Statistical Properties of the Zeros of Zeta Functions – Beyond the Riemann case

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Abstract: We investigate the statistical distribution of the zeros of Dirichlet $L$-functions both analytically and numerically. Using the Hardy–Littlewood conjecture about the distribution of primes we show that the two-point correlation function of these zeros coincides with that for eigenvalues of the Gaussian unitary ensemble of random matrices, and that the distributions of zeros of different $L$-functions are statistically independent. Applications of these results to Epstein’s zeta functions are shortly discussed.

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1 Introduction

Statistical reasoning and the modelization of physical phenomena by random processes have taken a major place in modern physics and mathematics. A physical example of this is the emergence of random behaviour from purely deterministic laws, as in classically chaotic Hamiltonian systems (see e.g. [1]). Randomness also enters in the quantum version of these systems (see e.g. [2]). In fact, the statistical properties of the semiclassical spectrum of fully chaotic systems are, in the universal regime, in good agreement with those obtained from an ensemble of random matrices [3, 4]. The Gaussian orthogonal ensemble (GOE) statistics applies to systems which are chaotic and have (generalized) time-reversal symmetry, while the Gaussian unitary ensemble (GUE) statistics are appropriate to describe systems which are chaotic and without time-reversal invariance. In particular, the GUE two-point correlation function is (after normalization of the average spacing between eigenvalues to unity)

\[ R_{2}^{GUE}(\epsilon) = 1 - \frac{\sin^{2}(\pi \epsilon)}{\pi^{2} \epsilon^{2}}, \quad (1.1) \]

or its Fourier transform, the two-point form factor, has the form:

\[ R_{2}^{GUE}(\tau) = \begin{cases} |	au| & \text{if } |	au| < 1 \\ 1 & \text{if } |	au| > 1. \end{cases} \quad (1.2) \]

A particularly interesting example of applying statistical considerations to a pure mathematical object is provided by the Riemann zeta function. This function is defined by a series

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (1.3) \]
which converges for \( \text{Re}(s) > 1 \) and can be analytically continued to the whole complex plane [5]. The region \( 0 < \text{Re}(s) < 1 \) is called the critical strip and it was proved [6] that in this region the Riemann zeta function has properties of a random function.

The Riemann hypothesis asserts that all the complex zeros of \( \zeta(s) \) lie on the critical line \( \text{Re}(s) = 1/2 \), which we henceforth denote by \( \mathcal{C} \). Assuming the Riemann hypothesis, Montgomery [7, 8] concluded that asymptotically the form factor of the critical Riemann zeros coincides, for \( |\tau| < 2 \), with the GUE result (1.2) (and conjectured that the agreement holds for arbitrary \( \tau \)). More recently, some spectacular numerical results by Odlyzko [9] strongly support that conjecture. Assuming a certain number theoretical hypothesis on the correlations between prime numbers to hold, Keating [10] showed that the main term of the two-point correlation function for the critical zeros of the Riemann zeta function does coincide with (1.2). In the context of "quantum chaos", an analogue of Montgomery's result was found by Berry [11], who also discussed the validity of the random matrix theory.

These two apparently disconnected physical and mathematical results have a common root in a formal analogy between the density of Riemann zeros expressed in terms of prime numbers (cf. Eqs.(2.7) below) and an asymptotic approximation of the quantum spectral density in terms of classical periodic orbits (the Gutzwiller trace formula [12]). This analogy has been fruitful for both mathematical and physical fields. For example, the correlations between prime numbers (the Hardy–Littlewood conjecture) inspired some work on the correlations between periodic orbits [13]. In the opposite direction, the statistical non-universalities of the spectral density, related to short periodic orbits, were successfully transposed to the Riemann zeta function [14].
It is thus clear that zeta functions are good models for investigating level statistics and the semiclassical trace formula. There are many generalizations of the Riemann zeta function [15], and since very little is known about their zeros it is of interest to investigate their distribution. In [16] the analog of Montgomery's result was proved for the average of all Dirichlet $L$-functions having the same modulus (which corresponds to the zeta function of a cyclotomic field). The purpose of this paper is to study the statistical properties of zeros of individual Dirichlet $L$-functions.

After a brief introduction (Section 2), in Section 3 we prove that the main asymptotic term of the two-point correlation function of the non-trivial zeros of Dirichlet $L$-functions with an arbitrary character agrees with GUE. We also prove the statistical independence of $L$-functions having different character and/or different modulus, i.e. the zeros of their product behave like the superposition of uncorrelated GUE-sets. The application of these results to the distribution of the zeros of the zeta function of positive binary quadratic forms, a particular case of the Epstein zeta function, is shortly discussed in Section 4.

2 Dirichlet $L$-functions

Dirichlet $L$-functions are natural generalizations of the Riemann zeta function (1.3). When $\text{Re}(s) > 1$ they are defined by a series (see, e.g. [17])

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$  \hspace{1cm} (2.1)

where the product is taken over all primes $p$.

Given an arbitrary integer $k$ (called the modulus), a function $\chi(n)$ (called a Dirichlet character mod $\chi$) is a complex function of positive integers satisfying: (i) $\chi(nm) = \chi(n)m^k$ and (ii) $\chi(n) = 1$ if $n \equiv 1 \pmod{k}$.
\( \chi(n) \chi(m) \), (ii) \( \chi(n) = \chi(m) \) if \( n \equiv m \mod k \), (iii) \( \chi(n) = 0 \) if \( (n, k) \neq 1 \), where \( (n, k) \) denotes the highest common divisor of \( n \) and \( k \).

A character is called principal and denoted by \( \chi_0 \) if \( \chi_0 = 1 \) when \( (n, k) = 1 \) and \( \chi_0 = 0 \) otherwise; the corresponding \( L \)-function essentially reproduces the Riemann zeta function. In fact

\[
L(s, \chi_0) = \zeta(s) \prod_{p|k} \left( 1 - p^{-s} \right),
\]

where the product is taken over all prime factors of \( k \). It also follows from the above definitions that \( \chi(1) = 1 \) and \( |\chi(k - 1)|^2 = |\chi(-1)|^2 = 1 \).

In general \( k \) can be any integer number. The total number of different characters modulo \( k \) is given by Euler's function \( \phi(k) \) (the number of positive integers prime to, and not exceeding \( k \)). The value \( \chi(n) \) is different from \( \chi_0 \) if \( (n, k) = 1 \) and its \( \phi(k) \) th power equals one. Table 1 provides a list of non-principal characters for \( k = 4 \) and \( 5 \), to be later on used in the numerical computations. (Detailed tables of characters can be found in [18]).

A character \( \chi \mod k \) is called nonprimitive if there is a divisor \( k' \) of \( k \) such that when \( n' \equiv n \mod k' \), \( \chi(n') = \chi(n) \). Otherwise the character is called primitive. For primitive characters Dirichlet \( L \)-functions satisfy the functional equation [19, 20]:

\[
\xi(s, \chi) = (-i)^a W_\chi \xi(1 - s, \bar{\chi})
\]

(2.2)

where

\[
\xi(s, \chi) = \left( \frac{k}{\pi} \right)^{s/2} \Gamma \left( \frac{s + a}{2} \right) L(s, \chi)
\]

and \( a = [1 - \chi(-1)]/2 \). \( W_\chi \) is a complex number of unit modulus which, for a given
character, is a constant
\[ W_k = \frac{1}{\sqrt{k}} \sum_{q=1}^{k-1} e^{2\pi i q/k} \chi(q). \] (2.3)

Like in the Riemann case, the functional equation allows to define a real function on the critical line \( L_c \) (where according to the generalized Riemann hypothesis should lie all non-trivial zeros of \( L \)-functions):
\[ Z(t, \chi) = e^{-\Omega_\chi(t)/2} L(1/2 - it, \chi) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \cos [t \ln n - \Theta_\chi(t)/2 + \arg \chi(n)] \] (2.4)

where the symbol \( \sum' \) indicates that the summation is done over all terms for which \( \chi(n) \neq 0 \) and
\[ \Theta_\chi(t) = \arg W_k + t \ln \left( \frac{k}{\pi} \right) - 2 \arg \left[ \Gamma \left( \frac{1 + 2\alpha}{4} - \frac{i}{2} \right) \right] - \frac{\alpha \pi}{2}. \]

Asymptotically
\[ \Theta_\chi(t) \sim_{00} \arg W_k + t \left[ \ln \left( \frac{kt}{2\pi} \right) - 1 \right] - \frac{\pi}{4} - O(t^{-1}). \] (2.5)

An approximate functional equation (the analogue of the Riemann–Siegel formula for \( \zeta(s) \)) also holds for \( L \)-functions in the large-\( t \) limit [21, 20]. It takes the form of a resummation of the series: instead of the infinite sum (2.4), \( Z(t, \chi) \) can be written as a truncated sum
\[ Z(t, \chi) \simeq 2 \sum_{n=1}^{N} \frac{1}{\sqrt{n}} \cos [t \ln n - \Theta_\chi(t)/2 + \arg \chi(n)] + O(t^{-1/4}), \] (2.6)

where \( N = \left[ \sqrt{\frac{t}{2\pi}} + \frac{1}{2} \right] k \) (the square brackets denote here integer part). An explicit form for the correction terms in Eq.(2.6) can be found in [21, 20]. This expression is particularly useful for numerical computations since we need to find the real zeros of a real function expressed as a finite sum of oscillating terms, their number being proportional to \( \sqrt{t} \).
As usual (see for example [10]), one can express the density of zeros lying on \( \mathcal{L} \) as a sum of an average part and a fluctuating part, \( d(t,\chi) = d_{av}(t,\chi) + d_{sc}(t,\chi) \). The result is

\[
d_{av}(t,\chi) = \frac{1}{2\pi} \frac{d\Theta_x(t)}{dt} \sum_{m=1}^{\infty} \frac{1}{2\pi} \ln (kt/2\pi)
\]

\[
d_{sc}(t,\chi) = -\frac{1}{\pi} \sum_p \sum_{m=1}^{\infty} \frac{\ln p}{m! \ln p} \cos [m (t \ln p + \arg \chi(p))] .
\]

This decomposition is analogous to the Gutzwiller trace formula [12], where the quantum spectral density is expressed as a sum of an average part and a fluctuating part. The average part of the level density is given by the so-called Weyl or Thomas–Fermi term and is related in the lowest order approximation to the derivative with respect to the energy of the volume of the classical phase space energy shell. The oscillating term is expressed as a sum over all the periodic orbits. Formally, Eq.(2.7b) is analogous to this latter term for a system whose periodic orbits are all isolated and unstable [12]. In this analogy, the prime numbers are identified with periodic orbits (whose lengths are given by the logarithm of the prime numbers) and the characters, being pure phase factors for \( (p,k) = 1 \), may be interpreted as Maslov indices, a quantity classically related to the focusing of a flux tube surrounding the periodic orbit. In part 5, however, we provide a different (and perhaps more relevant) interpretation of the characters in Eq.(2.7b) in terms of symmetries.

3 Two-point correlation function

Let us formally insert the density of zeros given by Eqs.(2.7) into the definition of the two-point correlation function

\[
R_2(\epsilon) = \langle d(t,\chi) d(t+\epsilon,\chi) \rangle
\]
and express $R_2(\epsilon)$ as a sum over prime numbers. In (3.1) the bracket ⟨ ⟩ denotes an average over an interval $\Delta t$ which is unavoidable when discussing statistical properties of a given function. Our purpose is to compute $R_2(\epsilon)$ in the large-$t$ limit and, accordingly, we choose the smoothing interval such that $1 \ll \Delta t \ll t$ [22]. Thus, any oscillating term with period smaller than $O(1)$ will be washed away by the averaging procedure. Then, from (3.1) and (2.7)

$$R_2(\epsilon) \approx d_{av}^2(t, \chi) + R_2^{osc}(\epsilon)$$

(3.2)

where

$$R_2^{osc}(\epsilon) = \langle d_{osc}(t, \chi) d_{osc}(t + \epsilon, \chi) \rangle$$

$$= \frac{1}{4\pi^2} \left\{ \sum_{p,p'} \chi(p)\chi(p') \ln p \ln p' \exp \left[ it(m \ln p - m' \ln p') - i\epsilon \ln p' \right] + \text{c.c.} \right\}$$

We must now show that $R_2^{osc}(\epsilon)$ reproduces the second term in the r.h.s. of Eq.(1.1). The main lines of our proof follow those developed in [10] for the Riemann zeta function.

The average in $R_2^{osc}(\epsilon)$ is not zero since the difference $m \ln p - m' \ln p'$ can be arbitrarily small (for large values of $p$ and $p'$) and can produce oscillating terms whose period is of order $\Delta t$ or bigger. Moreover, the sums are convergent for values of $m, m'$ bigger than one. So we restrict to $m = m' = 1$

$$R_2^{osc}(\epsilon) \approx \frac{1}{4\pi^2} \left\{ \sum_{p,p'} \chi(p)\chi(p') \ln p \ln p' \frac{1}{p^{1/2} (p')^{1/2}} \exp \left[ it(\ln p - \ln p') - i\epsilon \ln p' \right] + \text{c.c.} \right\}$$

(3.3)

Now we split the double sum into two parts $\sum_{p,p'} = \sum_{p=p'} + \sum_{p \neq p'}$, and first compute the diagonal part. Since $|\chi(p)| = 1$ if $(p, k) = 1$, then

$$R_2^{osc}(\epsilon)_{\text{diag}} = \frac{1}{4\pi^2} \sum_p \left( \frac{\ln^2 p}{p} e^{-i\epsilon \ln p} \right) + \text{c.c.}$$
Taking into account that only for a finite number of primes $(p,k) \neq 1$, it follows that one can replace the sum over primes for which $\chi(p) \neq 0$ by an integral, using the usual prime number theorem for the density of primes (see e.g. [5]). Putting $\tau = (\ln p)/2\pi$ one obtains:

$$R_{2}^{\text{off}}(\tau)_{\text{diag}} \approx \int_{0}^{\infty} d\tau \tau e^{-2\pi i \tau} + \text{c.c.} = -\frac{1}{2\pi^2 \tau^2}. \quad (3.4)$$

This contribution reproduces the non-fluctuating part of $R_{2}^{\text{GUE}}$. Alternatively, the contribution of (3.4) to the form factor is $\tau$, in agreement with Eq.(1.2) for $|\tau| < 1$.

In the off-diagonal part of (3.3) the smoothing will wash away terms for which $\Delta t \ln(p/p') \geq 1$, and $\Delta t \to \infty$ as $t \to \infty$. The main contribution then comes from large values of $p$ and $p \sim p'$. Putting $p' = p + h$ and assuming that $h \ll p$ the off-diagonal part reads

$$R_{2}^{\text{off}}(\tau)_{\text{off}} \approx \frac{1}{4\pi^2} \sum_{p} \ln^2 p e^{-i e \ln p} \left( \sum_{h} \chi(p) \chi(p + h) e^{-i h / p} \right) + \text{c.c.}, \quad (3.5)$$

where the external sum is taken over all primes and the internal one is taken over integers $h$ such that $p + h$ is a prime. To proceed further we need some information about the pair correlation function between primes. In Ref. [10] it was shown that for this purpose one can use the Hardy-Littlewood conjecture [23, 24], which gives the density $\Lambda_{h}(X)$ of primes $p$ lying between $X$ and $X + dX$ such that $p + h$ is also a prime.

According to this conjecture

$$\Lambda_{h}(X) \approx \frac{\alpha(h)}{\ln^2 X} \quad (3.6)$$

where

$$\alpha(h) = \alpha \prod_{p|h} \left( 1 + \frac{1}{p - 2} \right) \quad \text{and} \quad \alpha = 2 \prod_{q \geq 2} \left( 1 - \frac{1}{(q - 1)^2} \right).$$
The first product is taken over primes which divide \( h \) and the second one is taken over all primes (except 2).

\( \alpha(h) \) is an irregular number-theoretic function, and its main contribution comes from its average behavior, \( \langle \alpha(h) \rangle \). Moreover, we will only need the average behavior of \( \alpha(h) \) for large values of \( h \) (cf Eq.(B.2) below). If one averages over all integers \([10]\)

\[
\langle \alpha(h) \rangle \approx 1 - \frac{1}{2h} \quad \text{as } h \to \infty,
\]
expressing a repulsion between prime numbers for long distances.

Contrary to the Riemann case, to compute the two-point correlation function for Dirichlet L-functions one has to compute the mean value of \( \alpha(h) \) not averaging over all integers but over all equal integers mod \( k \) (i.e., all integers having the same remainder mod \( k \)). This is important because the quantity \( \chi(p)\overline{\chi}(p + h) \) which enters Eq.(3.5) depends only on \( h \mod k \). The details of this calculation are given in appendix A. The result is quite simple: for large \( h \)

\[
\langle \alpha(h) \rangle \approx \begin{cases} 
1 - 1/2h & \text{if } h \equiv 0 \mod k \\
1 & \text{otherwise.}
\end{cases}
\]

This equation is the essential ingredient of our proof. It expresses a non-trivial number-theoretic property of \( \langle \alpha(h) \rangle \) and completely eliminates the dependence on the character in (3.5). This is due to the fact that the \( h \)-independent (Poissonian) terms in (3.8) introduce no correlations between prime numbers. Then if in Eq.(3.5) \( \chi(p)\overline{\chi}(p + h) \) is replaced by its average, the sum over \( h \) of the Poissonian components vanishes. We thus only need to consider the terms \( h \equiv 0 \mod k \), and for them \( |\chi(p)|^2 = 1 \) if \( (p, k) = 1 \) and zero otherwise.
The computations are now exactly the same as for the Riemann case [10]. We briefly outline the main steps in appendix B, leading to

\[ R_2^{\text{off}}(\epsilon) \approx \frac{1}{2\pi^2} \frac{\cos(2\pi d_{av}\epsilon)}{\epsilon^2}. \]  

(3.9)

The final result for the two-point correlation function is, from (3.2), (3.4) and (3.9)

\[ R_2(\epsilon) = d_{av}^2 + R_2^{\text{diag}}(\epsilon) + R_2^{\text{off}}(\epsilon) \approx d_{av}^2 - \frac{\sin^2(\pi d_{av}\epsilon)}{\pi^2 \epsilon^2}, \]  

(3.10)

which coincides with the GUE two-point correlation function (1.1), once the average density is set to one. Eq.(3.10) holds for an arbitrary modulus and character.

In order to illustrate this result we have numerically computed, using the approximate functional equation (2.6), the zeros of Dirichlet \( L \)-functions for several values of \( k \) and, for each \( k \), for several characters (complex and real) and verified in all cases the agreement with the GUE statistics. For example, in Fig.1 we show \( R_2(\epsilon) \) and the nearest-neighbour spacing distribution for the approximately 18000 zeros lying in the interval \( 10^5 \leq t \leq 1.1 \times 10^5 \) for \( k = 5 \) and character \( \chi_2 \) of table 1. In part b of that figure the continuous curve represents the Wigner surmise \( p(s) = a s^2 \exp(-bs^2) \) where \( a = 32/\pi^2 \) and \( b = 4/\pi \).

Note that our result (3.10) for \( R_2(\epsilon) \) does not imply the Wigner surmise for \( p(s) \), since \( p(s) \) is a mixed \( n \)-point correlation function.

We shall now consider the correlations between zeros of different \( L \)-functions. For that purpose, let us consider the product of several \( L \)-functions having different characters \( \chi_i \mod k_i \). The total density is \( d_{av} = \sum_i d_{av}^i = \sum_i f_i d_{av} \), where the \( f_i \)'s are the relative densities and \( \sum_i f_i = 1 \). After averaging of exponential terms, the two-point correlation function

\[ R_2(\epsilon) = \sum_i R_2^{\text{diag}}(\epsilon) + \sum_i R_2^{\text{off}}(\epsilon) \approx \sum_i d_{av}^2 - \frac{\sin^2(\pi d_{av}\epsilon)}{\pi^2 \epsilon^2}. \]  

(3.11)
function for the product of $L$-functions can be written as

$$R_i(\varepsilon) = d_{av}^2 + \sum R_{i,j}^{sec}(\varepsilon) + \sum_{i \neq j} \langle d_{i}^{(i)}(t) d_{j}^{(j)}(t + \varepsilon) \rangle. \quad (3.11)$$

Now instead of $\chi(p)\bar{\chi}(p + h)$ in Eq.(3.5) we will find that $\langle d_{i}^{(i)}(t) d_{j}^{(j)}(t + \varepsilon) \rangle$ is proportional to $\chi_i(p)\bar{\chi}_j(p + h)$. Because of Eq.(3.8), only the terms $h \equiv 0 \mod k_j$ contribute and the result is therefore proportional to $\chi_i(p)\bar{\chi}_j(p)$. When $i \neq j$ the last quantity defines a non-principal character modulo the least common multiple of $k_i$ and $k_j$. All other terms are smooth in this scale and therefore the main contribution comes from the average value of this function. Since the mean value of any non-principal character is zero [17]

$$\frac{1}{k} \sum_{p \equiv 0} \chi_i(p) = 0,$$

it follows that

$$\langle d_{i}^{(i)}(t) d_{j}^{(j)}(t + \varepsilon) \rangle = 0 \quad \forall i \neq j$$

and sets of zeros for different $L$-functions (i.e., different $k$ and/or different character) are uncorrelated

$$R_2(\varepsilon) = d_{av}^2 - \sum \frac{\sin^2(\pi f_i d_{av} \varepsilon)}{\pi^2 \varepsilon^2}. \quad (3.12)$$

We have also numerically verified this prediction. Fig.2 plots the correlations between zeros for the product of the (three) non-principal characters for $k = 5$. The fluctuations are much smaller than in Fig.1 because of the better statistics. The continuous curve in part a) is the prediction (3.12), while the GUE result for $p(s)$ for a product of independent sets can be found in the appendix 2 of Ref.[3]. Similar numerical computations for Dirichlet $L$-functions were presented in [27].
4 Epstein zeta function

Let us consider briefly another interesting example of zeta function, namely Epstein's zeta function associated with a positive definite quadratic form \( Q(x, y) = ax^2 + bxy + cy^2 \)

\[
Z_Q(s) = \sum_{m,n} Q(m,n)^{-s},
\]

(4.1)

where the summation is done over all integers \( m, n \) except \( m = n = 0 \). \( Z_Q(s) \) is an analytic function, regular for \( \text{Re}(s) > 1 \), satisfying a functional equation and an approximate functional equation (see e.g. [25, 26]). In the following, we will consider the particular case \( a, b \) and \( c \) integers.

The properties of these functions strongly depend on the value of the discriminant \( \Delta = b^2 - 4ac \). If the class number of quadratic forms with a given discriminant is one, \( Z_Q(s) \) can be written as a product of two Dirichlet \( L \)-series, like for example

\[
\sum_{m,n} \frac{1}{(m^2 + n^2)^s} = 4 \zeta(s) L(s, \chi)
\]

(4.2)

where \( L(s, \chi) \) is the non-principal Dirichlet \( L \)-function for \( k = 4 \) (cf. table 1)

\[
L(s, \chi) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots.
\]

This type of Epstein zeta functions obviously have an Euler product and are assumed to satisfy the generalized Riemann hypothesis. For them, the results of the previous section hold, and we arrive at the conclusion that the statistical properties of their critical zeros are those of an uncorrelated superposition of two GUE sets, Eq.(3.12). Fig.3 illustrates this behaviour for the function (4.2).

When the class number is bigger than one, Epstein's functions cannot be factorized as in (4.2) and have no Euler product. Only a sum over all classes of forms with a given
discriminant is believed to have the above-mentioned properties. One (simple) member of this non-Euler class of functions is

\[ Z_Q(s) = \sum_{m,n} \frac{1}{(m^2 + 5n^2)^s}. \]  

(4.3)

For such functions with integer coefficients it is known [26]-[29] that

(i) there is an infinite number of zeros lying on \( L_c \);

(ii) many zeros lie off that line;

(iii) however, almost all the zeros lie on \( L_c \) or in its immediate neighbourhood.

Statement (iii) was recently made more precise. In fact, in Refs. [28, 29] it was proved that

\[ \frac{N_c(t)}{N(t)} \to 1 \text{ as } t \to \infty \],

where \( N(t) \) denotes the number of zeros of \( Z_Q(s) \) whose imaginary part is less or equal to \( t \) and \( N_c(t) \) the fraction of them lying on \( L_c \). To prove this the generalized Riemann hypothesis for Dirichlet \( L \)-functions was assumed to be valid as well as certain assumptions on the correlations between zeros.

Eq.(4.3) is an example of class number 2 Epstein's zeta function, which in general can be expressed as a sum of two terms \( L_1 L_2 \pm L_3 L_4 \), the \( L \) being appropriate Dirichlet \( L \)-functions. In [28, 29] it was proved that in sums of this type, in a certain range of \( t \), typically one of the two terms 'dominate'. This suggests that the statistical properties of the critical zeros for functions of the type (4.3) should be close to an uncorrelated superposition of two GUE sets.

5 Concluding Remarks

We have computed the two-point correlation function for the critical zeros of Dirichlet \( L \)-functions using the Hardy–Littlewood conjecture for the distribution of prime numbers
and showed that for any modulus and character the main term agrees with the statistics of the Gaussian Unitary Ensemble of random matrices. These results generalized those of Ref. [10] for the Riemann zeta function, and provide a unifying property for all L-functions.

The problem of estimating the next-to-leading terms (which should tend to zero as $t \to \infty$) is not simple, since it is connected to the short-range correlations between prime numbers [7, 8].

The Hamiltonian matrix of a quantum system having a discrete symmetry (like parity) splits, in the appropriate basis, into uncoupled submatrices, each of them corresponding to a symmetry class. It is well known (but not proved) that for a dynamical system with at least two degrees of freedom the eigenvalues belonging to different submatrices are uncorrelated. To each of these submatrices is associated a character of the symmetry group, which enters in the semiclassical trace formula. In fact, the symmetry-projected semiclassical spectral density includes the character of the symmetry group [30] exactly in the same way as in Eq.(2.7b) for Dirichlet L-functions. Thus, different characters in L-functions are the analog of different symmetry classes in quantum mechanics. In the light of these considerations, our result on the statistical independence of the zeros of different L-functions can be interpreted as the analog of the above-mentioned independence of the eigenvalues of different symmetry classes in quantum mechanics.

In all the cases we have investigated the zeros of zeta functions were distributed according to the statistics of the Gaussian Unitary Ensemble of random matrices, or a superposition of a few of them. If there exists a number-theoretic zeta function whose zeros obey the statistics of the Gaussian Orthogonal Ensemble remains an open problem.
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Appendix A

In this appendix we prove Eq.(3.8), concerning the average behaviour of $\alpha(h) \mod k$. The function $\alpha(h)$ in (3.6) can be expressed as [10]

$$\alpha(h) = \alpha \sum_{d|h} \beta(d)$$  \hfill (A.1)

where the sum runs over all divisors of $h$ and $\beta(d) = 0$ if $d$ is even or divisible by the square of a prime, otherwise

$$\beta(d) = \prod_{p|d} \frac{1}{p-2}.$$  

We are interested in the average over all integers $\mod k$. To do this let us define

$$\langle \alpha(h) \rangle = \frac{d}{dN} f(N)|_{N=h}$$  \hfill (A.2)

where

$$f(N) = \sum_{n=0}^{N} \alpha(nk+q).$$

From (A.1) it follows that

$$f(N) = \alpha \sum_{d=1}^{N} \mathcal{N}(N,d) \beta(d)$$  \hfill (A.3)

where $X = Nk+q$ and $\mathcal{N}(N,d)$ is the number of terms in the sequence $nk+q$, $n = 0, \ldots, N$ divisible by $d$. 

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Assuming that \( (d,k) = 1 \) it is easy to see that \( \mathcal{N}(N,d) = [(N + N_o)/d] \) where \( N_o = ((1/k)q)_d \) (henceforth, square brackets denote integer part, curly brackets fractional part and \( (x) \) means \( x \mod l \)).

Eq. (A.3) can be rewritten as the sum of two terms

\[
f(N) = \alpha \sum_{d=1}^{X} \left( \frac{N + N_o}{d} \right) \beta(d) - \alpha \sum_{d=1}^{X} \left\{ \frac{N + N_o}{d} \right\} \beta(d).
\]

(A.4)

The term proportional to \( N \) in the first sum can be computed from the relation [10]

\[
\sum_{d=1}^{\infty} \frac{\beta(d)}{d} = \frac{1}{\alpha};
\]

then, in the large-\( N \) limit

\[
\alpha N \sum_{d=1}^{X} \frac{\beta(d)}{d} \approx N. \tag{A.5}
\]

In the second sum, the fractional part depends only on \( N \mod d \) and its average value equals 1/2. Moreover, because as \( d \to \infty \) \( \beta(d) \) behaves, on average, like \( (\alpha d)^{-1} \) (see Ref.[10]) then

\[
\alpha \sum_{d} \left\{ \frac{N + N_o}{d} \right\} \beta(d) \approx \frac{\alpha}{2} \sum_{d=1}^{X} \beta(d) \approx \frac{1}{2} \log N. \tag{A.6}
\]

The most delicate term is the one involving \( N_o \) in the first sum, since \( N_o \) is a function of \( d \)

\[
\alpha \sum_{d=1}^{X} \frac{N_o \beta(d)}{d} = \alpha \sum_{d=1}^{X} (mq)_d \frac{\beta(d)}{d},
\]

where \( m = (1/k)_d \). Since by definition \( mk = 1 \mod d \), one can write

\[
mk = 1 + nd, \tag{A.7}
\]

\( n \) being an integer \( (< k) \) depending only on \( (d)_k \). Neglecting the contribution of \( q/k \), from (A.7) it follows that

\[
(mq)_d \approx \left\{ \frac{nq}{k} \right\} d. \tag{A.8}
\]
But again, on average, we can do the replacement

\[ \left\{ \frac{nq}{k} \right\} \to \frac{1}{2}. \]

Then

\[ \alpha \sum_{d=1}^{X} \left( \frac{1}{k} \right) q \frac{\beta(d)}{d} \approx \alpha \sum_{d=1}^{X} \left\{ \frac{nq}{k} \right\} \frac{\beta(d)}{d} \approx \frac{\alpha}{2} \sum_{d=1}^{X} \beta(d) \approx \frac{1}{2} \log N, \]

which exactly cancels the term (A.6). From this, Eq.(3.8) follows.

**Appendix B**

In this appendix we outline the main steps leading to Eq.(3.9). Replacing in (3.5), as for the diagonal term, the sums by integrals and from the considerations following Eq.(3.8) we can write

\[ R_2^{\text{off}}(e)_{\text{dt}} \approx \frac{1}{2\pi^2} \int_{1}^{\infty} \frac{dp}{p} e^{-ip\ln p} \int_{2}^{\infty} dh(h) \cos(\frac{th}{p}) + \text{c.c.} \quad (B.1) \]

We now make two changes of variables, \( w = (\ln p)/2\pi d_{av} \) and \( y = (2\pi)^w t^{1-w} h \). In the first one, we include the average density in order to set the mean level spacing between zeros to one. For the second, since \( p = \exp(2\pi d_{av} w) \) and from the asymptotic result (2.7a) for the average density, we have \( t/p \sim (2\pi)^w t^{1-w} \). Therefore

\[ R_2^{\text{off}}(e)_{\text{dt}} \approx \frac{d_{av}}{\pi} \int_{0}^{\infty} dw \exp(-2\pi id_{av} \epsilon w) \frac{\epsilon^{w-1}}{(2\pi)^w} \int_{2(2\pi)^{w-1-w}}^{\infty} dy \left( \frac{y^{w-1}}{(2\pi)^w} \right) \cos y + \text{c.c.} \quad (B.2) \]

The lower bound of the second integral has a different asymptotic behaviour according to whether:

i. \( w < 1; \text{ when } t \to \infty, t^{1-w} \to \infty. \text{ The integral vanishes.} \)

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ii. \( w > 1 \). In the limit, \( t^{1-w} \to 0 \).

As \( t \to \infty \), the argument of the averaged \( \alpha \)-function tends to infinity, and we can use the asymptotic result (3.7). As stated before, the Poissonian component does not contribute, since the integral over that term in (B.2) vanishes. The second term yields

\[
R_2^\text{ac}(\epsilon)_{\text{off}} \approx \frac{d_{av}}{2\pi} \int_1^\infty dw \exp(-2\pi id_{av}w) \int_{2(2\pi)^{1-w}}^\infty dy \cos y/y + \text{c.c.}.
\]  

(B.3)

But, for small \( x \), \( \int_1^\infty dy \cos y/y \simeq -C - \ln x + \mathcal{O}(x^2) \), where \( C \) is Euler's constant. This approximation and the asymptotic average density allow to rewrite (B.3) as

\[
R_2^\text{ac}(\epsilon)_{\text{off}} = \frac{d_{av}}{2\pi} \int_1^\infty dw e^{-2\pi id_{av}w} 2\pi d_{av} (1 - w) + \text{c.c.}
\]

or, by an evident transformation

\[
R_2^\text{ac}(\epsilon)_{\text{off}} = \int_0^\infty d\epsilon e^{-2\pi i \epsilon} (d_{av} - \tau) + \text{c.c.}
\]  

(B.4)

\[
= \frac{1}{2\pi^2} \cos \left( \frac{2\pi d_{av}\epsilon}{\epsilon^2} \right).
\]  

(B.5)

References


FIGURE CAPTIONS

Figure 1: a) Two-point correlation function and b) nearest-neighbor spacing distribution for the critical zeros of Dirichlet $L$-function with $k = 5$ and character $\chi_2$ of table 1. The continuous curves correspond to the theoretical prediction.

Figure 2: Same as in Figure 1 but for the critical zeros of the product of the three non-principal characters for $k = 5$.

Figure 3: nearest-neighbor spacing distribution for the critical zeros of Epstein's zeta function (4.2) lying in the interval $10^5 \leq t \leq 1.1 \times 10^5$. 
Table 1: Non-principal characters for $k = 4$ and 5.

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<th>$X_2$</th>
<th>$X_3$</th>
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</tr>
<tr>
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<td>0</td>
<td>$-1$</td>
<td>$i$</td>
</tr>
<tr>
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<td>$-1$</td>
<td>$-1$</td>
<td>$i$</td>
</tr>
<tr>
<td>4</td>
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</table>
Figure 2

Graph (b)

Graph (a)