q^4. \tag{3.1}$$

In the case of symmetrical potentials $V(q^2)$ the ground state wave function depends on q^2 only, i.e.,

$$\Psi = \Psi(q^2), \qquad \Psi'(0) = 0\ ,$$

so that we can write

$$\int_{-\infty}^{\infty} dq\ \Psi(q^2)\Big[-\frac{1}{2}\cdot\frac{d^2}{dq^2} + V(q^2) - mE\Big]\Psi(q^2) = 0$$

or

$$\int_0^{\infty} dq\ \Psi(q^2)\Big[-\frac{1}{2}\cdot\frac{d^2}{dq^2} + V(q^2) - mE\Big]\Psi(q^2) = 0, \tag{3.2}$$

$$V(q^2) = \frac{m^2\nu^2}{2}q^2 + m\lambda q^4.$$

Thus we can consider the wave equation on the positive semiaxis $0 \le q \le \infty$.

We are going to apply the oscillator representation to this Hamiltonian so that we should coordinate the Gaussian asymptotic behaviour of functions in OR with the true one. For large q this asymptotic is defined by the anharmonic term λq^4 and the wave function is proportional to

$$\Psi(q^2) \sim \exp(-q^3) \qquad \text{for} \quad q \to \infty.$$

However, it is clear that for small λ the true wave function is close more to the Gaussian wave function than to the anharmonic one. Thus, we can expect that the behaviour like

$$\Psi(q^2) \sim \exp(-q^\alpha) \qquad \text{for} \quad q \to \infty,$$

where $2 \le \alpha \le 3$ is a parameter, can be an acceptable approximation. Let us introduce the new variable

$$q = s^{2/\alpha} = s^{2\rho}, \qquad \rho = \frac{1}{\alpha}.$$

After some transformations the integral in (3.2) can be written as

$$\int_0^{\infty} ds s^{1-2\rho}\ \Psi(s)\left[-\frac{1}{2}\left[\frac{d^2}{ds^2} + \frac{1-2\rho}{s}\frac{d}{ds}\right] + W(s^2, E)\right]\Psi(s) = 0\ ,$$

where

$$W(s^2,E)=4\rho^2\left[\frac{m^2\nu^2}{2}\cdot(s^2)^{4\rho-1}+m\lambda(s^2)^{6\rho-1}-mE(s^2)^{2\rho-1}\right]. \tag{3.3}$$

Now we can identify the operator

$$\frac{d^2}{ds^2}+\frac{1-2\rho}{s}\frac{d}{ds}=\frac{d^2}{ds^2}+\frac{d-1}{s}\frac{d}{ds}\to\Delta_d,\qquad d=2-2\rho$$

and the measure

$$dss^{1-\rho}=dss^{d-1}\to(ds)^d$$

with "the Laplacian" Δ_d and "the measure" $(ds)^d$ in an auxiliary space R^d if these operators act on a function depending on the radius only. The relation (2.7) can be used and equation (3.3) looks as

$$\int(ds)^d\Psi(s)\Big[-\frac{1}{2}\Delta_d+W(s^2,E)\Big]\Psi(s)=0\,. \tag{3.4}$$

The wave function $\Psi(s)$ in (3.4) can be considered as a wave function of the ground state satisfying the Schrödinger equation

$$\begin{aligned}&[-\frac{1}{2}\Delta_d+W(s^2,E)]\Psi(s)=H\Psi(s)=\varepsilon(E)\Psi(s)\,,\\&H=\frac{p^2}{2}+W(s^2,E)\,,\end{aligned} \tag{3.5}$$

and the desired energy E is determined by the equation

$$\varepsilon(E)=0\,. \tag{3.6}$$

Now we can apply the oscillator representation method to the Hamiltonian (3.5). According to formula (2.6) the Schrödinger equation and the Hamiltonian (3.5) in the oscillator representation look as

$$\begin{aligned}&(H_0+H_I+\varepsilon_0)\Psi=\varepsilon(E)\Psi\,,\\&H=\frac{p^2}{2}+W(s^2,E)=H_0+H_I+\varepsilon_0\,,\end{aligned} \tag{3.7}$$

where H_0 and H_I are given by (2.11), and ε_0 (2.16) is

$$\varepsilon_0(E)=\min_{\{\omega,\rho\}}\varepsilon_0(E;\omega,\rho)\,,$$

with

$$\begin{aligned}\varepsilon_0(E;\omega,\rho)&=\frac{d\omega}{4}+\int_0^\infty\frac{du u^{\frac{d}{2}-1}e^{-u}}{\Gamma(\frac{d}{2})}W\Big(\frac{u}{\omega},E\Big)\\&=A(\omega,\rho)-EB(\omega,\rho)\,,\\A(\omega,\rho)&=\frac{1-\rho}{2}\omega+\frac{4\rho^2m\omega}{\Gamma(1-\rho)}\left[\frac{m\nu^2}{2x^2}\cdot\Gamma(3\rho)+\frac{\lambda}{x^3}\cdot\Gamma(5\rho)\right],\\B(\omega,\rho)&=\frac{4\rho^2m\omega}{x}\cdot\frac{\Gamma(\rho)}{\Gamma(1-\rho)}\,,\end{aligned}$$

where $x = \omega^{2\rho}$. The functions $\omega(E)$ and $\rho(E)$ are defined by the equations

$$\frac{\partial}{\partial\omega}\varepsilon_0(E,\omega,\rho) = 0. \qquad \frac{\partial}{\partial\rho}\varepsilon_0(E,\omega,\rho) = 0.$$

Let us calculate the second correction to the ground state energy. The first correction equals zero:

$$\varepsilon_1 = \langle 0|H_I|0\rangle = 0.$$

The second correction is defined by the standard formula

$$\varepsilon_2 = -\langle 0|H_I\frac{1}{H_0}H_I|0\rangle =$$

$$= -\frac{1}{\omega}\int\left(\frac{dk_1}{2\pi}\right)^d\int\left(\frac{dk_2}{2\pi}\right)^d\widetilde{W}_d(k_1^2)\widetilde{W}_d(k_2^2)\exp\Big(-\frac{k_1^2+k_2^2}{4\omega}\Big)S\Big(\frac{(k_1k_2)}{2\omega}\Big),$$

$$S(z) = \int_0^1\frac{dt}{t}\Big[\cosh(tz) - 1 - \frac{1}{2}z^2t^2\Big] = \sum_{n=2}^{\infty}\frac{z^{2n}}{2n(2n)!},$$

$$\varepsilon_2(E;\omega,\rho) = -\frac{8\rho^2m^2\omega}{\Gamma(1-\rho)}\sum_{n=2}^{\infty}\left(\frac{R_n}{n!}\right)^2\cdot\frac{\Gamma(n+1-\rho)}{\Gamma(n+2-2\rho)}, \tag{3.8}$$

where

$$R_n = \frac{m\nu^2}{6x^2}\cdot\frac{\Gamma(n+1-4\rho)\Gamma(1+3\rho)}{\Gamma(1-4\rho)} + \frac{\lambda}{5x^3}\cdot$$
$$\cdot\frac{\Gamma(n+1-6\rho)\Gamma(1+5\rho)}{\Gamma(1-6\rho)} - \frac{E}{x}\cdot\frac{\Gamma(n+1-2\rho)\Gamma(1+\rho)}{\Gamma(1-2\rho)}.$$

The ground state energy E in the zeroth perturbation order is defined by the equation (2.19)

$$E_0 = \min_{(\omega,\rho)}\frac{A(\omega,\rho)}{B(\omega,\rho)} = \frac{A(\omega_0,\rho_0)}{B(\omega_0,\rho_0)}, \tag{3.9}$$

where ω_0 and ρ_0 define the minimum. The energy in the second perturbation order equals

$$E_{(2)} = E_0 + E_2. \tag{3.10}$$

where

$$E_0 = \min_{(x,\rho)}\left[\frac{\Gamma(2-\rho)}{8m\rho\Gamma(1+\rho)}\cdot x + \frac{m\nu^2}{6}\cdot\frac{\Gamma(1+3\rho)}{\Gamma(1+\rho)}\cdot\frac{1}{x} + \frac{\lambda}{5}\cdot\frac{\Gamma(1+5\rho)}{\Gamma(1+\rho)}\cdot\frac{1}{x^2}\right].$$

$$E_2 = \frac{\varepsilon_2(E_0;\omega_0,\rho_0)}{B(\omega_0,\rho_0)} = -\frac{2m}{\Gamma(\rho)}\sum_{n=2}^{\infty}\left(\frac{R_n}{n!}\right)^2\cdot\frac{\Gamma(n+1-\rho)}{\Gamma(n+2-2\rho)}.$$

In Table 1 the numerical results for the background energy are given for the case $m = \frac{1}{2}$ and $m\nu = 1$ in the zeroth and second approximations.

Table 1. Results of calculation of the ground state energy of one-dimensional anharmonic oscillator for the case $m = \frac{1}{2}$ and $m\nu = 1$ as a function of the parameter λ. $E^{(0)}$ and $E^{(2)}$ are the energy of the zeroth and second approximations. E^{ex} is the exact value in Ref.[22]

λ	ρ	E		
		$E^{(0)}$	$E^{(2)}$	$E^{ex.}$
.02	2.02	1.015	1.015	
.1	2.07	1.065	1.065	
.2	2.12	1.119	1.118	1.118
.5	2.18	1.243	1.242	
1.	2.23	1.394	1.393	
1.5	2.25	1.511	1.510	
2.	2.27	1.610	1.609	1.608
5.	2.31	2.022	2.020	
10.	2.32	2.454	2.452	
20.	2.34	3.016	3.014	3.010
100.	2.36	5.009	5.008	

The accuracy of the zeroth approximation can be defined as

$$\delta = \frac{|E^{(0)} - E^{(2)}|}{E^{(0)}} \cdot 100\% ,$$

and from Table 1 one can see that it is less than 1 per cent, i.e., the perturbation series converges fairly fast.

If $\nu = 0$ and $m = 1$, then the ground state energy equals

$$E = c\lambda^{\frac{1}{3}}.$$

The constant c in the zeroth perturbation order is defined by equations (3.9) and equals

$$c_0 = \min_{\{\rho\}} \frac{3}{4\Gamma(1+\rho)} \cdot \left[\frac{\Gamma(5\rho)\Gamma^2(2-\rho)}{4\rho}\right]^{1/3} = .66933... .$$

The second approximation is done by (3.10). The result is

$$c_{(2)} = c_0 + c_2 = .66846..., \qquad c_2 = -.00087... .$$

The exact numerical value is $c = .667986...$ (see [15]).

3.2 Anharmonic potentials in R^3

In this section we consider the three-dimensional anharmonic potential. The Schrödinger equation looks as

$$\left[-\frac{1}{2mr}\left(\frac{d}{dr}\right)^2 r + \frac{l(l+1)}{2mr^2} + \frac{m\nu^2}{2}r^2 + \lambda r^4\right]\psi(r) = E\psi(r) . \tag{3.11}$$

According to (3.32) we do the transformation $r = s^{2\rho}$ and get the representation

$$H(E)\Phi(s) = \varepsilon(E)\Phi(s) ,$$
$$H(E) = \frac{1}{2}p^2 + W(s^2, E) ,$$
$$W(s^2, E) = 4\rho^2 m\left[\frac{m\nu^2}{2}(s^2)^{4\rho-1} + \lambda(s^2)^{6\rho-1} - E(s^2)^{2\rho-1}\right] , \qquad (3.12)$$

where $s \in \mathrm{R}^d$ with

$$d = 2\rho(2l+1) + 2.$$

Now we can apply the oscillator representation method. The Hamiltonian is

$$H = \frac{p^2}{2} + W(s^2, E) = H_0 + H_I + \varepsilon_0, \qquad (3.13)$$

where H_0 and H_I are given by (2.11) and ε_0 according to (2.16) equals

$$\varepsilon_0(E) = \min_{\{\omega,\rho\}} \varepsilon_0(E;\omega,\rho) ,$$

with

$$\begin{aligned}
\varepsilon_0(E;\omega,\rho) &= \frac{d\omega}{4} + \int_0^\infty \frac{du u^{\frac{d}{2}-1}e^{-u}}{\Gamma(\frac{d}{2})} W\left(\frac{u}{\omega}, E\right) \\
&= A(l,\omega,\rho) - EB(l,\omega,\rho) , \\
A(l,\omega,\rho) &= \frac{\rho(2l+1)+1}{2}\cdot\omega + \frac{4\rho^2 m\omega}{\Gamma(\rho(2l+1)+1)}\cdot \\
&\quad \left[\frac{m\nu^2}{2x^2}\cdot\Gamma(\rho(2l+5)) + \frac{\lambda}{x^3}\cdot\Gamma(\rho(2l+7))\right] , \\
B(l,\omega,\rho) &= \frac{4\rho^2 m\omega}{x}\cdot\frac{\Gamma(\rho(2l+3))}{\Gamma(\rho(2l+1)+1)} ,
\end{aligned}$$

where $x = \omega^{2\rho}$.

The ground state energy E_{0l} in the zeroth perturbation order is

$$\begin{aligned}
E_{0l} &= \min_{\{\omega,\rho\}} \frac{A(l,\omega,\rho)}{B(l,\omega,\rho)} = \min_{\{x,\rho\}} \left[\frac{\Gamma(\rho(2l+1)+2)}{\Gamma(\rho(2l+3))}\cdot\frac{x}{8m\rho^2} + \right. \qquad (3.14) \\
&\left. + \frac{m\nu^2}{2x}\cdot\frac{\Gamma(\rho(2l+5))}{\Gamma(\rho(2l+3))} + \frac{\lambda}{x^2}\cdot\frac{\Gamma(\rho(2l+7))}{\Gamma(\rho(2l+3))}\right] .
\end{aligned}$$

The ground state energy of the anharmonic oscillator has been extensively studied numerically and the exact result[6] for E_{00} in the case $\lambda = 1$, $m = 1/2$ and $\nu = 2$ equals

$$E_{00}^{\mathrm{ex}} = 4.64881\ldots .$$

Formula (3.14) gives for this case

$$E_{00} = 4.6511\ldots .$$

One can see that the oscillator representation method in the zeroth approximation gives a quite acceptable accuracy.

3.3 The power-low potentials

In this section the oscillator representation method will be applied to calculation of the ground, orbital and radial excitation energy spectrum of three-dimensional power-low potentails:

$$V(r) = \lambda r^{\nu} . \tag{3.15}$$

The Schrödinger equation looks as

$$\Big[-\frac{1}{2mr}\Big(\frac{d}{dr}\Big)^2 r + \frac{l(l+1)}{2mr^2} + \lambda r^{\nu}\Big]\psi(r) = E\psi(r) . \tag{3.16}$$

The transformation $r = s^{2\rho}$ leads to the representation

$$\begin{aligned} &H(E)\Phi(s) = \varepsilon(E)\Phi(s) , \\ &H(E) = \frac{1}{2}p^2 + W(s^2, E) , \\ &W(s^2, E) = 4\rho^2 m\Big[\lambda(s^2)^{\rho(2+\nu)-1} - E(s^2)^{2\rho-1}\Big] , \end{aligned} \tag{3.17}$$

where $s \in \mathrm{R}^d$ with

$$d = 2\rho(2l+1) + 2.$$

The Hamiltonian in the oscillator representation looks as

$$H = \frac{p^2}{2} + W(s^2, E) = H_0 + H_I + \varepsilon_0, \tag{3.18}$$

where H_0 and H_I are given by (2.12) and ε_0 according to (2.16) equals

$$\varepsilon_0(E) = \min_{\{\omega,\rho\}} \varepsilon_0(E;\omega,\rho) ,$$

with

$$\begin{aligned} \varepsilon_0(E;\omega,\rho) &= \frac{d\omega}{4} + \int_0^{\infty} \frac{du u^{\frac{d}{2}-1}e^{-u}}{\Gamma(\frac{d}{2})} W\Big(\frac{u}{\omega}, E\Big) \\ &= A(l,\omega,\rho) - EB(l,\omega,\rho) , \\ A(l,\omega,\rho) &= \frac{\rho(2l+1)+1}{2}\cdot\omega + \frac{4\rho^2 m\lambda\omega}{y^{2+\nu}}\cdot\frac{\Gamma(\rho(2l+3+\nu))}{\Gamma(\rho(2l+1)+1)} , \\ B(l,\omega,\rho) &= \frac{4\rho^2 m\omega}{y^2}\cdot\frac{\Gamma(\rho(2l+3))}{\Gamma(\rho(2l+1)+1)} , \end{aligned}$$

where $y = \omega^{\rho}$.

The ground state energy E_{0l} in the zeroth perturbation order is

$$\begin{aligned} E_{0l} &= \min_{\{\omega,\rho\}} \frac{A(l,\omega,\rho)}{B(l,\omega,\rho)} \\ &= \min_{\{y,\rho\}} \left[\frac{\Gamma(\rho(2l+1)+2)}{\Gamma(\rho(2l+3))}\cdot\frac{y^2}{8m\rho^2} + \frac{\lambda}{y^{\nu}}\cdot\frac{\Gamma(\rho(2l+3+\nu))}{\Gamma(\rho(2l+3))}\right]. \\ &= \min_{\{\rho\}} \frac{2+\nu}{\Gamma(\rho(2l+3))}\cdot\left[\frac{\Gamma(\rho(2l+1)+2)}{8\nu m\rho^2}\right]^{\frac{\nu}{2+\nu}}\cdot\left[\frac{\lambda}{2}\cdot\Gamma(\rho(2l+3+\nu))\right]^{\frac{2}{2+\nu}} . \end{aligned} \tag{3.19}$$

Now let us obtain the formula for radial excitations. According to (3.24), we have in the first approximation

$$\varepsilon_1^{(n)} = \varepsilon_0 + 2n\omega + \langle n|H_I|n\rangle \ .$$

where

$$\langle n|H_I|n\rangle = \int \left(\frac{dk}{2\pi}\right)^d \tilde{W}_d(k^2)\exp(-\frac{k^2}{4\omega})\langle n| : e_2^{ikq} : |n\rangle \ .$$

Let us define the polynomials

$$e_n(t,d) = \langle n| : e_2^{ikq} : |n\rangle = \sum_{m=2}^{2n} c_m(n,d)t^m, \qquad t = -\frac{k^2}{4\omega} \ ,$$

$$c_m(n,d) = \frac{n!\Gamma(\frac{d}{2})}{\Gamma(\frac{d}{2}+n)\Gamma(\frac{d}{2}+m)} \sum_p^{[\frac{m}{2}]} \frac{2^{m-2p}\Gamma(\frac{d}{2}+n+p)}{(n-m+p)!(m-2p)!(p!)^2} \ ,$$

where

$$p = \max(0, m-n).$$

The coefficients for $n = 1$ and $n = 2$ are

$$c_2(1,d) = \frac{2}{d} \ .$$

$$c_2(2,d) = \frac{4(d+8)}{d(d+2)}, \qquad c_3(2,d) = \frac{16}{d(d+2)}, \qquad c_4(2,d) = \frac{2}{d(d+2)}.$$

Three first polynomials are

$$c_0(t) = 0 \ .$$

$$c_1(t) = \frac{2}{d}t^2 \ .$$

$$c_2(t) = \frac{1}{d(d+2)}\left[(d+8)t^2 + 4t^3 + \frac{1}{2}t^4\right] \ .$$

If the potential has the form

$$W(s^2) = \sum_k w_k (s^2)^{\sigma_k},$$

then one can get

$$\langle n|H_I|n\rangle = \int \left(\frac{dk}{2\pi}\right)^d \tilde{W}_d(k^2)\exp(-\frac{k^2}{4\omega})\langle n| : e_2^{ikq} : |n\rangle =$$

$$= \sum_k \frac{w_k}{\omega^{\sigma_k}} \cdot C_k(d,\sigma_k) \ .$$

$$C_k(d,\sigma_k) = \frac{\Gamma(\frac{d}{2}+\sigma_k)}{\Gamma(\frac{d}{2}+1)} \cdot \sum_{s=2}^{2n} c_s(n,d) \cdot \frac{\Gamma(\sigma_k+1)}{\Gamma(\sigma_k+1-s)} \ .$$

Table 2.Results of calculation of the ground state energy power-low potentials for the $n = \ell = 0$ and $2m = 1$ of various ν. E_{00} is the zeroth approximation of the oscillator representation. The results of the numerical, Ref.[23], and $1/N$- expansion methods, Ref.[6], are also shown

ν	$1/N$	Num.	E_{00}
-1.5	-.29888	-.29609	-.29703
-1.25	-.22035	-.22029	-.22027
-1.0	-.25	-.25	-.25
.15	1.32795	1.32795	1.3279
.5	1.83341	1.83339	1.8335
.75	2.10815	2.10814	2.1082
1.5	2.70806	2.70809	2.7081
2.	3.0	3.0	3.0
3.0	3.45111	3.45056	3.4511
4.0	3.80139	3.79967	3.8024
5.0	4.09146	4.08916	4.0962
6.0	4.33801	4.33860	4.3524
7.0	4.54690	4.55866	4.5815
8.0	4.71772	4.75587	4.7901
10.	4.92220	5.09786	5.1607

The next step is to solve equations (2.23). The result is

$$E_{nl} = \lambda \min_{\rho} \frac{\Gamma(\rho(2l+3+\nu))}{\Gamma(\rho(2l+3))} \cdot \frac{F(\nu)}{F(0)} \cdot \left[\frac{F(0)}{8\rho^2 m \lambda J} \cdot \frac{\Gamma(\rho(2l+1)+1)}{\Gamma(\rho(2l+3+\nu))} \right]^{\frac{\nu}{2+\nu}} ,$$

with

$$F(\nu) = 4n[\rho(2-\nu)-1] + [\rho(2l+1)+1][\rho(2+\nu) + C_n(d, \rho(2+\nu)-1)] ,$$
$$J = \rho\nu + [\rho(2+\nu)-1]C_n(d, 2\rho-1) - (2\rho-1)C_n(d, \rho(2+\nu)-1) ,$$

where $d = 2\rho(2l+1)+2$. The numerical results are shown in Tables 2 and 3.

3.4 The Logarithmic potential

Now we consider the logarithmic potential

$$V(r) = \lambda \ln(r) . \tag{3.20}$$

Table 3. Results of calculation of the energy spectrum for a wide class of potentials in the zeroth approximation. The numerical ones are taken from Refs.[6], [23] (in parentheses)

		$V(r)$			
		$-\frac{2^{1.7}}{r^{0.2}}$	$-\frac{2^{.8}}{r^{0.8}}$	$2^{3.5}r$	$\ln r$
		$2m=1$	$2m=1$	$2m=1$	$m=1$
$n=0$	$l=0$	-2.686	-1.2186	9.353	1.045
		(-2.686)	(-1.218)	(9.35243)	(1.0443)
	$l=1$	-2.345	-0.5004	13.445	1.641
		(-2.345)	(-0.500)		(1.643)
	$l=2$	-2.156	-0.2947	16.993	2.014
		(-2.156)	(-0.295)		(2.015)
	$l=3$	-2.029	-0.2019	20.204	2.284
		(-2.029)	(-0.202)		(2.286)
$n=1$	$l=0$	-2.253	-0.462	16.355	1.848
		(-2.253)	(-0.462)	(16.3518)	(1.8474)
	$l=1$	-2.101	-0.281	19.540	2.151
		(-2.101)	(-0.281)		(2.151)
	$l=2$	-1.990	-0.195	22.521	2.388
		(-1.990)	(-0.195)		(2.388)
	$l=3$	-1.905	-0.146	25.330	2.580
$n=2$	$l=0$	-2.044	-0.265	22.084	2.290
		(-2.044)	(-0.265)	(22.08224)	(2.290)
	$l=1$	-1.951	-0.187	24.833	2.491
		(-1.951)	(-0.187)		(2.491)
	$l=2$	-1.875	-0.142	27.478	2.663
		(-1.875)	(-0.142)		(2.663)
	$l=3$	-1.812	-0.113	30.021	2.811

This is one of the potentials which has been used in heavy quarkonium spectroscopy [23]. The standard calculations according to formulas (3.12), give the following result for the energy E_{nl} in the lowest approximation:

$$E_{nl} = \lambda \min_{\rho} \left\{ \frac{\partial}{\partial \sigma} \ln \left[(\frac{d}{2} + 4n)\sigma + \frac{d}{2}(1 + C_n(d,\sigma)) \right] + \psi(\frac{d}{2} + \sigma) \right.$$

$$+\frac{1}{2\rho}\ln\left[\frac{1}{(\sigma+1)^3 m\lambda}\cdot\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d}{2}+\sigma)}\cdot\frac{(\frac{d}{2}+4n)\sigma+\frac{d}{2}(1+C_n(d,\sigma))}{1+C_n(d,\sigma)-\sigma\frac{\partial}{\partial\sigma}C_n(d,\sigma)}\right]\Bigg\}\,, \tag{3.21}$$

$$\frac{d}{2}=\rho(2l+1)+1, \qquad \sigma=2\rho-1, \qquad \psi(x)=\frac{d}{dx}\ln\Gamma(x).$$

The functions $C_n(d,\sigma)$ are defined in the previous section.

The numerical results are shown in Table 3. One can see that the first approximation of the OR method coincides with the exact values in four signs.

4 The relativized Schrödinger equation

The standard Schrödinger equation describes the behaviour of nonrelativistic particles. Nonrelativistic potential models turned out to give the successful description not only of heavy quarkonia but also of ordinary hadrons. One can say that these models work much better than we would naively expect. Nevertheless, the quark-quark systems are relativistic ones so that calculations of relativistic corrections are one of the important problems of quark bound states. The relativistic character of quark-quark interaction was studied in papers ([25]-[28]). The main point is that the complete quantum field theory of bound states is not yet formulated, so that we have different more or less motivated approaches like Bethe-Salpeter and Breit-Fermi equations and the so-called relativized Schrödinger equation,wich will be considered in this section.

There exists a voluminous literature where the bound states of quark-quark systems are studied in the framework of nonrelativistic and relativized Schrödinger equation (see, for example, [26]). Here we dare not discuss physical aspects of quarkonia. Our aim is to attract attention to the OR method and show its application to spectroscopy of the relativized Schrödinger equation.

The relativized Schrödinger equation is based on the simplest idea (see, for example,[28]) to take into account the kinematic relativistic corrections using the relativistic kinetic energy instead of the nonrelativistic one

$$\frac{p^2}{2m}\to\sqrt{p^2+m^2}\,.$$

Thus, we get the following relativized Schrödinger equation in the space R^3:

$$\left[\sqrt{p^2+m^2}+V(r^2)\right]\Psi=E\Psi\,. \tag{4.1}$$

The usual solution of this equation is done by numerical calculations on computers and by variational methods (see([25]-[27])).

We shall solve this equation by the OR method. Let the orbital moment be l; then the wave function is

$$\Psi(r,\theta,\phi)=Y_{l,m}(\theta,\phi)\Psi_{nl}(r)\,. \tag{4.2}$$

The radial Schrödinger equation becomes

$$\left[\sqrt{-\frac{1}{r}(\frac{d}{dr})^2 r + \frac{l(l+1)}{r^2} + m^2} + V(r^2)\right]\Psi_{nl} = E_{nl}\Psi_{nl} \ . \tag{4.3}$$

Introducing the function

$$\Psi_{nl}(r) = r^l \Phi(r),$$
$$L_d = \frac{d-3}{2} = l, \qquad d = 3 + 2l \ ,$$

one can get

$$\left[\sqrt{-\frac{1}{r^{\frac{d-1}{2}}}(\frac{d}{dr})^2 r^{\frac{d-1}{2}} + \frac{L_d(L_d+1)}{r^2} + m^2} + V(r^2)\right]\Psi_{nl} = E_{nl}\Psi_{nl} \ .$$

This equation coincides with the radial Schrödinger equation in the space R^d:

$$\left[\sqrt{-\Delta_d + m^2} + V(r^2)\right]\Phi(r) = E\Phi(r) \ . \tag{4.4}$$

Now we can apply the oscillator representation to the Hamiltonian

$$H = \sqrt{p^2 + m^2} + V(r^2)$$

in the space R^d. Let us rewrite this Hamiltonian in the form:

$$H = \frac{1}{2\mu}(p^2 + \vartheta r^2) + \left[\sqrt{p^2 + m^2} - \frac{p^2}{2\mu}\right] + \left[V(r^2) - \frac{\vartheta}{2\mu}r^2\right] \ , \tag{4.5}$$

where μ and ϑ are parameters, and introduce the oscillator canonical variables:

$$r_j = \frac{Q_j}{\sqrt{\vartheta}} \ , \qquad Q_j = \frac{a_j + a_j^+}{\sqrt{2}} \ ; \tag{4.6}$$
$$p_j = \sqrt{\vartheta}P_j \ , \qquad P_j = \frac{a_j - a_j^+}{i\sqrt{2}} \ ; \qquad (j = 1,\ldots,d) \ .$$

The vacuum is defined in the standard way:

$$\langle 0|0\rangle = 1 \ , \qquad a_j|0\rangle = 0 \ ,$$
$$\langle 0|r_i r_j|0\rangle = \frac{\delta_{ij}}{2\vartheta} \ , \qquad \langle 0|p_i p_j|0\rangle = \delta_{ij}\frac{\vartheta}{2} \ .$$

Let us substitute the representation (4.6) in (4.5), go over to the normal product of the operators a_i and a_i^+ in the Hamiltonian and require that the interaction Hamiltonian should not contain terms with $: p^2 :$ and $: r^2 :$. After some transformations one can get

$$H = H_0 + E_0 + H_I \ .$$

with

$$
\begin{aligned}
H_0 &= \omega a_j^+ a_j \,, \qquad \omega = \frac{\vartheta}{\mu} : \\
H_I &= \int \left(\frac{d\rho}{\sqrt{\pi}}\right)^d e^{-\rho^2} \left[\sqrt{\rho^2\vartheta + m^2} : \exp(-2p\rho - p^2) - 1 + p^2(1 - \frac{2}{d}\rho^2) : \right. \\
&\quad \left. + V(\frac{\rho^2}{\vartheta}) : \exp(-2Q\rho - Q^2) - 1 + Q^2(1 - \frac{2}{d}\rho) : \right] \\
&= \int \left(\frac{du}{\sqrt{\pi}}\right)^d \left[\tilde{K}(u^2) \exp(-\frac{u^2\vartheta}{4}) : e_2^{iup} : + \tilde{V}(u^2) \exp(-\frac{u^2}{4\vartheta}) : e_2^{iur} : \right] , \qquad (4.7)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{K}(u^2) &= \int (d\rho)^d \sqrt{\rho^2 + m^2} e^{iup} \,; \\
\tilde{V}(u^2) &= \int (d\rho)^d V(\rho^2) e^{iup} \,, \\
e_2^z &= e^z - 1 - z - \frac{z^2}{2} \,;
\end{aligned}
$$

$$
\begin{aligned}
E_0 &= \min_{\vartheta} \int \left(\frac{d\rho}{\sqrt{\pi}}\right)^d e^{-\rho^2} \left[\sqrt{\rho^2\vartheta + m^2} + V(\frac{\rho^2}{\vartheta}) \right] \\
&= \min_{\vartheta} \int_0^\infty \frac{du u^{d/2-1} e^{-u}}{\Gamma(\frac{d}{2})} \left[\sqrt{u\vartheta + m^2} + V(\frac{u}{\vartheta}) \right] .
\end{aligned}
$$

The parameters μ and ϑ are determined by condition of the oscillator representation, i.e., the interaction Hamiltonian H_I should not contain the quadratic terms with $: p^2 : \; : r^2 :$. These equations are

$$
\begin{aligned}
&\int_0^\infty du u^{d/2} e^{-u} \frac{d}{du} \left[\sqrt{u\vartheta + m^2} - V(\frac{u}{\vartheta}) \right] = 0 \,, \\
&\omega = \frac{2}{\Gamma(\frac{d}{2}+1)} \int_0^\infty du u^{d/2} e^{-u} \frac{d}{du} V(\frac{u}{\vartheta}) \,. \qquad (4.8)
\end{aligned}
$$

These formulas permit calculating the spectrum of the relativized Hamiltonian.

4.1 Examples

Here we would like to consider the Cornel potential[29] for the quark mass $m = 0$ because in this case all calculations can be performed analytically. The Cornel potential gives

the simplest interpolation from the Coulomb behaviour at short distances to the lineary growing confined potential at large distances. The Cornel potential is

$$H = \sqrt{p^2} - \frac{\kappa}{r} + hr \ . \tag{4.9}$$

This potential was used to give semirelativistic description of quark-quark bound states. Phenomenologically, the first term in the potential is connected with one-gluon exchange and describes short distances. The second term ensures the confinement of quarks.

Equations (4.8) for ϑ and ω can be solved easily:

$$\begin{aligned} \vartheta &= \frac{h}{1-\kappa_l} , \qquad \kappa_l = \frac{\kappa}{1+l} , \\ \omega &= \sqrt{\frac{h}{1-\kappa_l}} \frac{\Gamma(l+2)}{\Gamma(l+5/2)} . \end{aligned} \tag{4.10}$$

One can see that

1. if $h = 0$, i.e., for the pure Coulomb potential the relativized Schrödinger equation has no solutions describing any bound states;

2. for $\kappa \geq 1$ there exists "the downfall on the centre", i.e., there are no stable states in this system.

The Hamiltonian in OR looks as

$$H = \omega a_j^+ a_j + H_I + \varepsilon_0 \ ,$$

where

$$\begin{aligned} H_I &= -\sqrt{\frac{h}{1-\kappa_l}} \frac{\Gamma(2+l)}{\pi^{2+l}} \int \frac{(du)^d}{u^{2(2+l)}} \exp(-\frac{u^2}{4}) \Big[: e_2^{iuP} : \\ &\quad + (1 - \kappa_l + \frac{1}{2}\kappa_l u^2) : e_2^{iuQ} : \Big] , \\ E_0 &= 2\sqrt{h(1-\kappa_l)} \frac{\Gamma(l+2)}{\Gamma(l+3/2)} . \end{aligned} \tag{4.11}$$

The second correction can be calculated. We give this formula for the case $\kappa = 0$. It is

$$\begin{aligned} E &= E_0 + E_2 = E_0(1-\delta_l) = 2\sqrt{h} \frac{\Gamma(l+2)}{\Gamma(l+3/2)} (1-\delta_l) , \\ \delta_l &= \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{\Gamma(l+2)\Gamma(2n+1/2)}{\Gamma(2n+l+1/2)} \cdot \frac{2^{4n-1}\Gamma^2(2n-1/2)}{\Gamma(4n+2)} . \end{aligned}$$

The numerical values of δ_l are

$$\delta_0 = .006, \qquad \delta_1 = .004, \qquad \delta_l < .003 \quad \text{for } l \geq 2.$$

For asymptotically large l we obtain

$$E = E_{0l} \approx 2\sqrt{hl} \ . \tag{4.12}$$

For the same linear potential in the case of nonrelativistic kinetic energy in the Hamiltonian one can get

$$E = E_{0l} \approx \frac{3}{2}(hl)^{2/3}. \tag{4.13}$$

The phenomena of "the downfall on the centre" for $\kappa \geq 1$ leads to the sensitivity of the eigenvalues of the Hamiltonian to the mass m in the kinetic term. In particular, if κ is close to 1, the nonrelativistic limit comes for a quite large m. Let us demonstrate this statement. We have two Hamiltonians:

$$H_{rel} = \sqrt{p^2 + m^2} - m - \frac{\kappa}{r} + hr\ ,$$

$$H = \frac{p^2}{2m} - \frac{\kappa}{r} + hr\ .$$

Our aim is to find values of μ as functions of κ for which the ground state energies of these two Hamiltonians practically coincide. It is convenient to do the following substitutions:

$$r \to \frac{r}{\sqrt{h}}, \qquad \mu = \frac{m}{\sqrt{h}}, \qquad H \to H/\sqrt{h}\ .$$

We shall consider the Hamiltonians

$$H_{rel} = \sqrt{p^2 + \mu^2} - \mu - \frac{\kappa}{r} + r\ ,$$

$$H = \frac{p^2}{2\mu} - \frac{\kappa}{r} + r\ .$$

Table 4. Results of calculations of the ground state energy for the relativized and nonrelativized Hamiltonian as a function of the parameters κ and μ

	$\kappa = 0$		$\kappa = 0.1$		$\kappa = 0.5$		$\kappa = 0.9$	
μ	rel	nonrel	rel	nonrel	rel	nonrel	rel	nonrel
.01	2.247	8.614	2.131	8.591	1.586	8.500	0.704	8.409
.1	2.162	3.998	2.046	3.949	1.500	3.749	0.616	3.543
.5	1.878	2.338	1.757	2.254	1.186	1.896	0.256	1.503
1.	1.654	1.856	1.523	1.749	0.902	1.279	-0.134	0.730
2.	1.396	1.473	1.246	1.337	0.517	0.702	-0.802	-0.114
5.	1.068	1.085	0.877	0.896	-0.168	-0.120	-2.526	-1.706
10.	0.857	0.861	0.616	0.614	-0.934	-0.968	-5.226	-3.885
40.					-4.607	-4.925	-21.127	-16.158
100.							-52.850	-40.483

According to the above-stated, one can get for the ground state energy for both cases

$$E_{00}^{rel} = \min_x \frac{2}{\sqrt{\pi}} \left\{ \int_0^\infty dt \sqrt{t} e^{-t} [\sqrt{\mu^2 + tx^2} - \mu] - \kappa x + \frac{1}{x} \right\}\ ,$$

$$E_{00} = \min_{\rho,x} \left\{ \frac{\Gamma(2+\rho)}{\Gamma(3\rho) 8\rho^2 \mu} \cdot x^2 - \kappa x \cdot \frac{\Gamma(2\rho)}{\Gamma(3\rho)} + \frac{1}{x} \cdot \frac{\Gamma(4\rho)}{\Gamma(3\rho)} \right\}\ .$$

The numerical results are shown in Table 4. One can see that for $\kappa = 1/2$ the nonrelativistic approach comes approximately for $\mu^2 = m^2/h \approx 100 \div 500$ only.

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19.	Биофизика

Динейхан М., Ефимов Г.В. Е4-94-75

Ангармонический потенциал в осцилляторном представлении

Энергетический спектр широкого класса потенциалов, допускающих связанные состояния, вычислен методом осцилляторного представления в рамках нерелятивистского и релятивизованного уравнений Шредингера. Осцилляторное представление является регулярным методом описания и расчета уровней энергии как основного состояния, так и радиальных и орбитальных возбуждений для широкого класса потенциалов. В случае ангармонического потенциала результаты нулевого приближения находятся в очень хорошем согласии с точными вычислениями. Ряды модифицированной теории возмущений являются быстро сходящимися, так что высшие пертурбативные поправки к нулевому приближению оказываются малы.

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Dineykhan M., Efimov G.V. E4-94-75

Anharmonic Potential in the Oscillator Representation

In the nonrelativistic and relativized Schrödinger equation the Wick-ordering method called the Oscillator Representation is proposed to calculate the energy spectrum for a wide class of potentials allowing the existence of a bound state. The oscillator representation method gives a unique regular way to describe and calculate the energy levels of ground as well as orbital and radial excitation states for a wide class of potentials. The results of the zeroth approximation oscillator representation are in good agreement with the exact values for the anharmonic potentials. The oscillator representation method was applied to the relativized Schrödinger equation too. The perturbation series converges fairly fast, i.e., the highest perturbation corrections over the interaction Hamiltonian are small enough.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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