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Abstract

A seed signal and/or a prebunched electron beam may provide the start up of a FEL. During the last years, interest has grown around FEL's operating with prebunched electron beams, this paper is, therefore, devoted to the analysis of the dynamical features of FEL operating in such a configuration. We exploit a slightly modified form of the FEL high gain equation to derive quantities of practical interest like the dependence of the system growth rate on the bunching coefficients.

Riassunto

Un segnale di ingresso e/o un fascio di elettroni premodulato possono costituire le cause della generazione della radiazione coerente di un laser ad elettroni liberi (FEL). Durante gli ultimi anni è cresciuto un notevole interesse intorno ai progetti di FEL operanti con fasci di elettroni premodulati. Questo lavoro riguarda l'analisi teorica della dinamica di FEL operanti nelle condizioni di cui sopra. Si utilizzano inoltre forme modificate dell'equazione di alto guadagno per ottenere quantità di interesse pratico, come la dipendenza della crescita del segnale coerente dai coefficienti di modulazione.

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1. INTRODUCTION

The FEL dynamics is governed by the coupled pendulum and Maxwell equations. Limiting ourselves to the 1-D case the above quoted equations takes the very concise form [1]

$$\frac{d^2\zeta}{d\tau^2} = |a|\cos(\zeta+\phi)$$

$$\frac{d}{d\tau} a = -2\pi g_0 < e^{-i\zeta} >_{\zeta_0} \tag{1.1}$$

$\zeta(\tau)$ is the electron relative phase with respect to the radiation field, $|a|$ and ϕ are the amplitude and phase of the dimensionless optical field, g_0 is the small signal gain coefficient and $<.........>_{\zeta_0}$ represents an average over the electron initial phase ζ_0. We refer the reader to Ref. 1 for the notation and definition of the symbols used in this article. Equation (1.1) holds either in small and strong signal regime. The small signal regime, namely the dynamical configuration in which the electron motion is weakly perturbed by the field, is obtained from Eq. (1.1) expanding all the relevant quantities up to the first order in the amplitude a. Setting indeed [2]

$$\zeta=\zeta_0+\nu_0\tau+\delta\zeta \tag{1.2}$$

where ν_0 is the initial velocity and $\delta\zeta$, the electron phase variation, can be computed integrating the first of Eqs (1.1). Keeping the small signal contribution, we get

$$\delta\zeta \simeq \int_0^\tau d\tau'(\tau-\tau')\,\mathrm{Re}\left[a(\tau')\,e^{i(\zeta_0+\nu_0\tau')}\right] \tag{1.3}$$

Inserting (1.3) in the second of (1.1) and expanding the exponential up to the first order in $\delta\zeta$, we find the following integro differential equation, specifying the unsaturated behavior of the optical field

$$\frac{da}{d\tau} = -2\pi g_0 < e^{-i\zeta_0} >_{\zeta_0} e^{-i\nu_0\tau} + i\pi g_0 \int_0^\tau d\tau'(\tau-\tau')\, a(\tau')\, e^{-i\nu_0(\tau-\tau')}$$

$$+ i\pi g_0 < e^{-2i\zeta_0} >_{\zeta_0} \int_0^\tau d\tau'(\tau-\tau')\, a^*(\tau')\, e^{-i\nu_0(\tau+\tau')} \tag{1.4}$$

In the case of an initially unbunched beam the average over the initial phases is zero. We assume that the beam contains an initial bunching and write the input phase distribution as (*)

$$f(\zeta_0) = \sum_{n=-\infty}^{+\infty} b_n e^{in\zeta_0}$$

$$\left(\int_0^{2\pi} f(\zeta_0)\, d\zeta_0 = b_0 = 2\pi \right) \tag{1.5}$$

The coefficients b_n specify the harmonic content of the bunching. The averages apperaring in Eq. (1.4) can be therefore written as

$$< e^{-i\zeta_0} >_{\zeta_0} = \frac{1}{2\pi}\int_0^{2\pi} d\zeta_0\, f(\zeta_0)\, e^{-i\zeta_0} = b_1$$

$$< e^{-2i\zeta_0} >_{\zeta_0} = \frac{1}{2\pi}\int_0^{2\pi} d\zeta_0\, f(\zeta_0)\, e^{-2i\zeta_0} = b_2 \tag{1.6}$$

We make a preliminar simplification assuming that the bunching is *weak* enough that the coefficinets with $n \geq 2$ can be neglected. Equation (1.4) can be therefore written as

$$e^{i\nu_0\tau} D_\tau\, a(\tau) = i\pi g_0 D_\tau^{-2} \left[e^{i\nu_0 t}\, a(\tau) \right] - 2\pi g_0 b_1 \tag{1.7}$$

* We do not include energy distribution and consider spatial bunching only

The notation $D_\tau = \frac{d}{d\tau}$ has been adopted for brevity's sake. Negative derivatives of a function have been introduced within the context of differintegral calculus, according to the following definition [3]

$$D_\tau^{-n} g(\tau) = \frac{1}{(n-1)!} \int_0^\tau (\tau-\xi)^{n-1} g(\xi)\, d\xi \tag{1.8}$$

Multiplying both sides of Eq. (1.7) by the operator D_τ^2, we obtain

$$\left(D_\tau^3 + 2\, i\, \nu_0 D_\tau^2 - \nu_0^2 D_\tau \right) a(\tau) = i \pi g_0 a(\tau) \tag{1.9a}$$

The initial non homogeneous integrodifferential problem has been therefore turned into an ordinary third order equation with constant coefficients. The bunching contribution is not explicitly contained in Eq. (1.9a) but in its initial conditions, reported below

$$a(0) = a_0, \quad \left.\frac{da}{d\tau}\right|_{\tau=0} = -2\pi g_0 b_1, \quad \left.\frac{d^2 a}{d\tau^2}\right|_{\tau=0} = i\nu_0 g_0 2\pi b_1 \tag{1.9b}$$

The solution of (1.9a) can always be written in the form

$$a(\tau) = \sum_{j=1}^{3} a_j e^{-i(\nu_0 + \delta\nu_j)\tau} \tag{1.10}$$

where $\delta\nu_j$ are the roots of the algebraic equation

$$\delta\nu^2 (\nu_0 + \delta\nu) = \pi g_0 \tag{1.11}$$

and the amplitudes a_j are specified by the conditions

$$\sum_{j=1}^{3} a_j = a_0 \ , \quad \sum_{j=1}^{3} a_j \delta v_j = -2\pi i g_0 b_1 - v_0 a_0$$

$$\sum_{j=1}^{3} a_j \delta v_j^2 = 2\pi i v_0 g_0 b_1 + a_0 v_0^2 \tag{1.12}$$

In the next section we will exploit the above results to discuss the small signal dynamics of FEL's operating with a prebunched electron beam. In sect 2 we discuss the low gain case and the corrections due to the so called intermediate gain contributions. Section 3 is devoted to the analysis of the high gain regime and, finally, sect 4 contains concluding remarks.

2. LOW AND INTERMEDIATE GAIN REGIMES

For the purposes of this section it is convenient to rewrite Eq. (1.7) in the integrodifferential form

$$D_\tau a = -2\pi g_0 b_1 e^{-i v_0 \tau} + i \pi g_0 \int_0^\tau \tau' a(\tau - \tau') e^{-i v_0 \tau'} d\tau' \tag{2.1}$$

belonging to the class of non homogeneous Volterra equations. Iterative solutions leading to a Neumann series have been proposed in Ref. 4, and within this framework g_0 has been used as expansion parameter.

When the small signal gain coefficient is below 0.3, the first order contributions are sufficient to characterize the unsaturated dynamics of FEL. Non linear corrections start playing a role when g_0 exceeds 30%.

According to Ref. 4 we say that the system operates in

a) low gain regime, when $g_0 \leq 0.3$

b) intermediate gain regime, when $0.3 < g_0 \leq 10$

c) high gain regime, when $g_0 > 10$

In this section we discuss cases a) and b). When $g_0 \leq 0.3$ Eq. (2.1) can be further simplified, approximating $a(\tau - \tau')$ with $a(\tau)$ in the integral, and thus getting a simple first order differential equation whose solution at lowest order in g_0 writes

$$a(\tau) = a_0 \left[1 + \pi g_0 \frac{2(1 - e^{-i\nu_0 \tau}) - i\nu_0 \tau (e^{-i\nu_0 \tau} + 1)}{\nu_0^3} \right] - 2\pi g_0 b_1 \left(\frac{\sin(\nu_0 \tau/2)}{\nu_0/2} \right) e^{-i\nu_0 \tau/2} \tag{2.2}$$

According to the above equation, the field grows even for a vanishing input a_0 and the contribution proportional to the bunching coefficient can be understood as a coherent spontaneous emission.

If one defines the gain in the usual way

$$G = \frac{|a(1)|^2 - |a_0|^2}{|a_0|^2} \tag{2.3}$$

one gets from Eq. (2.2)

$$G = -\pi g_0 \frac{\partial}{\partial \nu_0} \left(\frac{\sin \nu_0/2}{\nu_0/2} \right)^2 - 4\pi g_0 \frac{|b_1|}{|a_0|} \left(\frac{\sin \nu_0/2}{\nu_0/2} \right) \cdot \cos(\nu_0/2 - \chi) \; ; \quad \chi = \arg(a_0^* b_1) \tag{2.4}$$

The first term is the usual FEL gain function, while the second contribution cannot be considered a gain in the strict sense, being dependent on the input field amplitude. An analogous correction has also been derived in Ref. 5. We stress however that the importance of the initial bunching is not associated to possible gain modifications, rather to the fact that is allows the signal start up even without any input seed and this is of crucial importance for a FEL amplifier.

When g_0 exceeds 0.3, contributions up to the first order g_0 are not sufficient to characterize the system dynamics. Higher order corrections can be calculated solving Eq. (2.1) recursively, namely setting

$$a(\tau) = \sum_{n=0}^{\infty} g_0^n a_n(\tau) \ , \quad a_n(0) = a_0 \delta_{n,0} \tag{2.5a}$$

thus finding for the various amplitudes

$$a_0(\tau) = a_0$$

$$\frac{d}{d\tau} a_1 = -2\pi b_1 e^{-i\nu_0 \tau} + i\pi a_0 \int_0^{\tau} \tau' e^{-i\nu_0 \tau'} d\tau' \tag{2.5b}$$

$$\frac{d}{d\tau} a_n = i\pi \int_0^{\tau} a_{n-1}(\tau-\tau')\tau' e^{-i\nu_0 \tau'} d\tau', \quad n \geq 2$$

The evaluation of the various terms is computationally rather cumbersome, we report here the amplitudes $a_{1,2}$ just to give an idea of the interplay between higher g_0 corrections and bunching contributions.

Assuming a_0=0 we find

$$a(\tau) \simeq -2\pi b_1 g_0 \left\{ \frac{\sin(\nu_0 \tau/2)}{\nu_0/2} e^{-i\nu_0 \tau/2} - \right.$$
$$\left. -\frac{1}{2}(\pi g_0) \frac{(i\nu_0^2\tau^2 + 4\nu_0\tau - 6i)\, e^{-i\nu_0\tau} + 6i + 2\nu_0\tau}{\nu_0^4} \right\} \tag{2.6}$$

In Fig. 1 we have plotted $|a(1)|^2$ vs ν_0 for different values of g_0. When the small signal gain coefficient increases there is a significant enhancement of $|a(1)|^2$ and, as we will see in the next section, Eq. (2.6) is a good approximation for g_0 up to 5.

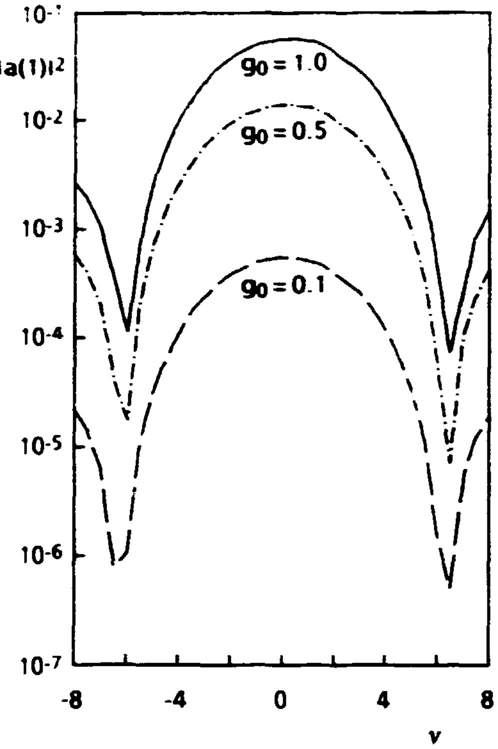


Fig. 1 $|a(1)|^2$ vs the detuning parameter for different values of g_0 and for the bunching coefficient b_1 specified by Re $b_1=-2.65\times10^{-2}$, $|\mathrm{Im}\ b_1|=2.8\times10^{-2}$. The plots have been derived using either Eq. (2.6) and the numerical solution of Eq. (2.1), no significant difference arises between the two procedures.

3. THE HIGH GAIN REGIME

In the previous section we have discussed low and intermediate gain regimes FEL operation, including electron beam prebunching. The high gain regime will be analyzed in this section and just to have a very first understanding of the prebunched high gain FEL dynamical behaviour, we solve Eq. (1.9a)) for $a_0=0$ and close to the resonance ($\nu_0=0$). With these assumptions the roots of the algebraic equation (1.11) are

$$\delta\nu_1 = (\pi g_0)^{1/3}, \quad \delta\nu_2 = -(\pi g_0)^{1/3} e^{-i\pi/3}, \quad \delta\nu_3 = -(\pi g_0)^{1/3} e^{+i\pi/3} \tag{3.1a}$$

and the amplitudes a_j are specified by

$$a_1 = -\frac{2ib_1(\pi g_0)^{2/3}}{3}, \qquad a_2 = \frac{2ib_1(\pi g_0)^{2/3}}{3} e^{i\pi/3} \tag{3.1b}$$

$$a_3 = \frac{2ib_1(\pi g_0)^{2/3}}{3} e^{-i\pi/3}$$

The field amplitude writes therefore

$$a = \frac{2ib_1(\pi g_0)^{2/3}}{3}\left\{-e^{-i(\pi g_0)^{1/3}\tau} + \right.$$
$$+ e^{+i\pi/3} e^{+\frac{1}{2}\sqrt{3}(\pi g_0)^{1/3}\tau} e^{+i/2(\pi g_0)^{1/3}\tau} + \tag{3.2}$$
$$\left. + e^{-i\pi/3} e^{-\frac{1}{2}\sqrt{3}(\pi g_0)^{1/3}\tau} e^{+i/2(\pi g_0)^{1/3}\tau}\right\}$$

and the relevant square modulus reads

$$|a|^2 = 4\frac{|b_1|^2(\pi g_0)^{4/3}}{9}\left\{-e^{-\sqrt{3}(\pi g_0)^{1/3}\tau} + e^{\sqrt{3}(\pi g_0)^{1/3}\tau}\right.$$
$$-2\left[e^{-\frac{1}{2}\sqrt{3}(\pi g_0)^{1/3}\tau} \cdot \cos\left(\frac{\pi}{3} - \frac{3}{2}(\pi g_0)^{1/3}\tau\right)\right. \tag{3.3}$$
$$\left. + e^{\frac{1}{2}\sqrt{3}(\pi g_0)^{1/3}\tau} \cos\left(\frac{\pi}{3} + \frac{3}{2}(\pi g_0)^{1/3}\tau\right)\right]$$

It is worth noticing that for small times ($\tau \ll 1$) the field grows quadratically with τ according to the simple relation

$$|a|^2 \simeq 4\pi^2|b_1|^2(g_0\tau)^2 \tag{3.4}$$

In practical units the above equation writes

$$I\left[\frac{MW}{cm^2}\right] \simeq 4.8\times10^{-10}\pi^2|b_1|^2\left(\frac{k}{\gamma}\right)^2\left(J_0(\xi)-J_1(\xi)\right)^2\cdot \qquad (3.5)$$

$$\cdot|J\,[A/m^2]|^2(z\,[m])^2$$

We have assumed that the prebunched beam with current density J and reduced relativistic energy γ is injected into a linear undulator. Furthermore K is the undulator parameter,

$$\xi=\frac{1}{4}\,\frac{K^2}{1+K^2/2} \qquad (3.6)$$

and z refers to the longitudinal coordinate of propagation inside the undulator ($\tau=z/L_u$).

Introducing the so called ρ parameter, namely

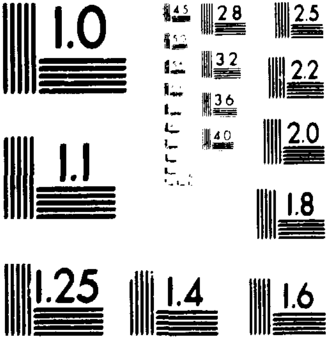
$$\rho=\frac{1}{4\pi}\left(\frac{\pi g_0}{N^3}\right)^{1/3} \qquad (3.7)$$

Eq. (3.8) can be written in practical units as

$$I\left[\frac{MW}{cm^2}\right]=1.15\,\pi^2\times10^{-6}\gamma\rho|b_1|^2|J\,[A/m^2]\cdot$$

$$\cdot\left\{\cosh\left(4\sqrt{3}\,\pi\rho\frac{z}{\lambda_u}\right)-\left[e^{-2\sqrt{3}\,\pi\rho z/\lambda_u}\cdot\right.\right. \qquad (3.8)$$

$$\left.\left.\cdot\cos\left(\pi/3-6\pi\rho\frac{z}{\lambda_u}\right)+e^{2\sqrt{3}\,\pi\rho z/\lambda_u}\cos\left(\frac{\pi}{3}+6\pi\rho\frac{z}{\lambda_u}\right)\right]\right\}$$



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The above analysis is independent of the condition of low or high gain regime. In the latter case the τ^2 (or equivalently the z^2) growth preceeds the exponential growth, dominated by the fastest root. an idea of the intensity evolution vs τ is offered by Fig. 2.

Regarding the exponential growth, it is interesting to notice that, when the fastest root dominates, Eq. (3.8) writes

$$I\left[\frac{MW}{cm^2}\right] \simeq I_0\,[MW/cm^2]\exp\left(4\sqrt{3}\,\pi\rho\,\frac{z}{\lambda_u}\right) \tag{3.9}$$

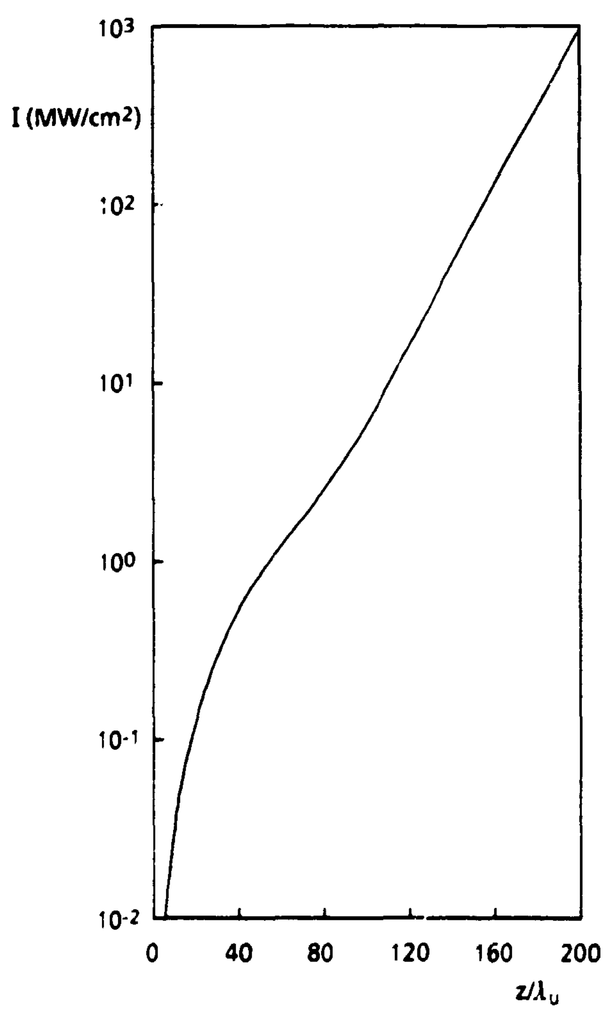

Fig. 2 Intensity growth vs the number of undulator periods (Eq. 3.8). Re b_1=-8.53×10^{-4}, Im b_1=1.01×10^{-3}, ρ=2.27×10^{-3}

where

$$I_0\left[\frac{MW}{cm^2}\right] = 5.77\,\pi^2 \times 10^{-7}\,|b_1|^2\,\gamma\,\rho\,|J\,[A/m^2]| \tag{3.10}$$

As is well known the efficiency of a constant parameters FEL amplifier is ρ, it is therefore possible to link saturation length and bunching coefficient imposing the condition

$$I_0\left[\frac{MW}{cm^2}\right]\cdot \exp\left[4\sqrt{3}\,\pi\rho\,z_s/\lambda_u\right] \simeq 0.511 \times 10^{-4}\gamma\,\rho\,|J\,[A/m^2]| \tag{3.11}$$

According to (3.9) we also get

$$|b_1| \simeq 3\,e^{-2\sqrt{3}\,\pi\rho N_s}\quad,\quad N_s = \frac{z_s}{\lambda_u} \tag{3.12}$$

The above equation provides an interesting relation between the beam bunching coefficient and the number of periods necessary to reach the onset of saturation.

To gain a deeper insight into the details of the problem it is worth discussing the topic from the numerical point of view. The method we use is that proposed in Ref. 4, which leads to a quite fast algorithm. Expanding the r.h.s. of (1.10), containing the root dependent part, we get

$$a(\tau) = e^{-i\nu_0}\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\,a_m\,\tau^m$$

$$a_m = \sum_{j=1}^{3}\delta\nu_j^m\,a_j \tag{3.13}$$

Furthermore according to Eqs (1.11) and (1.12) one finds that the coefficients a_m are specified by the following recurrence relations

$$a_m = \pi g_0 a_{m-3} - \nu_0 a_{m-1} \qquad (m \geq 3)$$

$$a_0 = \alpha_0, \qquad a_1 = -2\pi i g_0 b_1 - \nu_0 \alpha_0 \tag{3.14}$$

$$a_2 = 2\nu_0 \pi i g_0 b_1 + \alpha_0 \nu_0^2$$

According to the above equations the numerical solution of Eq. (1.9a) can be easily obtained and the results of the analysis are summarized in Figs 3.

It is important to add a few words of comment, better clarifying the physical content of the above results. Referring to Fig. 2, we notice that the evolution vs τ of an optical signal induced by a prebunched beam is characterized by three phases

a) quadratic growth

b) preexponential growth

c) expontential growth

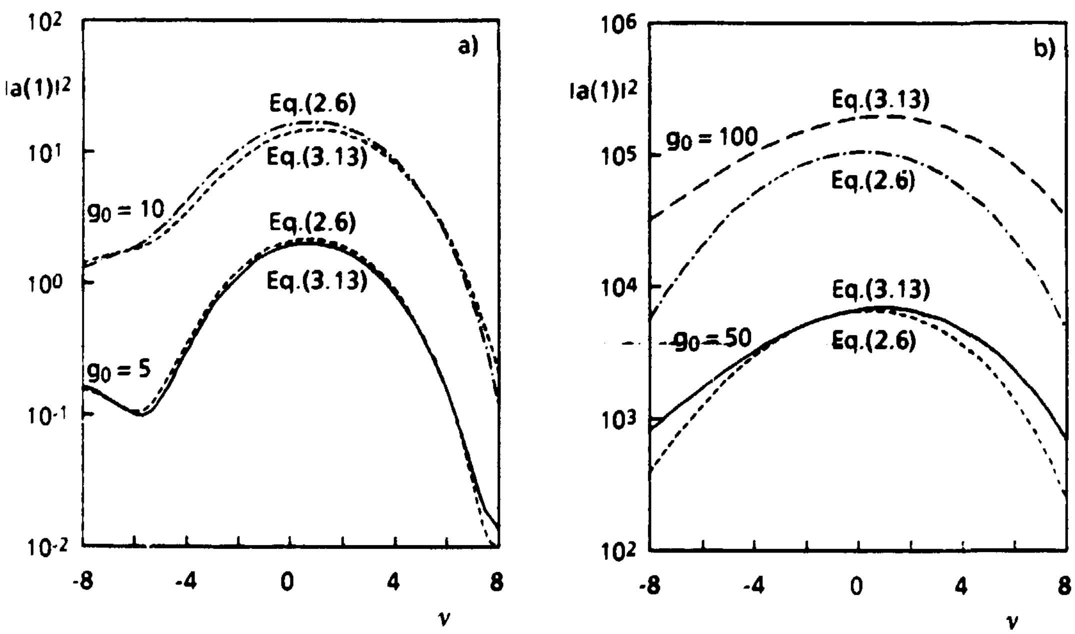

Fig. 3a,b Same as Fig. 1. $|a(1)|^2$ vs ν for different values g_0. The numerical solution, based on Eq. (3.13), is compared with the approximation (2.6)

In phase a) the evolution is dominated by the non homogeneous part of Eq. (2.1). In the second phase the field reaches a sufficiently large value, allowing the second term to play a non negligible role and coherence to develop. In the third step, when full coherence is estabilished, the system grows exponentially. One can get a further physical insight, comparing the different behaviour between the field evolution induced by prebunched beam or by an input seed (see Fig. 4).

In the latter case the phases a) and b) are replaced by a lethargy length in which the coherence develops. (A particularly illuminating comment on this point is offered in Ref. 1)

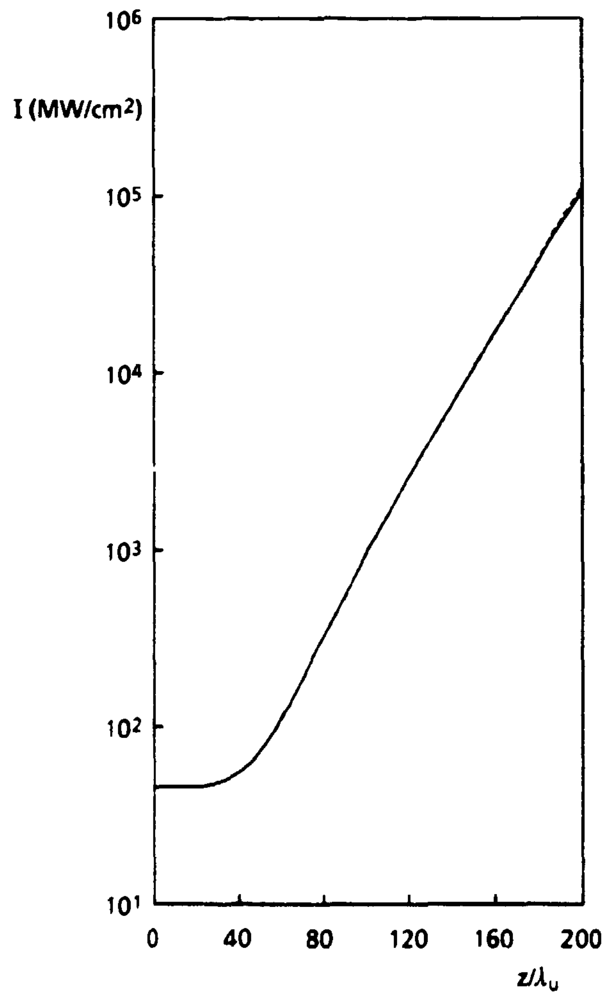

Fig. 4 Intensity evolution vs the number of periods for the same parameters of Fig. 2, b_1=0 and input seed corresponding to $I_0 = 45$ MW/cm^2

4. CONCLUDING REMARKS

We have assumed that the contributions coming from b_2 coefficient can be neglected in the analysis of the FEL dynamics driven by a prebunched e-beam. This assumption will be relaxed in this section and we discuss the physical consequences due to a non vanishing *higher* order bunching. Equation (1.4) can be transformed into an ordinary differential equation, using a procedure analogous to that leading to Eq. (1.9a). We get indeed

$$e^{i\nu_0\tau} D_\tau a = -2\pi g_0 b_1 + i\pi g_0 D_\tau^{-2}\left[a(\tau)\, e^{i\nu_0\tau}\right] + i\pi g_0 b_2 D_\tau^{-2}\left[a^*(\tau)\, e^{-i\nu_0\tau}\right] \tag{4.1}$$

and eliminating the negative derivatives in the r.h.s. of (4.1) we find

$$\left[D_\tau^3 + 2i\nu_0 D_\tau^2 - \nu_0^2 D_\tau\right] a(\tau) = i\pi g_0\left\{a(\tau) + b_2 a^*(\tau)\, e^{-2i\nu_0\tau}\right\} \tag{4.2}$$

whose initial conditions are identical to those given in (1.9b). The coefficient b_2 is not responsible as b_1 for a further contribution to the coherent spontaneous emission, rather it yields an extracontribution to the lowest order part of the stimulated emission.

Limiting the analysis to the low gain regime, we find that the new term containing b_2 provides the following contribution to the field evolution

$$a_{b_2}(\tau) = \frac{\pi g_0 b_2 a_0^*}{\nu_0^3}\left\{\left[\nu_0\tau - \sin(\nu_0\tau)\right]\sin(\nu_0\tau) - i\,\frac{\sin(2\nu_0\tau) - 2\nu_0\tau\cos(\nu_0\tau)}{2}\right\} \tag{4.3}$$

to be added to the r.h.s. of Eq. (2.2).

This further term is dependent on the complex conjugate of the input amplitude a_0 and, according to Eq. (2.3), provides the following relative intensity variation

$$G_{b_2}(\nu_0) = \frac{2\pi g_0}{\nu_0^3}\left[\cos\chi_1\,(\nu_0 - \sin\nu_0)\sin\nu_0 + \frac{1}{2}\sin\chi_1\left(\sin(2\nu_0) - 2\nu_0\cos\nu_0\right)\right] \quad , \quad \chi_1 = \arg\left(\frac{a_0^*}{a_0}\, b_2\right) \tag{4.4}$$

Even though the above quantity does not dependent on the modulus of a_0, it is a function of the input field phase. Its role as a genuine gain term is therefore doubtful. Higher order corrections in g_0 can be computed for the a_{b_2} contributions.This aspect of the problem will be discussed in a forthcoming paper, where we analyze the general solution of Eq. (4.2).

In this paper we have shown that the bunching coefficients $b_{1,2}$ play a direct role in the small signal dynamics of a FEL. The prebunched beam may be, however, specified by non zero $n>2$ Fourier components. A natural question is therefore "what is the role of the bunching coefficients $b_{n>2}$?"

To answer the above question we generalize the FEL integral equation including the contributions providing the onset of saturation. The method we follow is a slight extension of that proposed in Ref. 2.

Equation (1.4) has been derived expanding the exponential in the second of (1.1) up to the first order in $\delta\zeta$. If we do not perform any expansion and assuming that $\delta\zeta$ is still provided by Eq. (1.3), we get from the second of (1.1)

$$\frac{da}{d\tau} = -2\pi g_0 \left< e^{-i(\zeta_0 + \nu_0 \tau)} \cdot e^{-i|\rho(\tau)|\cos(\psi(\tau) + \zeta_0)} \right>_{\zeta_0} \tag{4.5}$$

where

$$\int_0^{\tau} (\tau-\tau')\, a(\tau')\, e^{i\nu_0\tau'} = |\rho|\, e^{i\psi} \tag{4.6}$$

Using the generating function

$$\sum_{n=-\infty}^{+\infty} e^{in\theta} I_n(x) = \exp[x\cos\theta] \tag{4.7a}$$

and the relation

$$I_n(ix) = i^n J_n(x) \tag{4.7b}$$

with $J_n(x)$ and $I_n(x)$ being first kind cylinder Bessel function and its modified version respectively, Eq. (4.5) can be cast in the form

$$\frac{da}{d\tau} = -2\pi g_0 \left\langle e^{-i\nu_0\tau} \sum_{m=-\infty}^{+\infty} e^{i[m\psi(\tau) + (m-1)\zeta_0]} I_m(-i|\rho|) \right\rangle_{\zeta_0} \tag{4.8}$$

Performing the average on the initial ζ_0 distribution we finally get

$$\frac{da}{d\tau} = -2\pi g_0 e^{-i\nu_0\tau} \sum_{m=-\infty}^{+\infty} e^{im\psi}\, b_{1-m} (-i)^m J_m(|\rho|) \tag{4.9}$$

The above equation can be recast into an integrodifferential form, using Eq. (4.6), namely

$$\frac{da}{d\tau} = -2\pi g_0 e^{-i\nu_0\tau} \sum_{m=-\infty}^{+\infty} (-i)^m\, b_{1-m} \frac{J_m(|\rho|)}{|\rho|^m} \left[\int_0^{\tau} d\tau' (\tau-\tau')\, a(\tau')\, e^{i\nu_0\tau'} \right]^m \tag{4.10}$$

The small signal limit is easily obtained from (4.10) recalling that $\lim_{x\to 0} \frac{J_1(x)}{x} = \frac{1}{2}$.

Going back to the original question we notice that the bunching coefficients play a rather intrigued role and that coefficients with n>2 are essentially linked to the saturated regime.

A more complete analysis including the mechanism leading to harmonic generation will be presented elsewhere.

Before closing the paper it is necessary to discuss a not yet touched point. We did not specify how the bunching may be achieved and we did not analyze the effect of the average over the detuning parameter ν, which is in turn an average over the beam energy distribution. The two points are not separate questions.

Experimental programs aimed at exploiting the emission mechanism quoted in the paper have been proposed [6-8] for FEL operating in the VUV region of the spectrum. The unifying feature of the proposed experiment is that of providing the prebunching, using a FEL type interaction. In the case of Ref. 8 an oscillator tripler scheme has been proposed. In that scheme a high quality electron beam drives a FEL oscillator at 240 nm. The radiation stored in the optical cavity, reinteracts with the e-beam producing bunching and modifying the natural e-beam energy distribution. The e-beam passes successively through a second undulator, whose parameters are adjusted in such a way that the resonance frequency is just the third harmonic of the first. In the second undulator (the tripler) coherent radiation at 80 nm develops owing to the bunching induced in the first undulator. We have however remarked that not only bunching is produced in the oscillator but also a variation of the energy distribution. To understand the role of this last effect we have plotted in Fig. 5 the field evolution in the second undulator with and without the inclusion of the energy spread induced in the first section. There is a significant difference between the two behaviours. The quadratic growth is the same, but when the proper energy distribution is included the system has not to spend any time to organize its own coherence, thus passing to the exponential regime without any intermediate step.

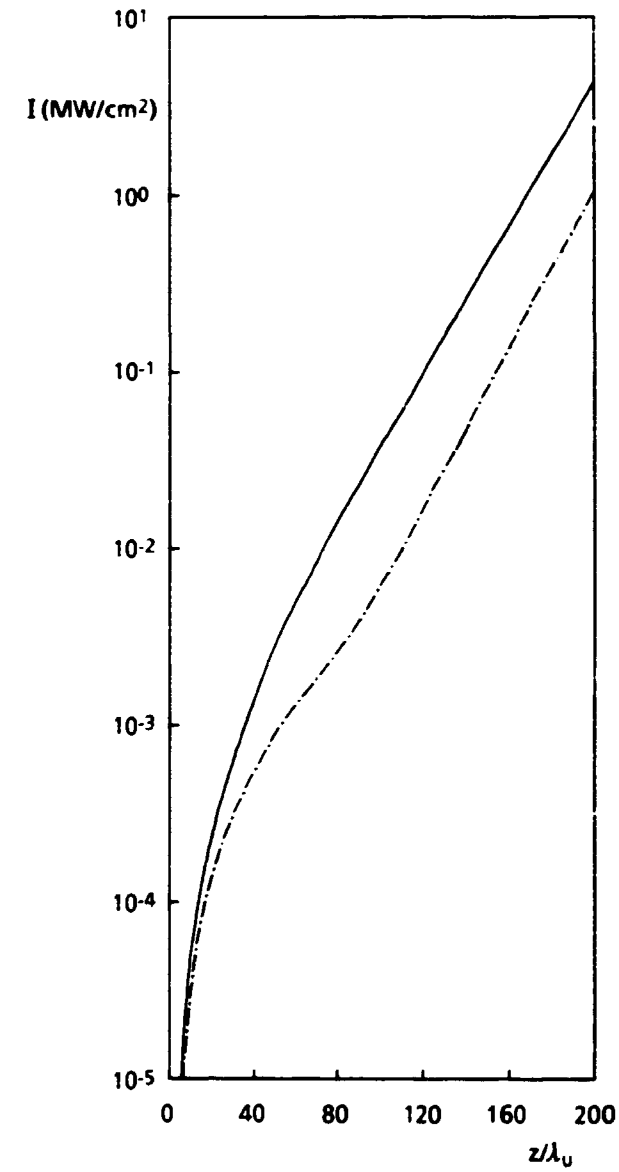

Fig. 5 Intensity evolution vs number of periods ($\nu_0=3$). (———) prebunched e-beam with inclusion of induced energy spread and distribution. (———) prebunched e-beam without the inclusion of induced energy spread and distribution, in both cases Re $b_1=-2.65\times10^{-2}$, Im $b_1=2.8\times10^{-2}$ and $\rho=2.27\times10^{-3}$

This aspect of the problem, along with the extension of these effects to the pulse propagation dynamics, will be further analyzed elsewhere.

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