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M. Zabiégo and X. Garbet

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M. Zabiégo and X. Garbet
DRFC/SPPF, Association Euratom-CEA
CE Cadarache, BP n°1
F-13108 Saint-Paul-lez-Durance, France

ABSTRACT

The non linear behavior of low and large wave number tearing modes is studied. The emphasis is layed on diamagnetic effects. A kinetic equation, including transport processes associated with a background of microturbulence, is used to describe the electron component. Such transport processes are shown to play a significant rôle in the adjustment of density and temperature profile and also in the calculation of the island rotation frequency. The fluctuating electric potential is calculated self-consistently, using the differential response of electrons and ions. Four regimes are considered, related to island width (smaller or larger than an ion Larmor radius) and transport regime (electron-ion collisions or electro-viscosity dominated). It is shown that diamagnetism does not influence the island stability for small island width in the viscous regime, as long as the constant A constraint is maintained. It turns out that the release of this constraint may strongly modify the previously calculated stability thresholds. Finally, it is found that diamagnetism is destabilizing (stabilizing) for island width smaller (larger) than an ion Larmor radius, in both resistive and viscous regimes. A typical island evolution scenario is studied which shows that even large scale tearing modes with positive Δ' could saturate to island width of order of a few ion Larmor radii. Illustrative Δ' threshold and island saturation size are calculated.

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I. INTRODUCTION

The stability of magnetic islands is of crucial importance for magnetohydrodynamic (MHD) stability and transport in tokamaks[1]. As far as the MHD stability is concerned, it is relevant for large scale tearing modes such as m=1, n=1 (when the plasma β does not exceed the MHD critical value[2]) and m=2, n=1 modes. On the other hand, small scale tearing modes with negative Δ', i.e. microtearing modes[3-6], can be excited by diamagnetism[1,7-12], provided that their width is smaller than an ion Larmor radius. The interaction of many of these modes may generate a microturbulence which could account for anomalous transport in tokamaks. In this paper, we consider the stability of such modes in the non-linear regime, observable magnetic islands having size larger than the linear layer width in present day tokamaks, as pointed out by Scott and Hassam[13].

Many physical processes are known to affect the island stability: thermal effects[14-17], toroidal curvature[18,19], bootstrap current[20,21], plasma flow[22] and diamagnetism[1,7-12]. We will focus here on the effect of diamagnetism on the stability since it is generally the dominant process. Moreover, there is some apparent contradiction in the litterature since the same effect is sometimes found to be destabilizing[7] or stabilizing[13]. It is shown in this work that both results are indeed possible, depending on the radial scale of the island (with respect to ion Larmor radius). We obtain results similar to those of Smolyakov[1,10] concerning island width evolution. However, transport processes due to a background of turbulence[7-9], which are usually neglected in the litterature, are included in our work. This has two important consequences. First, these processes determine the shape of density and temperature, which are generally arbitrarily obtained[1,10-12]. Second, the rotation frequency of the island, or equivalently the radial electric field in the frame where the latter is static, is also fixed by the energy dependence of the quasi-linear operator associated with the microturbulence bath.

Furthermore, these studies are most often achieved within the frame of the constant A approximation[23] and in the resistive regime, where fluid equations can be used. When the island width is small, this approximation is questionable, since the viscous term (electro-viscosity) in the Ohm's law can dominante the resistive term[24,25] (electron-ion collisions). In this case, a kinetic calculation is more appropriate than the use of fluid equations. Moreover, it will be shown that the constant A approximation fails in some cases.

The remainder of this paper is organized as follows. The basic equations are presented in sec. **II**. In sec. **III**, the electron distribution function is calculated. First, the perturbative method for the resolution of the electron kinetic equation is described. It allows to separate the problem in elementary sub-problems, associated with each of the physical mechanisms influencing the island stability (parallel dynamics, transport processes, inductive field and diamagnetism), which are then treated independently according to the same technique. The fluctuating electric potential is self-consistently obtained in sec. **IV**, using the ion response.

These preliminary results are used in sec. V to calculate both the island rotation frequency and the island width evolution equation in various regimes (small/large island width and resistive/viscous regime), within the frame of the constant A approximation. It is shown that, for small island width, the viscous regime is degenerate with respect to diamagnetism. Thus, the constant A constraint must be released. This peculiar situation is treated numerically in sec. VI. Sec. VII is devoted to the study of typical scenarios of island width evolution. Finally, the main results are summarized and discussed in sec. VIII.

II BASIC EQUATIONS

A. Magnetic topology

The magnetic field in a tokamak can be written

$$\mathbf{B}_{eq} = f(\psi_{eq})\, \nabla\varphi + \nabla\psi_{eq} \times \nabla\varphi \tag{1}$$

where φ is the toroidal angle and ψ_{eq} the poloidal flux normalized to 2π. We consider, in the following, a simplified equilibrium of nested magnetic surfaces, with circular cross section. Each magnetic surface is then labelled by its minor radius r and

$$\begin{cases} d\psi_{eq} = -\dfrac{r}{q(r)}\, B_0\, dr \\ f(r) = B_0 R_0 \end{cases} \tag{2}$$

where B_0 is the toroidal field at the major radius R_0 and q(r) is the safety factor. Adding to this equilibrium a vector potential fluctuation

$$\begin{cases} \delta A(r,\theta,\varphi,t) = 2\,\widetilde{A}(r,t)\cos u\, \dfrac{\mathbf{B}_{eq}}{B_{eq}} \\ u = m\theta - n\varphi - \omega t \end{cases} \tag{3}$$

an island chain topology is found near the resonant surface $r = r_0$, where $q(r_0) = m/n$. The new set of surfaces is labelled by the helical flux ψ defined by

$$\psi \equiv \frac{B_0}{2\,L_s}(r-r_0)^2 + 2\widetilde{A}(r,t)\cos u \tag{4}$$

where $L_s = q_0 R_0 / s_0$ is the shear length, $s_0 = (r/q\, dq/dr)_{r=r_0}$ is the local shear and q_0 is the local safety factor. In this perturbed topology, an island is characterised by its half-width δ_I and its poloidal wave number K_θ

$$\left\{ \begin{array}{l} \delta_I(t) = \sqrt{\dfrac{8\,\tilde{A}(r_0,t)\,L_s}{B_0}} \\ K_\theta = \dfrac{m}{r_0} \end{array} \right. \quad (5)$$

In the following, all the calculations will be performed within the frame of the thin island approximation $|K_\theta|\delta_I \ll 1$. In particular, $\partial/r\partial\theta \ll \partial/\partial r$ everywhere in the non linear layer.

For convenience, we define the dimensionless variables

$$\left\{ \begin{array}{l} \rho = \dfrac{r - r_0}{\delta_I} \\ a(\rho,t) = \dfrac{\tilde{A}(r,t)}{\tilde{A}(r_0,t)} \end{array} \right. \quad (6)$$

and

$$\overline{\psi}(\rho,u,t) \equiv \frac{\psi}{2\tilde{A}(r_0,t)} = 2\rho^2 + a(\rho,t)\cos u \quad (7)$$

B. Electron kinetic equation

Passing electrons (which account for the essential of the physics involved in this problem, apart from bootstrap effects[20,21] that we do not consider in this paper) may be characterized by a distribution function $F(H,\mu,\varepsilon_{//},\psi,u)$, where $H \approx m_e v^2/2 + eU(\psi,u)$ (e is the electron charge and U is the electric potential), $\mu = mv_\perp^2/2B_{eq}$ is the adiabatic invariant, and $\varepsilon_{//}$ is the sign of the parallel velocity $v_{//}(H,\mu,\varepsilon_{//},\psi,u)$. The kinetic equation which determines F is

$$\frac{\partial F}{\partial t} + v_{//}\nabla_{//}F + v_D.\nabla F - e\,v_{//}\frac{\partial \delta A}{\partial t}\frac{\partial F}{\partial H} = T(F) + S(F) \quad (8)$$

where v_D is the drift velocity that is restricted to the electric drift velocity v_E in the following

$$v_E = \frac{\mathbf{B}_{eq} \times \nabla U}{\mathbf{B}_{eq}^2} \quad (9)$$

Doing this, we neglect average curvature effects[19] which appear to be less efficient diamagnetic effects that we consider in this paper (see **V.B.2** and **4**), to saturate large islands. The operator $T(F) = C(F) + D(F)$ is associated with transport processes. The collision operator

C(F) is restricted to an electron-ion collision term, characterized by the (energy dependent) electron-ion collision frequency ν_C and defined as

$$\begin{cases} C(F_e) \equiv 0 \\ C(F_o) \equiv -\nu_C F_o \end{cases} \qquad (10)$$

where F_e (resp. F_o) is the even (resp. odd) part of the distribution function F with respect to $v_{//}$. The operator D(F) is associated with diffusion phenomena induced by a background microturbulence. Assuming that the turbulent diffusion mainly acts in the radial direction, an estimate of the operator D(F) is

$$D(F) \cong \frac{D(H,\mu)}{\delta_I^2}\left(\frac{\partial^2 F}{\partial \rho^2}\right)_u \qquad (11)$$

where $D(H,\mu)$ depends on the kind of turbulence responsible for the transport (for instance $D=D_M|v_{//}|$ for a magnetic microturbulence). Source terms represented by S(F), such as bootstrap current[20,21] and radiative[14-17] effects, will be neglected in the following.

On the other hand, calculations will be performed in the frame where the island chain is static. In this frame, there exists a radial electric field $E_r = - dU_{eq}/dr$ which is linked to the frequency ω of the mode , in the usually considered frame where there is no radial electric field, through the relationship

$$\omega = -\frac{K_\theta}{B_{eq}}\frac{dU_{eq}}{dr}\bigg|_{r=r_0} \qquad (12)$$

Consequently, the $\partial F/\partial t$ term will be dropped in the following. We define a Poisson bracket operator

$$[X,Y] \equiv 2\tilde{A}(r_0,t)\frac{\mathbf{B}_{eq}}{B_{eq}^2}\cdot(\nabla X \times \nabla Y) = \frac{G(\overline{\psi},u)}{4L_{//}}\{X,Y\} \qquad (13)$$

where $L_{//} = L_S/K_\theta\delta_I$ is the typical length of a field line and

$$\{X,Y\} \equiv \left(\frac{\partial X}{\partial \overline{\psi}}\right)_u\left(\frac{\partial Y}{\partial u}\right)_{\overline{\psi}} - \left(\frac{\partial Y}{\partial \overline{\psi}}\right)_u\left(\frac{\partial X}{\partial u}\right)_{\overline{\psi}} \qquad (14)$$

$$G(\overline{\psi},u) \equiv \left|\left(\frac{\partial \overline{\psi}}{\partial \rho}\right)_u\right| = \left|4\rho + \frac{\partial a}{\partial \rho}\cos u\right| \qquad (15)$$

so that the parallel gradient along the perturbed magnetic field lines ($\mathbf{B} = \mathbf{B}_{eq} + \mathbf{rot}\ \delta\mathbf{A}$) of any function Z(r,u) writes

$$\nabla_{//}Z \equiv \frac{\mathbf{B}}{B}\cdot\nabla Z = [Z,\overline{\psi}] = -\frac{G(\overline{\psi},u)}{4L_{//}}\left(\frac{\partial Z}{\partial u}\right)_{\overline{\psi}} \qquad (16)$$

Moreover, if $eU/T_e \ll 1$, i.e. if the island half-width δ_I is much smaller than the gradient lengths, then $v_E.\nabla F = [U , F]/ 2\tilde{A}(r_0,t) = -v_{//}[F , U/2v_{//}\tilde{A}(r_0,t)]$, so that the electron kinetic equation may be written under the more compact form:

$$v_{//}[F , \overline{\psi} - \overline{U}] - e\, v_{//} \frac{\partial \delta A}{\partial t} \frac{\partial F}{\partial H} = T(F) = C(F) + D(F) \qquad (17)$$

where $\overline{U}$ is the dimensionless quantity

$$\overline{U} \equiv \frac{U}{2v_{//}\tilde{A}(r_0,t)} \qquad (18)$$

C. Ion response

To compute the self-consistent electric potential, it is necessary to determine the ion density fluctuation. Two cases may be considered which consist in comparing the island width δ_I with the thermal ion Larmor radius $\rho_S = \sqrt{2m_iT_e}/ e_iB_0$. Note that the definition we use in this paper for the thermal ion Larmor radius involves the electron temperature T_e instead of the ion one T_i. The reason for this is simply that it allows to treat the limit case T_i=0 (cold ion hypothesis) more easily.

- *small island width: $\delta_I \ll \rho_S$*

In this case, where the island half-width is smaller than the thermal ion Larmor radius, the ion density response is adiabatic, i.e., stating $n_i \equiv n_{eq} + \tilde{n}_i$ and $U \equiv U_{eq} + \tilde{U}$ then

$$\tilde{n}_i/ n_0 = - e_i\tilde{U} / T_i = e\tilde{U} / \tau\, T_e \qquad (19)$$

where $\tau = T_i/T_e$ (note that the cold ion hypothesis corresponds to the limit τ=0).

- *large island width: $\delta_I \gg \rho_S$*

In this case, the ion density may be calculated by means of the ion fluid equation

$$e_i\, \nabla.(n_i v_E) + \nabla.(j_{pol}) = e_i\, D_i\, \Delta\left(n_i + n_0 \frac{e_i U}{T_i}\right) \qquad (20)$$

where the ions are assumed to be static along the field lines and D_i is the ion particle diffusivity. The polarization current $\mathbf{j}_{pol}$ (including both ion inertia and finite Larmor radius (FLR) effects) is such that[26]

$$\nabla.(j_{pol}) = - \nabla.\left(\frac{n_i e^2}{T_e} \frac{\rho_S^2}{2} \left[\left(v_E + v_i^*\right) . \nabla\right] \nabla U\right) \qquad (21)$$

where v_i^* is the ion diamagnetic velocity

$$v_i^* = \frac{\mathbf{B}_{eq} \times \nabla p_i}{n_i e_i B_{eq}^2} \tag{22}$$

D. Self-consistency

Once the perturbed electron density and current are calculated, by solving the kinetic equation, the self-consistent electric potential can be found by expressing the quasi-electroneutrality condition

$$n_e = n_i \tag{23}$$

The self-consistency of the island topology must be verified by solving the Ampère equation

$$-\frac{\partial^2 \tilde{A}}{\partial r^2} + K_\theta^2 \tilde{A} = \mu_0 \tilde{j}_{//} \tag{24}$$

$$\text{where} \quad \begin{cases} \tilde{j}_{//}(r,t) = \displaystyle\int \frac{du}{2\pi} \exp(-iu)\, \delta j_{//}(r,u,t) \\ \delta j_{//} = \displaystyle\int d^3p \; e v_{//} F \end{cases} \tag{25}$$

This provides the perturbed vector potential, which must be matched to the outer solution. If the vector potential behaves smoothly within the layer, i.e., within the frame of the constant A approximation, this matching is simply achieved by defining from the outer solution the Δ' parameter

$$\Delta' \equiv \frac{\left(\partial\tilde{A}/\partial r\right)_{r_0^+} - \left(\partial\tilde{A}/\partial r\right)_{r_0^-}}{\tilde{A}(r_0,t)} \tag{26}$$

For large n,m magnetic modes, this quantity reduces to $\Delta' = -2|K_\theta|$. Then, the matching condition is

$$\Delta' \tilde{A}(r_0,t) = -\mu_0 \int dr\ \tilde{j}_{//}(r,t) \tag{27}$$

This condition provides the island rotation frequency and the island evolution equation (i.e. a stability threshold) (see details in V).

III. ELECTRON DISTRIBUTION FUNCTION

A. Method for the resolution of the electron kinetic equation

Considering the electron kinetic equation (17), we can see that the electron distribution function F is determined by five different mechanisms, each of these being associated with a characteristic frequency

- the parallel dynamics ($v_{//}[F,\overline{\psi}]$), associated with the transit frequency $v_{Te}/L_{//}$, where v_{Te} is the electron thermal velocity.
- the transverse dynamics induced by the electric drift ($-v_{//}[F,\overline{U}]$), characterized by a typical frequency $K_\theta v_{E\theta}$.
- the inductive field related to the vector potential perturbation ($-e\,v_{//}\,\partial\delta A/\partial t\;\partial F/\partial H$), related to a frequency $e/m_e v_{Te}\,\partial\delta A/\partial t \approx \delta_I^2/L_S\rho_e\,\partial\delta_I/\delta_I\partial t$, where $\rho_e = \sqrt{2m_eT_e}\,/|e|B_0$ is the electron Larmor radius and $\partial\delta_I/\delta_I\partial t$ is the island growth rate
- the electron-ion collisions (C(F)), characterized by the frequency v_C
- the diffusion induced by the background turbulence (D(F)), with a frequency $v_D \simeq D_e/\delta_I^2$, where D_e is the thermal diffusion coefficient

$$D_e \equiv \int d^3p\; D\,F_{eq} \Big/ \int d^3p\; F_{eq} \qquad (28)$$

Given these last two frequencies and the rôle played by transport processes (which will be clarified later), we will consider two distinct transport regimes in this paper:

- the resistive regime, corresponding to the case $v_C \gg v_D$
- the viscous regime, corresponding to the opposite case $v_C \ll v_D$

Usually, except for simple investigations[24,25], only the resistive regime is considered. However, for typical values of collision frequency v_C ($\approx 10^4$–10^5 s^{-1}) and anomalous diffusion coefficient D_e (≈ 1 m^2s^{-1}) in present day tokamaks, islands thinner than millimetric size are in the viscous regime.

In the following, we will study the island evolution in the non linear regime where the parallel electron transit frequency is larger than the other four frequencies involved in the problem. Note that this hypothesis obviously excludes the skin depth regime, recently investigated by Smolyakov and Hirose[27], for which the parallel dynamics is dominated by electron inertia.

Then, in order to solve the electron kinetic equation perturbatively, we define the following small parameters

- three which are linked to transport processes

$$\varepsilon_D \equiv D_e L_{//}/\delta_I^2 v_{Te} \qquad (29a)$$

$$\varepsilon_C \equiv v_C L_{//}/v_{Te} \qquad (29b)$$

$$\varepsilon_T \equiv \mathrm{Max}\{\varepsilon_C, \varepsilon_D\} \qquad (29c)$$

- one associated with the transverse drift

$$\varepsilon_U \equiv K_\theta v_{E\theta} L_{//} / v_{Te} \tag{29d}$$

- one related to the inductive field $\partial\delta A/\partial t$

$$\varepsilon_A \equiv \varepsilon_T^{-1}\ eL_{//}/T_e\ \partial\delta A/\partial t \approx \delta_I^2/\rho_e L_S\ L_{//}/v_{Te}\ \partial\delta_I/\delta_I\partial t \tag{29e}$$

Thus, the following orderings are satisfied

$$\frac{T(F)}{v_{//}[F, \overline{\psi}]} \sim \varepsilon_T \tag{30a}$$

$$\frac{[F, \overline{U}]}{[F, \overline{\psi}]} \sim \varepsilon_U \tag{30b}$$

$$\frac{e\, v_{//} \dfrac{\partial\delta A}{\partial t} \dfrac{\partial F}{\partial H}}{v_{//}[F, \overline{\psi}]} \sim \varepsilon_A\, \varepsilon_T \tag{30c}$$

This allows to develop the electron distribution function as

$$F \equiv \hat{F} + F_T + F_A + F_U + F_{TA} + F_{TU} + \ldots \tag{31}$$

where $\hat{F}$ is the lowest order distribution function and the subscripts indicate the order of the perturbation (for instance, F_T is of order one in ε_T). Introducing this development in the kinetic equation (17), we can separate the resolution in three identical elementary sub-problems consisting of the ensuing couples of equations

$$[\hat{F}, \overline{\psi}] = 0 \tag{32a}$$

$$v_{//}[F_T, \overline{\psi}] = T(\hat{F}) \tag{32b}$$

$$[F_A, \overline{\psi}] = 0 \tag{33a}$$

$$v_{//}[F_{TA}, \overline{\psi}] - ev_{//}(\partial\delta A/\partial t)(\partial\hat{F}/\partial H) = T(F_A) \tag{33b}$$

$$-[\hat{F}, \overline{U}] + [F_U, \overline{\psi}] = 0 \tag{34a}$$

$$v_{//}[F_{TU}, \overline{\psi}] - v_{//}[F_T, \overline{U}] = T(F_U) \tag{34b}$$

which determine the functions $\hat{F}$, F_T, F_A and F_U.

The first equation of each couple is used to determine the shape of the corresponding distribution function ($\hat{F}$ for (32), F_A for (33) and F_U for (34)). Introducing the averaging operator $\langle \ldots \rangle_\psi$ as

$$\langle Z \rangle_\psi = \oint_\psi \frac{du}{2\pi} \frac{Z(\overline{\psi}, u)}{G(\overline{\psi},u)} \Bigg/ \oint_\psi \frac{du}{2\pi} \frac{1}{G(\overline{\psi},u)} \tag{35}$$

(where the contour in the integral indicates an integral over u between two successive turning points (if any), defined by $G(\overline{\psi},u) = 0$ (see Eq. (15)), on a magnetic surface labelled by $\overline{\psi}$), the second equation is averaged to adjust arbitrary functions arising from the resolution of the first equation, consistently with transport. It must be stressed that this procedure is generally replaced in the litterature [1,10-12] by approximate arguments that only lead to a partial release of the arbitrariness. Thanks to the assumption that the lowest order solution $\hat{F}$ is not far from a local maxwellian distribution function, Eq. (32b) allows to determine the transport induced perturbation F_T. The three couples of equations are solved according to the above technique: Eqs. (32) in **III.B** and **C**, Eqs. (33) in **III.D** and Eqs. (34) in **III.E.** The modular structure of the resolution greatly simplifies the resolution of Eq. (17) and considerably clarifies the influence of each physical mechanism.

B. Lowest order solution

Eq. (32a) implies that $\hat{F}$ is a function of H, and $\overline{\psi}$ (it does not depend on the sign of the parallel velocity because the equilibrium current is neglected in the layer). Assuming that $\hat{F}$ is not far from a maxwellian, it can be written under the form:

$$\hat{F}(H,\overline{\psi}) = \frac{N(\overline{\psi})}{(2\pi m_e T_e(\overline{\psi}))^{3/2}} \exp\left\{-\frac{H}{T_e(\overline{\psi})}\right\} \tag{36}$$

where

$$N(\overline{\psi}) = n_e(x) \exp\left\{\frac{eU(x)}{T_e(\overline{\psi})}\right\} \approx n_e(x) + n_0 \frac{eU(x)}{T_{e0}} \tag{37}$$

It must be stressed at this point that $N(\overline{\psi})$ and $T_e(\overline{\psi})$ are usually rather arbitrary in the litterature[1,10-12]. In fact, the shape of $N(\overline{\psi})$ and $T_e(\overline{\psi})$ is determined by the transport processes: remarking that $\langle ... \rangle_\psi$ is an annihilator of the operator $[\ldots, \overline{\psi}]$, the flux average of Eq. (32b) yields

$$\langle D(\hat{F}) \rangle_\psi = 0 \tag{38}$$

which leads to

$$\frac{\partial \hat{F}}{\partial \overline{\psi}} = \varepsilon_p \, \delta_I \left. \frac{dF_{eq}}{dr} \right|_{r=r_0} Q(\overline{\psi}) \, Y(\overline{\psi} - \overline{\psi}_s) \tag{39}$$

where $Y(\overline{\psi} - \overline{\psi}_S)$ is an Heaviside function which accounts for plateau effects in the island and $Q(\overline{\psi})$ is such that

$$Q^{-1}(\overline{\psi}) = \oint_{\psi} \frac{du}{2\pi} G(\overline{\psi}, u) \tag{40}$$

ε_ρ is the sign of ρ , and $\overline{\psi}_S$ is the value of $\overline{\psi}$ at the separatrix. Using the expression (11) for D(F) and separating the energy dependent terms, the following result is obtained[7]

$$\begin{cases} \dfrac{dN}{d\overline{\psi}} = \varepsilon_\rho\, \delta_I \left.\dfrac{dN_{eq}}{dr}\right|_{r=r_0} Q(\overline{\psi})\, Y(\overline{\psi} - \overline{\psi}_s) \\ \dfrac{dT_e}{d\overline{\psi}} = \varepsilon_\rho\, \delta_I \left.\dfrac{dT_{e,eq}}{dr}\right|_{r=r_0} Q(\overline{\psi})\, Y(\overline{\psi} - \overline{\psi}_s) \end{cases} \tag{41}$$

C. Perturbation induced by transport

Once the function $\hat{F}$ is calculated, Eq. (32b) provides the perturbed function F_T. This equation may be written

$$\frac{\partial F_T}{\partial u}(H,\mu,\varepsilon_{//},\overline{\psi},u) = -4\, \varepsilon_\rho \frac{D\, L_{//}}{\delta_I\, v_{//}} \left.\frac{dF_{eq}}{dr}\right|_{r=r_0} \frac{\partial}{\partial \overline{\psi}}[G(\overline{\psi},u)Q(\overline{\psi})]\, Y(\overline{\psi} - \overline{\psi}_s) \tag{42}$$

where $[Z,\overline{\psi}] = -G(\overline{\psi},u)/4L_{//}\,(\partial Z/\partial u)_{\overline{\psi}}$ has been used. D, F_{eq} and $v_{//}$ are functions of H, μ, $\varepsilon_{//}$ and $\overline{\psi}$. Note that the function F_T is an odd function of $\varepsilon_{//}$ and is in quadrature of phase with the vector potential perturbation.

D. Perturbation associated with the inductive field

Eq. (33a) implies that the function F_A does not depend on u=mθ+nφ. This function is obtained by averaging Eq. (33b)

$$\langle T(F_A)\rangle_\psi = -e\, v_{//} \left\langle \frac{\partial \delta A}{\partial t} \right\rangle_\psi \frac{\partial \hat{F}}{\partial H} \tag{43}$$

which implies that F_A is an odd function of $v_{//}$. Two cases are then to be considered:

- *resistive regime* $v_C \gg v_D = D_e/\delta_I^2$

In this case $T(F_A) = C(F_A) = -v_C F_A$ and within the frame of the constant A approximation

$$F_A\left(H,\mu,\varepsilon_{//},\overline{\psi}\right) = -\frac{e\,v_{//}}{\nu_C\,T_e}\,\hat{F}\left\langle\frac{\partial\delta A}{\partial t}\right\rangle_\psi = -4\,\frac{R(\overline{\psi})}{S(\overline{\psi})}\,\frac{e\,v_{//}}{\nu_C\,T_e}\,\hat{F}\,\frac{\partial\delta_I}{\delta_I\partial t}\,\widetilde{A}(r_0,t) \tag{44}$$

where

$$R(\overline{\psi}) = \oint_\psi \frac{du}{2\pi}\,\frac{\cos u}{G(\overline{\psi},u)} \tag{45}$$

$$S(\overline{\psi}) = \oint_\psi \frac{du}{2\pi}\,\frac{1}{G(\overline{\psi},u)} \tag{46}$$

- *viscous regime* $\nu_C \ll \nu_D = D_e/\delta_I^2$

In this case $T(F_A) = D(F_A)$ and within the frame of the constant A approximation

$$F_A\left(H,\mu,\varepsilon_{//},\overline{\psi}\right) = L(\overline{\psi})\,\frac{\delta_I^2}{D}\,\frac{e\,v_{//}}{T_e}\,\hat{F}\,\frac{\partial\delta_I}{\delta_I\partial t}\,\widetilde{A}(r_0,t) \tag{47}$$

with

$$\frac{\partial L(\overline{\psi})}{\partial\overline{\psi}} = Q(\overline{\psi})\,W(\overline{\psi}) \tag{48}$$

$$W(\overline{\psi}) = \oint_\psi \frac{du}{2\pi}\,\cos u\,G(\overline{\psi},u) \tag{49}$$

$$\frac{\partial W(\overline{\psi})}{\partial\overline{\psi}} = 4\,R(\overline{\psi}) \tag{50}$$

E. Perturbation induced by the electric drift

Since $\hat{F}$ is a flux function, Eq. (34a) implies that

$$F_U = -\frac{\partial\hat{F}}{\partial\overline{\psi}}\left(\overline{U} - K(\overline{\psi})\right) \tag{51}$$

where $K(\overline{\psi})$ must be calculated by averaging Eq. (34b) over flux surfaces

$$\left\langle T(F_U)\right\rangle_\psi = \left\langle v_{//}\left[\overline{U},F_T\right]\right\rangle_\psi \tag{52}$$

Note that F_U is an odd function of $v_{//}$. Again two cases occur:

- resistive regime $v_C \gg v_D = D_e/\delta_I^2$

Since F_T is of the order of $\varepsilon_D = D_e L_{//}/\delta_I^2 v_{Te}$ and $T(F_U) = - v_C F_U$, at lowest order in $\varepsilon_D/\varepsilon_C = D_e/\delta_I^2 v_C$, Eq. (52) implies that

$$\langle F_U \rangle_\psi = 0 \tag{53}$$

so that

$$K(\overline{\psi}) = \langle \overline{U} \rangle_\psi \tag{54}$$

- viscous regime $v_C \ll v_D = D_e/\delta_I^2$

In this case, $K(\overline{\psi})$ is determined by the equation

$$\langle D(F_U) \rangle_\psi = \langle v_{//} [\overline{U}, F_T] \rangle_\psi \tag{55}$$

and it can be shown (see appendix) that

$$Q(\overline{\psi})\, K(\overline{\psi}) = \int_{+\infty}^{\overline{\psi}} d\overline{\psi}'\, Q^2(\overline{\psi}') \oint_{\overline{\psi}'} \frac{du}{2\pi}\, G(\overline{\psi}';u)^2 \frac{\partial}{\partial \overline{\psi}'}\left(\frac{\overline{U}}{G(\overline{\psi}',u)}\right) \tag{56}$$

Three particular situations may be considered:

- The case of an electric potential which is in quadrature of phase with the vector potential perturbation δA. In both resistive (54) and viscous (56) regimes, it implies that $K(\overline{\psi})=0$, so that

$$F_U = -\frac{\partial \hat{F}}{\partial \overline{\psi}} \overline{U} \tag{57}$$

- The case of an electric potential which is a flux function, that corresponds to an electric drift velocity tangent to flux surfaces. It turns out that, for both resistive and viscous regimes again,

$$F_U = 0 \tag{58}$$

so that no effect is due to the transverse electric drift.

- The case of a constant radial electric field, which corresponds to a poloidal drift velocity, i.e. $\overline{U} = \delta_I\, d\overline{U}/dr|_{r=r_0}\, \rho$

In the resistive regime, Eqs. (51) and (54) yield

$$F_U = -\frac{\partial \hat{F}}{\partial \overline{\psi}}\, \delta_I \frac{\partial \overline{U}}{\partial r}\Big|_{r=r_0} (\rho - \langle \rho \rangle_\psi) \tag{59}$$

In the viscous regime, Eq. (56) implies that $K(\overline{\psi})=0$ so that (after Eq. (51))

$$F_U = -\frac{\partial \hat{F}}{\partial \overline{\psi}} \delta_I \frac{\partial \overline{U}}{\partial r}\Big|_{r=r_0} \rho \tag{60}$$

IV. CALCULATION OF THE ELECTRIC POTENTIAL

The electric potential U (or its dimensionless associated quantity $\overline{U} \equiv U / 2v_{//}\tilde{A}(r_0,t)$) is calculated self-consistently using the ion response. As in **II.C**, the cases of small and large islands must be considered separately.

A. Small island width ($\delta_I \ll \rho_S$)

Since no electron density fluctuation is associated with F_T, F_A, and F_U, which are all even functions of $v_{//}$, the definition of N provides the electron density fluctuation

$$N = n_{eq} + \tilde{n}_e + n_0 \frac{e\left(U_{eq} + \tilde{U}\right)}{T_e} = N_{eq} + \tilde{N} \tag{61}$$

Using the ion density fluctuation Eq. (19) and the quasi-electroneutrality constraint Eq. (23), the following expression for the electric potential is obtained

$$U = U_{eq} + \tilde{U} = U_{eq} + \frac{\tau}{1+\tau} \frac{T_e}{en_0} \left(N\text{-}N_{eq}\right) \tag{62}$$

Given that N is a flux function , $U_{eq} \simeq \delta_I \left(dU_{eq}/dr\right)_{r=r_0} \rho$ and $N_{eq} \simeq \delta_I \left(dN_{eq}/dr\right)_{r=r_0} \rho$, Eq. (62) together with Eqs. (58), (59) and (60) provides the response F_U (see sec. **V**).

B. Large island width ($\delta_I \gg \rho_S$)

For large island width, the ion response is given by Eq. (20). This fluid equation is solved perturbatively with a technique similar to that applied to electrons in **III.** At lowest order in ρ_S^2/δ_I^2, Eq. (20) is

$$\nabla.\left(n_i v_E\right) = D_i \Delta\left(n_i + n_0 \frac{e_i U}{T_i}\right) = v_{//}\left[\overline{U}, n_i\right] \tag{63}$$

If $D_i/\delta_I^2 \gg K_\theta v_{E\theta} \approx \omega$, it is found again that the ion response is adiabatic. We will discard this case and consider the opposite regime: $D_i/\delta_I^2 \ll K_\theta v_{E\theta} \approx \omega$. Eq. (63) at lowest order in $\varepsilon_{Di}/\varepsilon_U = D_i/\delta_I^2 K_\theta v_{E\theta}$ implies that the ion density $\hat{n}_i$ is a function of the electric potential $\hat{U}$. Since $\hat{n}_e + n_0 e\hat{U}/T_{e0}$ is a flux function and $\hat{n}_e = \hat{n}_i$, this means that $\hat{n}_e$ and $\hat{U}$ are flux functions.

Using the two constraints (associated with the average of Eq. (63) at first order in $\varepsilon_{D_i}/\varepsilon_U$ and its equivalent for electrons)

$$\begin{cases} \left\langle D_e \Delta\left(\hat{n}_e + n_0 \frac{e\hat{U}}{T_e}\right)\right\rangle_\psi = 0 \\ \left\langle D_i \Delta\left(\hat{n}_i + n_0 \frac{e_i\hat{U}}{T_i}\right)\right\rangle_\psi = 0 \end{cases} \tag{64}$$

it is found that

$$\begin{cases} \frac{d\hat{n}_e}{d\overline{\psi}} = \varepsilon_p\, \delta_I \left.\frac{dn_{eq}}{dr}\right|_{r=r_0} Q(\overline{\psi})\, Y(\overline{\psi} - \overline{\psi}_s) \\ \frac{d\hat{U}}{d\overline{\psi}} = \varepsilon_p\, \delta_I \left.\frac{dU_{eq}}{dr}\right|_{r=r_0} Q(\overline{\psi})\, Y(\overline{\psi} - \overline{\psi}_s) \end{cases} \tag{65}$$

It has been shown in **III.E** that a potential $\hat{U}(\overline{\psi})$ does not produce any perturbed distribution function F_U and consequently does not induce any parallel current. Thus, it is necessary to investigate the next order in $\varepsilon_{D_i}/\varepsilon_U$.

At next order in $\varepsilon_{D_i}/\varepsilon_U$, Eq (63) reads

$$\frac{d\hat{n}_i}{d\overline{\psi}} v_{/\!/}[\overline{U}_{D_i},\overline{\psi}] - \frac{d\hat{U}}{d\overline{\psi}} v_{/\!/}[n_{D_i},\overline{\psi}] = D_i \Delta\left(\hat{n}_i + \frac{n_0 e_i \hat{U}}{T_{i0}}\right) = v_{/\!/} \frac{d}{d\overline{\psi}}\left(\hat{n}_e + \frac{n_0 e \hat{U}}{T_0}\right)[\overline{U}_{D_i},\overline{\psi}] \tag{66}$$

since N and $\hat{N}$ are flux functions, which implies that $N_{D_i} \equiv n_{D_i} + n_0 e U_{D_i}/T_{e0}$ is also a flux function. This shows, by analogy with the calculation of F_T (see **III.C**) that the potential U_{D_i} associated with the dissipation is such that

$$\left[v_{/\!/}\frac{\partial \overline{U}_{D_i}}{\partial u}\right](\overline{\psi},u) = -\frac{4D_i\, L_{/\!/}}{\delta_I^2}\left(\frac{\left.\frac{dn_{eq}}{n_e dr}\right|_{r=r_0} - \frac{1}{\tau}\frac{e}{T_e}\left.\frac{dU_{eq}}{dr}\right|_{r=r_0}}{\left.\frac{dn_{eq}}{n_e dr}\right|_{r=r_0} + \frac{e}{T_e}\left.\frac{dU_{eq}}{dr}\right|_{r=r_0}}\right)$$

$$\times \frac{1}{Q(\overline{\psi})}\frac{\partial}{\partial \overline{\psi}}[G(\overline{\psi},u)Q(\overline{\psi})]\, Y(\overline{\psi} - \overline{\psi}_S) \tag{67}$$

As this quantity is in phase with the vector potential perturbation, it turns out that the electric potential U_{D_i} is in quadrature of phase with the latter and thus, as a result of Eq. (57), produces a response

$$F_{U_{D_i}} = -\frac{\partial \hat{F}}{\partial \overline{\psi}} \overline{U}_{D_i} \tag{68}$$

This potential is related to perturbed distribution function and current in quadrature of phase with the vector potential. To obtain a non vanishing current in phase with δA, it is then necessary to investigate the next order in ρ_S^2/δ_I^2.

Setting

$$U(\overline{\psi},u) \equiv \hat{U}(\overline{\psi}) + U_{D_i}(\overline{\psi},u) + U_{in}(\overline{\psi},u) \tag{69}$$

where $\hat{U}$ is given by Eq. (65) and $U_{in}/\hat{U}$ is of order one in ρ_S^2/δ_I^2, Eq. (20) provides the potential U_{in}

$$e_i v_{//}\left(\left[\overline{U}_{in}, \hat{N}\right] + \left[\overline{\hat{U}}, N_{in}\right]\right) = -\nabla.\mathbf{j}_{pol} \tag{70}$$

The functions N and $\hat{N}$ are flux functions so that N_{in} is also a flux function. Using

$$\nabla.\mathbf{j}_{pol} = -\frac{n_0 e^2}{T_{e0}} v_{//}\left[\hat{U} + \frac{p_i}{n_0 e_i}, \frac{\rho_S^2}{2}\Delta\overline{\hat{U}}\right] \tag{71}$$

(see Eq. (21)), the following expression of $\overline{U}_{in}$ is then obtained

$$\overline{U}_{in} = \frac{\rho_S^2}{2} \frac{\dfrac{e}{T_e}\left.\dfrac{dU_{eq}}{dr}\right|_{r=r_0} - \tau \left.\dfrac{dp_{eq}}{p_{eq}dr}\right|_{r=r_0}}{\left.\dfrac{dN_{eq}}{n_{eq}dr}\right|} \Delta\overline{\hat{U}} \tag{72}$$

Together with Eqs. (51), (54) and (56), Eq. (72) provides the response $F_{U_{in}}$ (see sec. V).

V. ISLAND STABILITY

In this section, we will restrict ourselves to the case of the constant A approximation. The self-consistency is then expressed by using Eq. (27) where the current $\tilde{j}_{//}$ is calculated with the electron distribution function F

$$-2\mu_0 \delta_I \int_{-1}^{+\infty} d\overline{\psi} \oint_{\overline{\psi}} \frac{du}{2\pi} \frac{\cos u}{G(\overline{\psi},u)} \int d_3\mathbf{p}\, e\, v_{//} F(H,\mu,\varepsilon_{//},\overline{\psi},u) = \Delta' \tilde{A}(r_0,t) \tag{73a}$$

$$\int_{-1}^{+\infty} d\overline{\psi} \oint_{\overline{\psi}} \frac{du}{2\pi} \frac{\sin u}{G(\overline{\psi},u)} \int d_3\mathbf{p}\, e\, v_{//} F(H,\mu,\varepsilon_{//},\overline{\psi},u) = 0 \tag{73b}$$

where it has been used

$$\int_{-1}^{+\infty} d\overline{\psi} \oint_{\mathrm{v}} \frac{du}{2\pi} \frac{\bullet}{G(\overline{\psi},u)} = \frac{1}{2} \int_{0}^{2\pi} \frac{du}{2\pi} \int_{-\infty}^{+\infty} d\rho \,. \tag{74}$$

Equation (73b) is associated with the current perturbation in quadrature of phase with the vector potential perturbation δA, namely those related to F_T (see **III.C**) and, as far as the large island case is concerned, to the part of F_U related to U_{Di} (see **IV.B**). Given Eq. (27), the radial integral of the latter current must vanish. This condition provides the equilibrium radial electric field and consequently (Eq. (12)) the rotation frequency of the island chain, in terms of diamagnetic frequencies.

Equation (73a) is associated with the current perturbation in phase with δA , namely those related to F_A (see **III.D**) and to the remaining of F_U (see **III.E** and **IV**). This self-consistency constraint leads to an evolution equation over the island width. Due to Eq. (27), any in-phase current perturbation which radial integral is zero does not contribute to the island stability.

A. Radial electric field and rotation frequency

In order to treat simultaneously small ($\delta_I \ll \rho_S$) and large ($\delta_I \gg \rho_S$) island width cases, we define:

$$\varepsilon_{SL} \equiv \begin{cases} 0 & \text{if small} \\ 1 & \text{if large} \end{cases} \tag{75}$$

Thus, Eq. (42), (67) and (68) imply that (73b) is verified if

$$\int d_3\mathbf{p}\ D \left.\frac{\partial F_{eq}}{\partial r}\right|_{r=r_0} - \varepsilon_{SL} D_i \left\{ \frac{\left.\frac{dn_{eq}}{n_e dr}\right|_{r=r_0} - \frac{1}{\tau}\frac{e}{T_e}\left.\frac{dU_{eq}}{dr}\right|_{r=r_0}}{\left.\frac{dn_{eq}}{n_e dr}\right|_{r=r_0} + \frac{e}{T_e}\left.\frac{dU_{eq}}{dr}\right|_{r=r_0}} \right\} \int d^3p\ \left.\frac{\partial F_{eq}}{\partial r}\right|_{r=r_0} = 0 \tag{76}$$

which imposes the constraint

$$D_e\left\{ \left.\frac{dn_{eq}}{n_{eq}\,dr}\right|_{r=r_0} + \left.\frac{e\,dU_{eq}}{T_{eq}\,dr}\right|_{r=r_0} + \alpha \left.\frac{dT_{eq}}{T_{eq}\,dr}\right|_{r=r_0} \right\} - \varepsilon_{SL} D_i \left\{ \left.\frac{dn_{eq}}{n_{eq}\,dr}\right|_{r=r_0} - \frac{1}{\tau} \left.\frac{e\,dU_{eq}}{T_{eq}\,dr}\right|_{r=r_0} \right\} = 0 \tag{77}$$

$$\text{with} \quad \begin{cases} D_e \equiv \int d^3p\ D\, F_{eq} \Big/ \int d^3p\ F_{eq} \\ \alpha \equiv \int d^3p\ \left(\frac{H}{T} - \frac{3}{2}\right) D\, F_{eq} \Big/ \int d^3p\ D\, F_{eq} \end{cases} \tag{78}$$

Eq. (77) fixes the value of the radial electric field in the island frame. We will suppose in the following that the ion diffusion coefficient is much smaller than the electron diffusion coefficient, i.e. $D_i \ll D_e$. In this case, in both small and large island width cases, the radial electric field is determined by

$$\left.\frac{dn_{eq}}{n_{eq}dr}\right|_{r=r_0} + \left.\frac{e\,dU_{eq}}{T_{eq}dr}\right|_{r=r_0} + \alpha \left.\frac{dT_{eq}}{T_{eq}dr}\right|_{r=r_0} = 0 \tag{79}$$

providing (via Eq. (12)) the rotation frequency of the island chain in the usual reference frame where there is no radial electric field

$$\omega = \omega^*_{n_e} + \alpha\,\omega^*_{T_e} \tag{80}$$

where $\omega^*_{n_e} \equiv K_\theta T_e/eB_0\,\partial n/n\partial r$ and $\omega^*_{T_e} \equiv K_\theta T_e/eB_0\,\partial T_e/T_e\partial r$ are the density and temperature electron diamagnetic frequencies.

According to its definition (Eq. (78)), α can be positive or negative, depending on the background turbulence (i.e. on the dependency of D on H and μ). However, most of the models provide positive values for α. Thus, we will restrict the following discussion to this case and more specifically to the case of a background of magnetic turbulence, for which $\alpha=1/2$.

B. Island growth rate

Four cases have to be considered

1. *Resistive regime and small island width* $(v_C \gg v_D \equiv D_e/\delta_I^2$ *and* $\delta_I \ll \rho_S)$

For resistive regime and small island width, the constraint (73a) and the expressions (39), (44), (58), (59), (62) and (79) lead to

$$F_U = -\frac{\partial \hat{F}}{\partial \psi}\,\frac{\delta_I}{2v_{//}\tilde{A}(r_0,t)}\left\{\left.\frac{dU_{eq}}{dr}\right|_{r=r_0} - \frac{\tau}{1+\tau}\,\frac{T_e}{e}\,\left.\frac{dN_{eq}}{n_0 dr}\right|_{r=r_0}\right\}\,(\rho - \langle\rho\rangle_\psi) \tag{81}$$

and

$$K_R\,\frac{\mu_0}{\eta}\,\frac{\partial \delta_I}{\partial t} = \Delta' + K_{dia}\,\frac{\beta_p^*}{\delta_I}\,\frac{\alpha\eta_e(1+\tau+\alpha\eta_e)}{(1+\tau)^2(1+\eta_e)^2} \tag{82}$$

or, given Eqs. (12) and (80),

$$K_R\,\frac{\mu_0}{\eta}\,\frac{\partial \delta_I}{\partial t} = \Delta' + K_{dia}\,\frac{\beta_p^*}{\delta_I}\,\frac{(\omega-\omega^*_{n_e})(\omega-\omega^*_{n_i})}{\omega_p^{*2}} \tag{83}$$

where

$$\beta_p^* \equiv \frac{2\,\mu_0\, n_0\, T_{e0}\,(1+\tau)}{B_0^2}\left(\frac{L_s}{L_p}\right)^2 \tag{84}$$

$$\tau \equiv T_i/T_e \tag{85}$$

$$\frac{1}{\eta} \equiv \frac{e^2}{m_e}\int d^3p\, \frac{2\, v_{/\!/}^2/v_{Te}^2}{v_C}\, F_{eq} \tag{86}$$

$$\eta_e = d\mathrm{Log}T_e/d\mathrm{Log}n_e \tag{87}$$

$$\omega_{x_j}^* = K_\theta T_j/e_j B_0\, \partial x/x\partial r \quad (x = n\,,\, p\,;\, j = e\,,\, i) \tag{88}$$

$$\omega_p^* \equiv \omega_{p_e}^* - \omega_{p_i}^* \tag{89}$$

$$K_R = 8\int_{-1}^{+\infty} d\overline{\psi}\, \frac{R^2(\overline{\psi})}{S(\overline{\psi})} = 1.7 \tag{90}$$

$$K_{dia} = 8\int_{1}^{+\infty} d\overline{\psi}\, \frac{R(\overline{\psi})\,Q(\overline{\psi})}{S(\overline{\psi})} = 1.6 \tag{91}$$

The first part of Eqs. (82) and (83) corresponds to the resistive growth rate obtained by Rutherford[23]. The r.h.s. corresponds to the diamagnetism which is found to be destabilizing, since we restrict α to positive values (of course, diamagnetism may be stabilizing for $\alpha<0$). The result of Samain[7] is therefore recovered. Setting the l.h.s to zero, one can show that it exists a marginal stability state for

$$\beta_p^* = -\frac{\Delta'\,\delta_I\,\omega_p^{*2}}{K_{dia}\left(\omega - \omega_{n_e}^*\right)\left(\omega - \omega_{n_i}^*\right)} = -\frac{\Delta'\,\delta_I\,(1+\tau)^2\,(1+\eta_e)^2}{K_{dia}\,\alpha\eta_e\,(1+\tau+\alpha\eta_e)} \tag{92}$$

if this threshold value is positive.

2. *Resistive regime and large island width* $\left(v_C \gg v_D \equiv D_e/\delta_I^2 \text{ and } \delta_I \gg \rho_S\right)$

For resistive regime and large island width, the constraint (73a) and the expressions (39), (44), (51), (54), (65) (72) and (79) lead to

$$F_U = -\frac{\partial \hat{F}}{\partial \overline{\psi}} \frac{1}{2\, v_{//} \tilde{A}(r_0,t)} \frac{\rho_S^2}{2} \frac{1+\alpha\eta_e + \tau(1+\eta_e)}{\alpha\eta_e} \Delta\hat{U} \tag{93}$$

(where it has been used $\langle \Delta\hat{U} \rangle_\psi \approx 0$), and

$$K_R \frac{\mu_0}{\eta} \frac{\partial \delta_I}{\partial t} = \Delta' - K_{flr} \frac{\beta_p^*}{\delta_I} \frac{(1+\tau+\eta_e(\tau+\alpha))(1+\alpha\eta_e)}{(1+\tau)(1+\eta_e)^2} \frac{\rho_S^2}{\delta_I^2} \tag{94}$$

or, given Eqs. (12) and (80), :

$$K_R \frac{\mu_0}{\eta} \frac{\partial \delta_I}{\partial t} = \Delta' - K_{flr} \frac{\beta_p^*}{\delta_I} \frac{\omega\,(\omega - \omega_{p_i}^*)}{\omega_p^*\, \omega_{p_e}^*} \frac{\rho_S^2}{\delta_I^2} \tag{95}$$

with

$$\rho_S \equiv \sqrt{2 m_i T_e} \,/\, e_i B_0 \tag{96}$$

$$K_{flr} = 64 \int_1^{+\infty} d\overline{\psi}\; Q^2 (R\text{-}QSW) = 1.4 \tag{97}$$

where the relations $dW/d\overline{\psi}=4R$ and $dQ/d\overline{\psi}=-4Q^2S$ have been used. The diamagnetism is then found to be stabilizing through the ion inertia and FLR effects, since we restrict α to positive values (of course, these effects may be destabilizing for $\alpha<0$). A result similar to Eq. (82) has been found by Smolyakov[1,10].

3. *Viscous regime and small island width* $(\nu_C \ll \nu_D \equiv D_e/\delta_I^2$ *and* $\delta_I \ll \rho_S)$

For viscous regime and small island width, it turns out , after Eqs. (58), (60), (62) and (74), that the radial integral of the current associated with F_U vanishes:

$$F_U = -\frac{\partial \hat{F}}{\partial \overline{\psi}} \frac{\delta_I}{2\, v_{//} \tilde{A}(r_0,t)} \left\{ \left.\frac{dU_{eq}}{dr}\right|_{r=r_0} - \frac{\tau}{1+\tau} \left.\frac{T_e dN_{eq}}{e n_0 dr}\right|_{r=r_0} \right\} \rho \propto |\rho|\, Q(\overline{\psi})\, Y(\overline{\psi} - \overline{\psi}_s) \tag{98}$$

This implies (see remark at the begining of V) that diamagnetic effects do not influence the island stability in the viscous regime, as long as the constant A constraint is maintained. Then, the constraint (73a) and Eqs. (39) and (47) lead to the island evolution equation

$$K_v \frac{\mu_0 n_0 e^2}{m_e D_v} \delta_I^2 \frac{\partial \delta_I}{\partial t} = \Delta' \tag{99}$$

where D_V is defined by

$$\frac{1}{D_V} \equiv \frac{1}{n_{eq}} \int d^3p \, \frac{m_e v_{//}^2}{T_e D} F_{eq} \tag{100}$$

and

$$K_V = \frac{1}{2} \int_{-1}^{+\infty} d\overline{\psi} \, W^2(\overline{\psi}) \; Q(\overline{\psi}) = 3.5 \tag{101}$$

This is the growth rate obtained by Kaw and coworkers[25]. The fact that diamagnetic effects do not play any rôle with respect to the island width evolution is the consequence of a general result which states that the radial integral of the current associated with any uniform poloidal drift cancels. This may be shown as follows. The solution $F(H,\mu,\varepsilon_{//},r,u)$ of the equation

$$v_{//} [F \, , \overline{\psi} - \overline{U}] = D(F) \tag{102}$$

where $\overline{U}$ varies linearly with $r(\overline{\psi},u)$, can be deduced from the solution $F^{\circ}(H,\overline{\psi}(r,u))$ of the same equation without electric potential. It is indeed the same with r (resp. $\overline{\psi}$) shifted by

$$\delta r = \frac{\delta_I^2}{4} \frac{d\overline{U}}{dr}\Big|_{r=r} \quad \left(\text{resp.} \quad \delta\overline{\psi} \approx \delta_I \frac{d\overline{U}}{dr}\Big|_{r=r} \rho\right) \tag{103}$$

so that

$$F_U = \delta_I \frac{d\overline{U}}{dr}\Big|_{r=r} \frac{\partial F^{\circ}}{\partial \overline{\psi}} \rho \tag{104}$$

which varies as $Q(\overline{\psi}) \, |\rho|$ radially and thus clearly leads to zero when radially integrated. This general property is violated in two cases

- the case where the collisional friction term is dominant over the electron viscosity. This is the situation which has been previously investigated (**V.B.1**).

- the failure of the constant A approximation for sufficiently large values of $\beta_p = 2\mu_0 p_{eq}/B_p^2$. This case will be studied in details in sec. **VI**.

4. *Viscous regime and large island width* $\left(\nu_C \ll \nu_D \equiv D_e/\delta_I^2 \text{ and } \delta_I \gg \rho_S\right)$

For viscous regime and large island width, Eqs. (51), (56), (65), (72) and (79) yield

$$F_U = - \frac{\partial \hat{F}}{\partial \overline{\psi}} \frac{1}{2 \, v_{//} \tilde{A}(r_0,t)} \frac{\rho_S^2}{2} \frac{1+\alpha\eta_e + \tau(1+\eta_e)}{\alpha\eta_e} \tag{105}$$

:

$$\times\left\{\Delta\hat{U} - \frac{1}{Q(\overline{\psi})}\int_{+\infty}^{\overline{\psi}} d\overline{\psi}' Q^2(\overline{\psi}')\oint_{\overline{\psi}'} \frac{du}{2\pi} G^2 \frac{\partial}{\partial\overline{\psi}}\left(\frac{\Delta\hat{U}}{G}\right)\right\}$$

which, together with Eqs. (47) and (73a) leads to

$$K_V \frac{\mu_0 n_0 e^2}{m_e D_V} \delta_I^2 \frac{\partial\delta_I}{\partial t} = \Delta' - K'_{flr} \frac{\beta_p^*}{\delta_I} \frac{(1+\tau+\eta_e(\tau+\alpha))(1+\alpha\eta_e)}{(1+\tau)(1+\eta_e)^2} \frac{\rho_S^2}{\delta_I^2} \tag{106}$$

or, given Eqs. (12) and (80),

$$K_V \frac{\mu_0 n_0 e^2}{m_e D_V} \delta_I^2 \frac{\partial\delta_I}{\partial t} = \Delta' - K'_{flr} \frac{\beta_p^*}{\delta_I} \frac{\omega(\omega-\omega_{p_i}^*)}{\omega_p^* \omega_{p_e}^*} \frac{\rho_S^2}{\delta_I^2} \tag{107}$$

with

$$K'_{flr} = 64 \int_1^{+\infty} d\overline{\psi}\, Q^2 (R\text{-}QSW)\left\{1 + 8Q\left[2QS(W_s\text{-}W) + R\right]\right\} = 3.3 \tag{108}$$

where $W_S = W(\overline{\psi}_S)$. The conclusion in the viscous regime is the same as in the resistive one (**V.B.2**): depending on the value of α, τ and η_e, ion inertia and FLR effects can stabilize or destabilize the islands, but are strictly stabilizing for $\alpha>0$, i.e. in the usual case.

:

VI. RELEASE OF THE CONSTANT A CONSTRAINT

In the previous section, the island stability has been studied in the frame of the constant A approximation. In particular, it has been shown in **V.B.3** that diamagnetic effects do not influence the small island stability in the viscous regime $(\delta_I \ll \rho_S$ and $\nu_C \ll \nu_D \equiv D_\zeta/\delta_I^2)$, so that, contrary to the resistive regime $(\nu_C \gg \nu_D \equiv D_\zeta/\delta_I^2)$, it cannot exist any finite β_p^* threshold in this regime. However, it is well known[5,6] that for large values of β_p^* the constant A approximation is no longer valid. The aim of this section is thus to calculate β_p^* thresholds outside of the constant A approximation, particularly in the viscous regime for small island width, which is marginal with respect to diamagnetic stabilization in the frame of the constant A approximation. For the sake of simplicity, we restrict ourselves to the case of large wave numbers, for which $\Delta' = -2|K_\theta|$.

:

A. Numerical procedure

In the case where the potential $\bar{A}(\rho)$ is allowed to vary in the layer, the full Ampère law (Eq. (24)) has to be solved. Using the normalized quantities $a(\rho,t)=\bar{A}(\rho,t)/\bar{A}(r_0,t)$ and $\rho = (r - r_0) / \delta_I$, it can be written as

$$\frac{\partial^2 a}{\partial \rho^2} + \bar{\beta}_p \; \sigma(a,\rho) \; a = -\left(K_\theta \, \delta_I\right)^2 a \tag{109}$$

where

$$\bar{\beta}_p = 16 \, \beta_p^* \frac{\alpha\eta_e\left(1+\tau+\alpha\eta_e\right)}{(1+\tau)^2\left(1+\eta_e\right)^2} \tag{110}$$

and, as far as the case of small island width $(\delta_I \ll \rho_S)$ is concerned,

$$\sigma(a(\rho),\rho)\, a(\rho) = \int_{\bar{\psi}>\bar{\psi}_s} \frac{du}{2\pi} \cos u \; Q(2\rho^2 + a(\rho)\cos u)\left\{|\rho| - 1/4S(2\rho^2 + a(\rho)\cos u)\right\} \tag{111a}$$

in the resistive regime $\left(\nu_C \gg \nu_D \equiv D_e/\delta_I^2\right)$, and

$$\sigma(a(\rho),\rho)\, a(\rho) = \int_{\bar{\psi}>\bar{\psi}_s} \frac{du}{2\pi} \cos u \, |\rho| \, Q(2\rho^2 + a(\rho)\cos u) \tag{111b}$$

in the viscous regime $\left(\nu_C \ll \nu_D \equiv D_e/\delta_I^2\right)$ (the integration is performed at constant ρ and outside the island, due to plateau effects). The case of large island width may be treated similarly, but this would overweight the paper since the physical conclusions are qualitatively the same. Eq. (109) is similar to a Schrödinger equation with a potential σ which depends not only on ρ but also on $a(\rho)$.

Integrating Eq. (109) over the layer $(|\rho| \le \rho_L)$, one finds the $\bar{\beta}_p$ threshold

$$\bar{\beta}_p = -K_\theta\delta_I \frac{a(\rho_L) + K_\theta\delta_I \int_0^{\rho_L} d\rho \, a(\rho)}{\int_0^{\rho_L} d\rho \, \sigma(\rho)\, a(\rho)} \tag{112a}$$

The non linear layer half width ρ_L corresponds to the radius ρ where the asymptotic behavior $a(\rho) \propto \exp(-|K_\theta\delta_I\rho|)$ is reached. The constant A approximation consists in considering that $a(\rho) \approx 1$ in the layer, which is valid if and only if $K_\theta\delta_I\rho_L \ll 1$. In this case, one obtains the constant-A $\bar{\beta}_p$ threshold

$$\bar{\beta}_p(a=1) = \frac{-K_\theta\delta_I}{\int_0^{\rho_L} d\rho \, \sigma^{(0)}(\rho)} \approx \frac{-K_\theta\delta_I}{\int_0^{+\infty} d\rho \, \sigma^{(0)}(\rho)} \tag{112b}$$

with $\sigma^{(0)}(\rho)$ given by Eqs. (111a and b), where $a(\rho)=1$. Together with Eqs. (110) and (111a) (where $a(\rho)=1$), this expression effectively leads to the resistive threshold provided by Eq. (92). Moreover, given the expression for σ in the viscous regime (Eq. (111b) where $a(\rho)=1$), the radial integral of $\sigma^{(0)}(\rho)$ vanishes and β_p is infinite, which yields the result established in **V.B.3**: diamagnetism does not affect small island stability in the viscous regime as long as the constant A constraint is maintained. However, the validity of the constant A approximation: $K_\theta\delta_I\rho_L \ll 1$, must be questionned. Indeed, $\sigma^{(0)}(\rho)$ asymptotically scales as $1/K_\theta\delta_I$, so that $K_\theta\delta_I\rho_L$ is not necessarily much smaller than 1. The exact Ampère equation (109) then has to be solved numerically, with the constant A constraint released. The numerical calculation consists in solving this equation, so that the dependency of σ on $a(\rho)$ is removed, by means of the following iterative procedure.

The first stage consists of the (analytic) computation of a potential $\sigma^{(0)}(\rho)$ in the frame of the constant A approximation (i.e. with $a(\rho) = a^{(0)}(\rho) = 1$ in Eq. (111a or b)).

At the second stage, this potential $\sigma^{(0)}(\rho)$ is injected in Eq. (109) which, being integrated numerically, provides a solution $a^{(1)}(\rho)$ and a $\beta_p^{(1)}$ threshold. Then, the varying solution $a^{(1)}(\rho)$ is injected in Eq. (111), allowing the computation of an iterated potential $\sigma^{(1)}(\rho)$. (The problem at this stage is to compute the integral of Eq. (111), the relationship between $\overline{\psi}$ and ρ being no more invertible analytically outside of the constant A approximation (see Eq. (7)). This procedure, when iterated, is found to be rapidly convergent and yields the desired β_p threshold.

Before going to numerical results, we should mention a few things about the similarity of Eq. (109) with a Schrödinger equation. Once the above procedure is applied, Eq. (109) becomes, at the nth stage,

$$-\frac{d^2a^{(n)}}{d\rho^2} + \beta_p^{(n)}\,\sigma^{(n-1)}(\rho)\,a^{(n)}(\rho) = -(K_\theta\delta_I)^2\,a^{(n)}(\rho) \tag{113}$$

that is merely a one dimensional stationary Schrödinger equation, where $-(K_\theta\delta_I)^2$ and $\beta_p^{(n)}\,\sigma^{(n-1)}(\rho)$ respectively stand for the energy and the potential. Usually, given the potential, an energy spectrum is obtained from this equation. However, our goal being to compute $\beta_p^{(n)}$ thresholds, we reverse the problem. Indeed, we fix the energy $-(K_\theta\delta_I)^2$ and then adjust $\beta_p^{(n)}$ so that the solution $a^{(n)}(\rho)$ (calculated via a shooting code) corresponds to the fundamental eigenfunction of our Schrödinger equation. Note that we choose the fundamental eigenvalue for $\beta_p^{(n)}$ because it is the lowest one, i.e. the most easily excited solution. Nevertheless, this solution, which is associated with the zero-node eigenfunction (i.e. the closest from the usual constant A approximation), is not the only one. Actually, given the energy $-(K_\theta\delta_I)^2$, it exists a $\beta_p^{(n)}$ spectrum associated with a set of eigenfunctions $a^{(n)}(\rho)$. It is interesting to remark that, even if these solutions are a priori harder to excite, due to their higher $\beta_p^{(n)}$ threshold (for a given

$K_\theta \delta_I$), they could be excited at very low $K_\theta \delta_I$ (i.e. small δ_I) and then non linearly interact with other modes. Thus, these apparently harmless solutions, which have never been considered, could strongly affect island stability. Nonetheless, they are far beyond the scope of this paper.

B. Numerical results

The release of the constant A constraint is expected to strongly modify the β_p threshold for small islands ($\delta_I \ll \rho_S$) in the viscous regime ($\nu_C \ll \nu_D \equiv D_\sigma/\delta_I^2$), since the constant A threshold is found to be infinite. Nevertheless, it is interesting, first, to check the validity of this approximation for the resistive regime ($\nu_C \gg \nu_D \equiv D_\sigma/\delta_I^2$) which is usually studied in this frame. The numerical procedure converges very fast; an example is shown in fig. 1 for $K_\theta \delta_I$=0.1 (the fast convergence also appears on β_p : $\beta_p^{(1)} \approx 2.61$, $\beta_p^{(2)} \approx 2.62$). Note that the radial interval is not the same in figures 1.(a) and 1.(b) and that the shape of $\sigma(\rho)$ and $a(\rho)$ is typical of resistive regime for small island width. The evolution of the β_p threshold as a function of $|K_\theta| \delta_I$ is shown in table I. It must be compared with the threshold obtained within the frame of the constant A approximation. Setting K_{dia}=1.6 (Eq. (91)) and $\Delta'=-2|K_\theta|$ in Eq. (92) and using Eq. (110) yields $\beta_p=20|K_\theta|\delta_I$ for the constant A threshold. The latter result obviously underestimates the more realistic threshold provided by the release of the constant A constraint. This fact is illustrated in fig. 2, where the typical values α=1/2, τ=1 and η_c=2 are chosen to give the β_p^* threshold as a function of $|K_\theta|\delta_I$. It is clear from fig. 2 that the constant A approximation fails for the highest $|K_\theta|\delta_I$ values: the destabilizing influence of diamagnetism may be strongly attenuated by the release of the constant A constraint. This has important practical consequences in the high β_p^* regimes of present day tokamaks. Indeed, consider a small island ($\delta_I \ll \rho_S$) which is growing in the resistive regime, in a plasma with a given β_p^*. It evolves in the fig. 2 diagram along an horizontal line, from the left to the right, until it reaches the $|K_\theta|\delta_I$ threshold (i.e. until it crosses the continuous or dashed line, depending on whether the constant A constraint is released or not). Such an island, if it saturates in this regime (it could fall in the large island regime ($\delta_I \gg \rho_S$) before saturation), obviously saturates to smaller δ_I if the constant A constraint is released than if it is not, provided that the value of β_p^* is sufficient, as illustrated for β_p^*=10 in fig. 2 (dotted lines).

As mentioned above, in the viscous regime, for small island width, the constant A approximation is unable to provide any finite β_p threshold: the radial integral over the layer of the potential $\sigma^{(0)}(\rho)$ calculated under this constraint vanishes. The release of the constant A constraint turns out to be essential to remove this problem. Indeed, the numerical calculations show that the small island viscous regime exhibits hollow profiles of the vector potential that strongly affect the denominator of Eq. (112a), which, no more being equal to zero, leads to finite β_p threshold. Again, the convergence of the iterative procedure is very fast; an example is shown in fig. 3 (the convergence also appears on β_p: $\beta_p^{(1)} \approx 20$, $\beta_p^{(2)} \approx 21.2$ and $\beta_p^{(3)} \approx 21.1$; this example corresponds to $|K_\theta|\delta_I = 0.1$). The shape of $\sigma(\rho)$ and $a(\rho)$ is typical of small islands in the viscous regime. The evolution of the β_p^* threshold as a function of $|K_\theta|\delta_I$ is shown in table II. It is obviously different from the constant A threshold since it is finite. On the other hand, it is interesting to compare it with its equivalent in the resistive regime (see table I). The viscous threshold is clearly greater than the resistive one. This situation is illustrated in fig. 4, where the typical values α=1/2, τ=1 and η_e=2 are chosen to provide the β_p^* threshold as a function of $|K_\theta|\delta_I$. The comparison of the viscous and resistive thresholds indicates that small islands may be sustained by diamagnetic effects in the viscous regime, even if the required β_p^* threshold is much higher than in the resistive regime. However, one should bear in mind that, for given mode (i.e. K_θ) and background plasma (i.e. ν_C and D_e), the viscous regime is related to smaller island width δ_I, i.e. smaller $|K_\theta|\delta_I$, than the resistive one. Thus, one has to be cautious when making comparisons since resistive and viscous regimes do not compete with each other for a given mode. Finally, one can remark that the extrapolation of the numerically calculated values suggests, in fig; 4, that the β_p^* threshold does not vanish for $|K_\theta|\delta_I = 0$ (similar result is found in ref. 9). This would mean that for low β_p^* plasmas (i.e. for β_p^* lesser than ≈3.5 for the parameters of fig. 4), diamagnetism could be always stabilizing in the viscous regime, so that any island could not grow nor saturate unless it is created in the resistive regime (i.e. with large enough δ_I, so that $\nu_C \gg \nu_D \equiv D_e/\delta_I^2$), by a sufficiently large external perturbation (large error fields for instance).

VII. ISLAND EVOLUTION

Summarizing the above results, the effects of the diamagnetism are indicated in table III. Diamagnetism is then found to be destabilizing for small islands ($\delta_I \ll \rho_S$) and stabilizing for large islands ($\delta_I \gg \rho_S$) (at least for the case α>0 that we are considering here: see discussion in V.A). Unfortunately, it is not possible to describe in details the transition near ($\delta_I \approx \rho_S$). The island behavior may be tentatively described by interpolating the results obtained in the two limits $\delta_I \ll \rho_S$ and $\delta_I \gg \rho_S$.Considering for instance the resistive regime $(\nu_C \gg \nu_D \equiv D_e/\delta_I^2)$ and the the constant A approximation (for which the results are analytical), Eqs. (83) and (95) lead to

$$\frac{\partial \delta}{\partial \zeta} = \gamma_R \begin{cases} +\dfrac{\beta_p^{dia}}{\delta} & \text{if } \delta_I << \rho_S \\ -\dfrac{\beta_p^{flr}}{\delta^3} & \text{if } \delta_I >> \rho_S \end{cases} \tag{114}$$

where

$$\zeta \equiv \frac{\eta}{K_R \mu_0 \rho_s^2} t \tag{115}$$

$$\delta \equiv \frac{\delta_I}{\rho_s} \tag{116}$$

$$\beta_p^{dia} \equiv \frac{K_{dia}\, \alpha\eta_e (1 + \tau + \alpha\eta_e)}{(1+\tau)^2 (1+\eta_e)^2} \beta_p^* \quad , \quad K_{dia} = 1.6 \tag{117a}$$

$$\beta_p^{flr} \equiv \frac{K_{flr} (1 + \tau + \eta_e(\tau + \alpha))\,(1 + \alpha\eta_e)}{(1+\tau)\,(1+\eta_e)^2} \beta_p^* \quad , \quad K_{flr} = 1.4 \tag{117b}$$

$$\gamma_R \equiv \Delta' \rho_s \tag{118}$$

Then, we may interpolate these to regimes by the model evolution equation

$$\frac{\partial \delta}{\partial \zeta} = \gamma_R + \beta_p^{dia} \frac{1 - (\delta/\delta_0)^2}{\delta \left(1 + (\delta/\delta_0)^2\right)^2} \tag{119}$$

with

$$\delta_0 \equiv \sqrt{\beta_p^{flr} / \beta_p^{dia}} = \left(\frac{K_{flr} (1 + \tau + \eta_e(\tau + \alpha))(1 + \alpha\eta_e)(1 + \tau)}{K_{dia}\, \alpha\eta_e (1 + \tau + \alpha\eta_e)} \right)^{1/2} \tag{120}$$

This equation is integrable and leads to

$$\zeta = \int_0^\delta d\delta'\, \delta' \frac{\left(1 + (\delta'/\delta_0)^2\right)^2}{\gamma_R \delta' \left(1 + (\delta'/\delta_0)^2\right)^2 + \beta^{dia}\left(1 - (\delta'/\delta_0)^2\right)} \tag{121}$$

Then, two situations occur (depending on whether the denominator of the integrand of Eq. (121) goes through zero or not):

- $\gamma_R \delta_0 / \beta_p^{dia} < 0.08$

In this case, the island width grows with time and saturates to a value less than $\delta_{Isat} \approx 1.5 \delta_0 \rho_s$. The value of δ_{Isat}/δ_0 may be determined graphically by looking for the intersections of the curves $y = (x^2 - 1) / x\,(x^2 + 1)^2$ and $y = \gamma_R \delta_0 / \beta_p^{dia}$.

- $\gamma_R \delta_0 / \beta_p^{dia} > 0.08$

The island is not fully stabilized by FLR effects. It can still be stabilized by average curvature effects[19]. Basically, the latter effect is efficient, to stabilize positive Δ', in the ratio $\sqrt{R/L_p}$

with respect to FLR effects. Ultimately, if average curvature effects are not sufficient, the island width saturates because of quasilinear effects[28].

The situation is described in fig. 5. The problem may be viewed as that of a particle in a potential. The island is growing from zero size (i.e. from the left to the right along the horizontal scale), with an available free energy given by $\gamma_R\delta_0/\beta_p^{dia}$, in the potential figured by the curve $y = (x^2 - 1) / x (x^2 + 1)^2$. If this free energy is sufficiently high, i.e. if $\gamma_R\delta_0/\beta_p^{dia} >$ 0.08, the island can overcome the potential barrier and grow, until it is saturated by a mechanism not taken into account in this diagram. On the contrary, if $\gamma_R\delta_0/\beta_p^{dia} < 0.08$, the available free energy is not sufficient to drive the island over the potential barrier, against the stabilizing diamagnetic effects, so that it saturates to a width less than $\delta_{Isat} \approx 1.5\ \rho_s\delta_0$, given by the intersection of the curves $y = (x^2 - 1) / x (x^2 + 1)^2$ and $y = \gamma_R\delta_0/\beta_p^{dia}$. The Δ' saturation threshold is: $\Delta'_{sat} = 0.08\ \beta_p^{dia} / \rho_s\delta_0$.

We must precise that the values (0.08 and 1.5) that arise from this calculation are related to the model of the transition $\delta_I = \rho_S$ (Eq. (119)) and to the fact that we used the constant A approximation.

The conclusion of this chapter is that the width of a magnetic island in a tokamak will not exceed a few times ρ_s unless the Δ' is sufficient. For instance, using an estimate of $\delta_0 \approx 3$, (which corresponds to the typical values $\alpha=1/2$, $\tau=1$ and $\eta_e=2$) one finds that an island is stabilized by FLR effects if

$$\Delta' < 0.002\ \beta_p^*/\rho_s \tag{122}$$

For $\rho_s \approx 10^{-3}$m, $\beta_p^* \approx 2$, one finds $\Delta'_{crit.} \approx 4$. This means that, in this case, a m=2 mode, for instance, needs a Δ' greater than 4 to overcome the diamagnetic saturation and grow to a scale larger than ρ_s. The observation on Text[29] of a quasicoherent mode could correspond to such a saturated island chain (with m > 2). The stabilizing effect is even stronger for the m=1 mode. This case has been studied by Migliuolo et al.[30] and Zakharov et al.[31] in the linear regime. Note however that the Δ' is also large for this mode in standard conditions. Nevertheless, this could explain why in large tokamaks, where the β_p^* is large, the m=1 mode is not always observed. A saturation width of order ρ_s is indeed generally beyond the spatial resolution of the soft X rays or the ECE diagnostics. Note, however, that the above stability analysis has been performed assuming that the vector potential perturbation $\tilde{A}(r,t)$ is symetric with respect to the resonant surface $r=r_0$, which is not realistic for the m=1 mode[1]. Although the main ideas of this work are qualitatively correct for the m=1 case, its precise treatment will be discussed in a forthcoming paper. Similar results are obtained in the viscous regime $(\nu_C \ll \nu_D \equiv D_c/\delta_I^2)$.

APPENDIX

Eqs. (42) and (55) can be written

$$\frac{\partial X}{\partial u} = - \frac{\partial}{\partial \overline{\psi}}[GQ] \tag{A1}$$

$$\langle \Delta (Q (\overline{U} - K)) \rangle_{\psi} = - \left\langle G \left(\frac{\partial \overline{U}}{\partial \overline{\psi}} \frac{\partial X}{\partial u} - \frac{\partial \overline{U}}{\partial u} \frac{\partial X}{\partial \overline{\psi}} \right) \right\rangle_{\psi} \tag{A2}$$

where

$$X = \frac{\delta_I^2}{4 D L_{//}} \frac{1}{\delta_I \left. \frac{dF_{eq}}{dr} \right|_{r=r_0}} v_{//} F_T$$

Eqs. (A1) and (A2) lead to

$$\frac{\partial}{\partial \overline{\psi}} \oint_{\psi} \frac{du}{2\pi} G \frac{\partial (Q (\overline{U} - K))}{\partial \overline{\psi}} = - \frac{\partial}{\partial \overline{\psi}} \oint_{\psi} \frac{du}{2\pi} \overline{U} \frac{\partial X}{\partial u}$$

or equivalently

$$\oint_{\psi} \frac{du}{2\pi} G \frac{\partial (Q (\overline{U} - K))}{\partial \overline{\psi}} = \oint_{\psi} \frac{du}{2\pi} \overline{U} \frac{\partial}{\partial \overline{\psi}} (GQ)$$

so that

$$\frac{\partial (Q K)}{\partial \overline{\psi}} = Q^2 \oint_{\psi} \frac{du}{2\pi} G^2 \frac{\partial}{\partial \overline{\psi}} \left(\frac{\overline{U}}{G} \right)$$

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VIII. CONCLUSION

In this paper, microtearing and large scale tearing modes have been studied in the non linear regime where the parallel electron dynamics provides the fastest time scale. Electrons are treated via a kinetic equation including transport processes induced by a background of microturbulence. This equation is solved (see sec. **III**) according to a perturbative technique that allows to clearly display the rôle played by each of the physical mechanisms involved in the problem (parallel dynamics, transport, inductive field and diamagnetism). Turbulent transport processes appear to be essential to determine density and temperature profiles (see **III.B**) as well as island rotation frequency. Indeed, it is clear, from **III.C, IV.B** and **V.A**, that the part of the response which determines the island rotation frequency , is fixed by the quasi-linear diffusion operator related to the background of turbulence. Moreover, turbulent transport characteristics appear in the definition of the rotation frequency ω (Eq. (80)).

It has been shown in this work (see **V.B** and **VI.B**) that diamagnetism destabilizes a magnetic island if its width is smaller than an ion Larmor radius and stabilizes it in the opposite case (for microturbulence bath representative of present day tokamaks as discussed at the end of **V.A**). This has practical consequences. First, for low m perturbations with a positive Δ', the island width will saturate to a value of order ρ_s unless the value of Δ' is sufficient, i.e. unless $\Delta'>0.002\beta_p^*/\rho_s$ (see sec. **VII**). This effect is similar to the stabilization by the curvature effect[19], but is more efficient. Second, diamagnetism is destabilizing for modes with large m wave number and small island width ($\delta_I \ll \rho_S$) , but the island width saturates to a value of the order of an ion Larmor radius. This means that such a microturbulence should exhibit radial scales of order ρ_s, as it was indeed numerically found in reference 9. However, the actual radial correlation length could be ultimately determined by toroidal coupling which is not included here. The next step will be therefore to take the toroidal mode coupling into account in this model.

Finally, it has been remarked in sec. **VI** that the β_p^* stability threshold arising from the calculations within the frame of the constant A approximation (**V.B**) may be significantly modified by the release of this latter constraint. This implies that, although qualitative results provided by the constant A approximation are correct (except for small islands ($\delta_I \ll \rho_S$) in the viscous regime $\left(\nu_C \gg \nu_D \equiv D_e/\delta_I^2\right)$), any precise quantitative result has to be calculated within the appropriate frame where this constraint is released.

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TABLE CAPTIONS

Table I: Evolution of the $\bar{\beta}_p$ threshold as a function of $|K_\theta| \delta_I$ for small islands ($\delta_I \ll \rho_S$) in the resistive regime ($\nu_C \gg \nu_D \equiv D_\perp/\delta_I^2$), the constant A constraint being released.

Table II: Evolution of the $\bar{\beta}_p$ threshold as a function of $|K_\theta| \delta_I$ for small islands ($\delta_I \ll \rho_S$) in the viscous regime ($\nu_C \ll \nu_D \equiv D_\perp/\delta_I^2$), the constant A constraint being released.

Table III: Summary of the results obtained in **V.B** and **VI** for the various regimes and island sizes.

$\lvert K_\theta \rvert \delta_I$	0.025	0.05	0.1	0.25	0.5	1
$\overline{\beta}_p$	0.55	1.12	2.6	8.5	23	68

Tab. I

$\lvert K_\theta \rvert \delta_I$	0.0175	0.025	0.05	0.1	0.175	0.25
$\overline{\beta}_p$	6.1	7.5	12.5	21	33	45

Tab. II

resistive regime $\nu_C \gg \nu_D = D_e/\delta_I^2$		viscous regime $\nu_C \ll \nu_D = D_e/\delta_I^2$	
small island $\delta_I \ll \rho_S$	large island $\delta_I \gg \rho_S$	small island $\delta_I \ll \rho_S$	large island $\delta_I \gg \rho_S$
destabilizing (V.B.1)	stabilizing (V.B.2)	destabilizing for *non constant* A (V.B.3 and VI)	stabilizing (V.B.4)

Tab. III

FIGURE CAPTIONS

Figure 1: Convergence of radial profiles over the layer, in the resistive regime for small island width, with $|K_\theta|\delta_I = 0.1$:

(a) iterated potential $\sigma^{(n)}(\rho)$:

line $\rightarrow \sigma^{(0)}(\rho)$, circles $\rightarrow \sigma^{(1)}(\rho)$.

(b) iterated vector potential perturbation $a^{(n)}(\rho)$:

line $\rightarrow a^{(1)}(\rho)$, circles $\rightarrow a^{(2)}(\rho)$.

Figure 2: Evolution of the β_p^* threshold as a function of $|K_\theta|\delta_I$ with α=1/2, τ=1 and η_e=2. The mode is unstable (stable) above (under) the threshold. The threshold computed numerically with the constant A constraint released (circles + continuous fit) is compared with the constant A threshold (dashed line).

Figure 3: Convergence of radial profiles over the layer, in the viscous regime for small island width, with $|K_\theta|\delta_I = 0.1$:

(a) iterated potential $\sigma^{(n)}(\rho)$:

dashed line $\rightarrow \sigma^{(0)}(\rho)$, solid line $\rightarrow \sigma^{(1)}(\rho)$, circles $\rightarrow \sigma^{(2)}(\rho)$

(b) iterated vector potential perturbation $a^{(n)}(\rho)$:

dashed line $\rightarrow a^{(1)}(\rho)$, solid line $\rightarrow a^{(2)}(\rho)$, circles $\rightarrow a^{(3)}(\rho)$

Figure 4: Evolution of the β_p^* threshold as a function of $|K_\theta|\delta_I$ with α=1/2, τ=1 and η_e=2. The mode is unstable (stable) above (under) the threshold. The viscous threshold (circles + continuous fit) is compared with the resitive one (stars + dashed fit).

Figure 5: Island evolution in the resistive regime. The two typical scenarios are displayed: saturating (dashed lines) and growing (dashed-dotted lines) islands. The curve $y = (x^2 - 1) / x\,(x^2 + 1)^2$ (solid line) provides both the saturation island width and the Δ' saturation threshold (see dotted lines):

- the horizontal scale is associated with the island width: $x \rightarrow \delta / \delta_0$
- the vertical scale is related to the available free energy (Δ'): $y \rightarrow \gamma_R \delta_0 / \beta_p^{dia}$

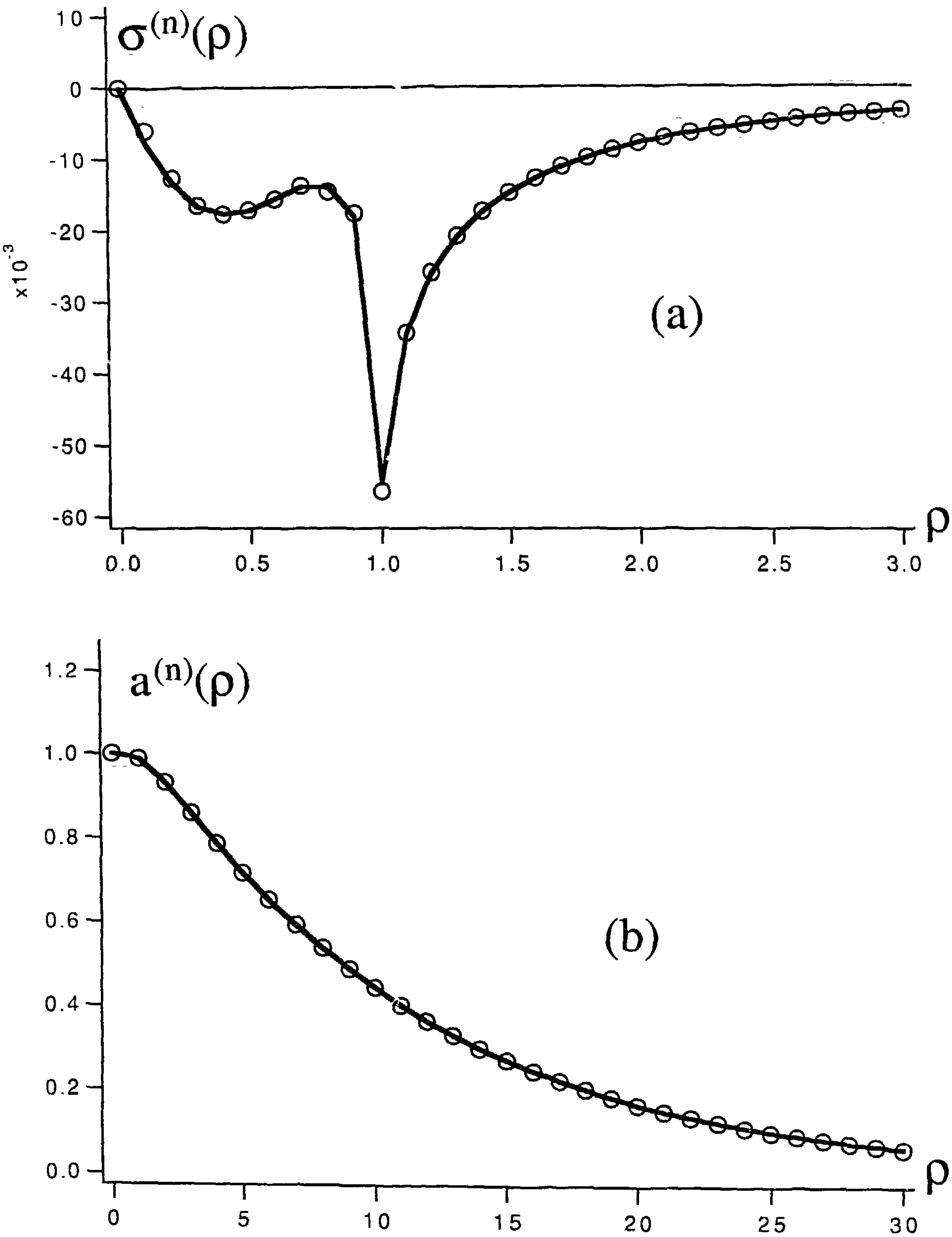


Fig.1

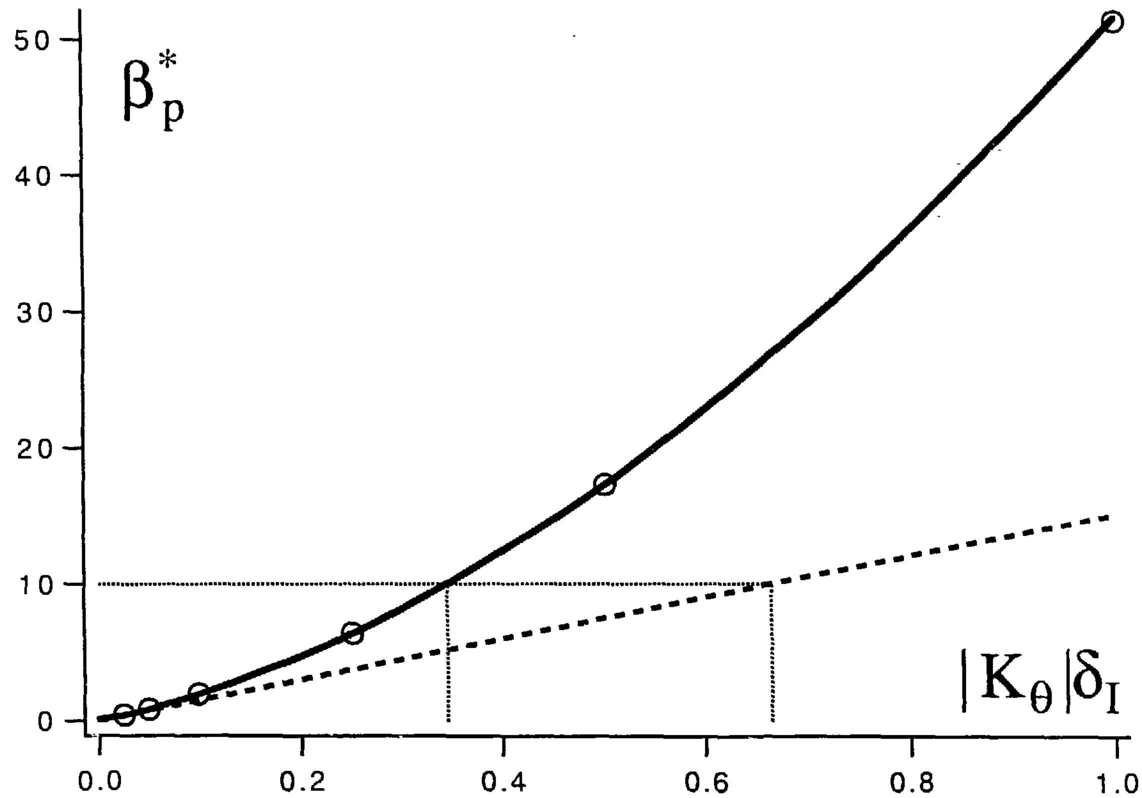
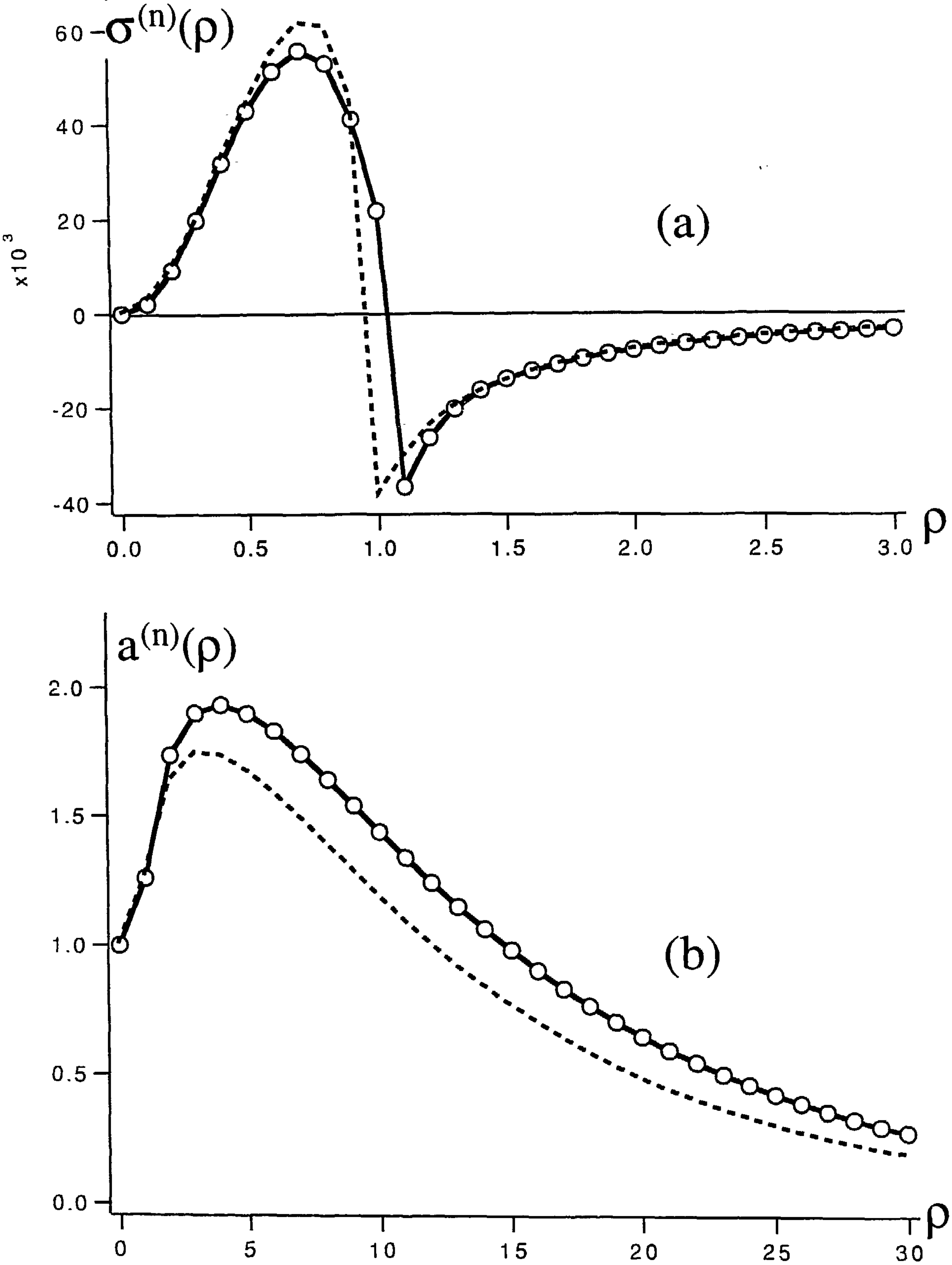


Fig. 2



$\sigma^{(n)}(\rho)$
60
40
20
x10^3
0
-20
-40
(a)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
ρ
$a^{(n)}(\rho)$
2.0
1.5
1.0
0.5
0.0
(b)
0
5
10
15
20
25
30
ρ

Fig. 3

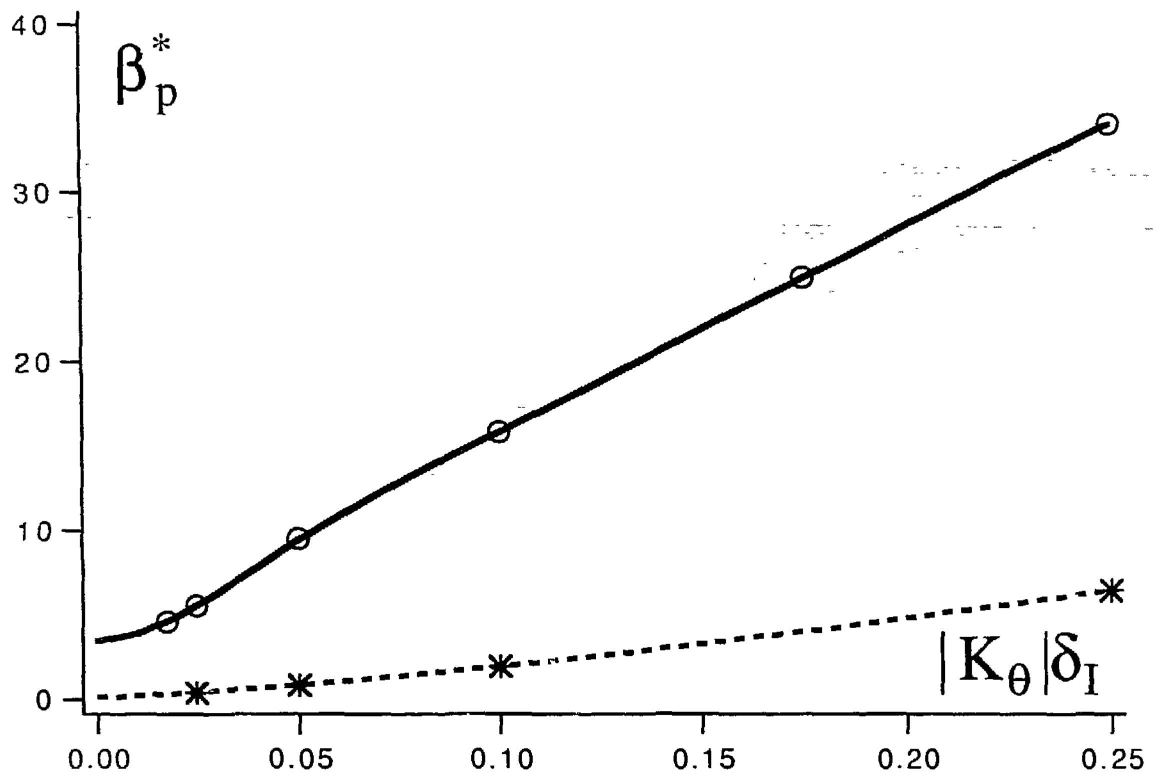
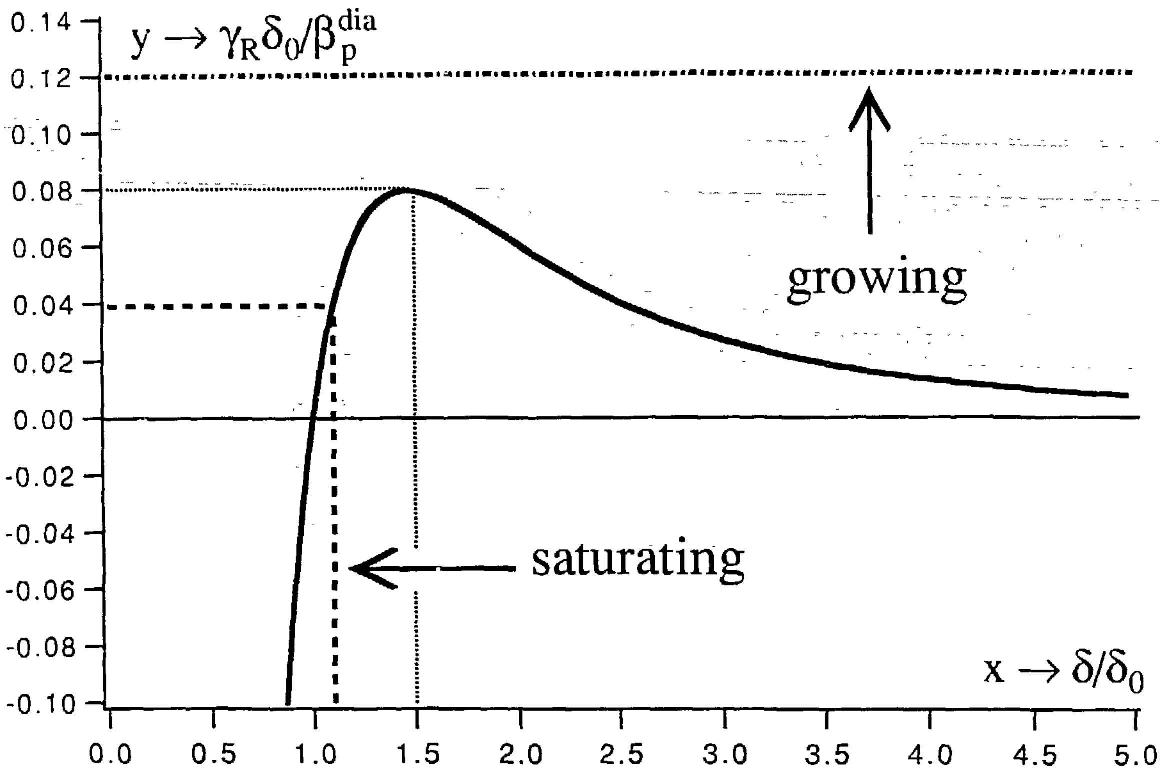


Fig. 4



$y \rightarrow \gamma_R \delta_0 / \beta_p^{dia}$
growing
saturating
$x \rightarrow \delta / \delta_0$
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
-0.02
-0.04
-0.06
-0.08
-0.10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0

Fig. 5