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INVOLUTIVE CO-DISTRIBUTIONS PRESERVED BY TRANSITIVE FAMILIES OF VECTOR FIELDS

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INVOLUTIVE CO-DISTRIBUTIONS PRESERVED BY TRANSITIVE FAMILIES OF VECTOR FIELDS[1]

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ABSTRACT

This paper leads with integrability conditions of involutive co-distributions defined on co-tangent bundle of a differentiable manifold M. Via Frobenius's integrability theorem, the analysis is aimed at the search for conditions so that this type of co-distributions preserved by transitive families of vector fields in M. We rely on the work of Lobry, Sussmann, Matsuda and Stefan. The type of situation studies comes up naturally in weak-observability problems and weakly-minimal realizations of arbitrary control systems.

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1 Introduction

Let M be a differentiable manifold and θ a co-distribution on the co-tangent bundle T^*M, that is, for each $x \in M$, $\theta(x)$ is a vector subspace of the dual of the tangent space to M in x. Frobenius's theorem ensures that the integrability of θ if this co-distribution is involutive and regular, [10]. In this paper we are interested in finding conditions that warrant regularity of the involutive co-distribution. As we shall see, the type of situation to be analyzed appears in the study of weak-observability and weakly-minimal realization of control systems.

A control system Σ is determined by the following data:

$$\Sigma = (M, D, \mathbb{R}^s, h)$$

where M is a differentiable manifold, $D \subset X(M)$ is a family of vector fields in M, (strategies), and $h : M \to \mathbb{R}^s$ is a differentiable function.

Let us denote by X_t the flow associated to the vector field X. Then D induces a pseudo-group

$$G_\Sigma = \{X^1_{t_1} \circ X^2_{t_2} \circ \ldots \circ X^k_{t_k} | X^j \in D, t_j \in \mathbb{R}\}$$

and a semi-pseudo-group

$$S_\Sigma = \{X^1_{t_1} \circ X^2_{t_2} \circ \ldots \circ X^k_{t_k} | X^j \in D,\ t_j \geq 0\}$$

of local diffeomorphisms on M.

G_Σ acts over M on such a way that the orbits

$$G_\Sigma(x) = \{\rho(x) | \rho \in G_\Sigma\},\ x \in M$$

form a foliation with singularities, [7].

Σ is said to be transitive if there is a single orbit and in this case for each $x \in M$, $G_\Sigma(x) = M$.

Each orbit $G_\Sigma(x)$ has a structure of a differentiable manifold, however the topology can be strictly finer than the induced topology. It is possible to restrict Σ to each orbit $G_\Sigma(x)$, thus obtaining a natural environment of the system for the initial condition x, [9]. Particularly, it is always possible to consider Σ as being transitive (on its orbits).

The accessibility set of Σ from the point x is given by its positive orbit under Σ, that is:

$$S_\Sigma(x) = \{\rho(x) | \rho \in S_\Sigma\}\ .$$

If $S_\Sigma(x) = M$, we say that the system Σ is controllable from x.

Two states $x, y \in M$ are said to be undistinguishable (by Σ) if

$$h \circ \rho(x) = h \circ \rho(y), \quad \forall \rho \in S_\Sigma\ .$$

Σ is said to be observable if there are not two undistinguishable states in M, and weakly-observable in $x \in M$ if there exists a neighbourhood U of x such that for each $y \in U$, x and y are not undistinguishable, i.e. the family of local diffeomorphisms S_Σ restricted to U separates x from any point $y \in U$.

A minimal realization for Σ is a transitive and observable system $\Sigma_1 = (M_1, d_1, \mathbb{R}^s, h_1)$ and a submersion $\pi : M \to M_1$ such that: each vector field $X \in D$ is projectable on M_1 via the differential $d\pi = \pi_*$, $\pi_*(D) = D_1$ and $h = h_1 \circ \pi$.

Two states $x, y \in M$ are said to be quasi-undistinguishable (by Σ) if:

$$h \circ \rho(x) = h \circ \rho(y), \ \forall \rho \in G_\Sigma \ .$$

The concepts of quasi-observability and quasi-weakly observability for quasi-minimal realization are defined in an analogous way.

The study of the observability of Σ naturally leads to the analysis of the curves $h_i \circ \phi(x)$ for $1 \leq i \leq s$ and $\rho \in S_\Sigma$.

Hermann and Krener, with the purpose of studying sufficient conditions of weak-observability introduced in [4] the co-distribution defined by:

$$\Lambda = \text{Span}\{L_{X^1} \circ L_{X^2} \circ \ldots \circ L_{X^r}(dh_i) | X^j \in D, 1 \leq i \leq s\}$$

where $L_X(h_i) = dh_i(X)$ is a Lie's derivative of h_i in the direction of the vector field X. Since this derivative commutes with the exterior derivative [10], i.e. $dL_X(f) = L_X(df)$ for any function $f : \text{dom}(f) \subset M \to \mathbb{R}$, Λ is, to a certain extent, an infinitesimal measure of the behaviour of h with respect to all possible concatenations of the system strategies.

In [4] it is proved the so-called weak-observability rank condition:

$$\dim \Lambda(x) = \dim M \Rightarrow \Sigma \text{ is weakly-observable in } x \ .$$

Basto Goncalvez studies in [3] the problem of the quasi-observability of systems and defines Δ as the smallest Σ invariant co-distribution containing the dh_i, $1 \leq i \leq s$, and builds quasi-minimal realization of Σ, through the integral manifolds of the regular co-distribution Δ.

In [1] Ayala and San Martin introduce the co-distribution

$$\Delta^+ = \text{Span } \{\rho^*(dh_i) | \rho \in S_\Sigma, \ 1 \leq i \leq s\}$$

where the pull-back is defined by $\rho^*(dh_i) = d(h_i \circ \rho)$ obtaining weakly-minimal realization of controllable systems in the process of building minimal realization of this type of systems.

As we shall see for transitive systems, Δ will always be integrable, however this may not be true for Λ and Δ^+. on the other hand $\Lambda \subset \Delta^+ \subset \Delta$ and since Λ is in general less difficult to build, it is convenient to specify the situations such that $\Lambda = \Delta^+, \Lambda = \Delta^+ = \Delta$. Under these circumstances, the study of the observability of control system leads to the search for integrability conditions of involutive co-distributions in the sense of differential ideals, in fact Λ, Δ^+ and Δ are generated by families of exact 1-forms. Consequently, the condition of regularity (that is, that the dimension of this type of co-distribution remains constant on M) is sufficient to determine integrability in the classical sense by a direct application of Frobenius theorem [10].

For this purpose we rely on the techniques used by Lobry [5], Sussmann [9], Matsuda [6] and Stefan [8], in relation to the study of the integrability of distribution on the tangent bundle TM.

2 Co-distributions

Let us begin with some basic concepts. For each differentiable manifold M and for each x in M, T_x^*M, will denote the dual of the vector space T_xM, that is

$$T_xM = \{\lambda : T_xM \to \mathbb{R} | \lambda \text{ linear}\}$$

The co-tangent bundle on M is by definition the manifold

$$T^*M = \bigcup_{x \in M} T_x^*M \ .$$

A codistribution Θ on M is a map such that for each $x \in M$, $\Theta(x)$ is a vector sub-space of T_x^*M. Θ will be said to be regular if $\dim \theta(x)$ is independent of x in M.

An immersion (N, i) is said to be a (maximal) integral manifold of θ through the point x in M, if

a) $\theta(i(x))(di_x(T_xN)) = 0$

b) dim (N)=dim(M)$-$dim$\theta(i(x))$.

θ is integrable in x, (in M), if there exists an integral manifold of θ containing x, denoted by $In_\theta(x)$, $(\forall x \in M)$.

Let us look at this concept under the light of the co-distributions, Λ, Δ^+ and Δ.

Examples 2.0.

1. Let $\Sigma = (\mathbb{R}, D = \{u\frac{d}{dx} | u \in \mathbb{R}\}, \mathbb{R}^2, h = (\cos x, \text{sen} x))$. Then

$$\Lambda(x) = \text{Span } \{\text{sen } x dx, \cos \ x \ dx\} = T_x^*\mathbb{R} \ .$$

But $T_x^*\mathbb{R}$ can be nullified only at the origin x. Thus, the integral manifold Λ at $x \in \mathbb{R}$ is $In_\Lambda(x) = \{x\}$.

Evidently, i is the inclusion. Since $\Lambda \subset \Delta^+ \subset \Delta$ we have $\Lambda = \Delta^+ = \Delta$.

2. Let $\Sigma = (\mathbb{R}^2, D = \{u\frac{\partial}{\partial x} | u \in \mathbb{R}\}, \mathbb{R}, h)$ be a control system such that

$$h(x, y) = \int_{-\infty}^{x} f(t) dt$$

where f is a function of C^∞ class such that is zero on $(-\infty, 0]$ and strictly increasing on $(0, \infty)$. For each $\omega = (x, y) \in \mathbb{R}^2$ we have

$$\Lambda(\omega) = \text{Span } \{f^{(k)}(x) dx | k \geq 0\} \ .$$

Then

$$\Lambda(\omega) = \begin{cases} \text{Span}\{dx\} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \ . \end{cases}$$

Thus, if π_1 denotes the projection on x axis we have:

a) $J = \{w \in \mathbb{R}^2 | \pi_1(\omega) < 0\}$ is the integral manifold of Λ through each $\omega \in J$.

b) Span $\{\frac{\partial}{\partial y}|_\omega\}$ is the integral manifold of Λ containing ω for each ω satisfying $\pi_1(\omega) < 0$.

c) There is no integral manifolds of Λ containing elements with null first coordinate. In fact, since all the manifolds considered are boundaryless, each integral manifold of Λ "passing" over these points will have to "go through" the y axis. This situation is not possible because of the difference of the dimension of Λ in any neighbourhood of this type of state.

The integrability of Δ^+ comes from de equality

$$\Delta^+(x, y) = \text{span}\{f^{(k)}(x + t) dx | t > 0, k \geq 0\} = \text{span } \{dx\} \ .$$

In particular, Δ^+ is regular and for each $\omega = (x,y)$ in $\mathbb{R}^2$ we have

$$In_{\Delta^+}(x,y) = \text{Span}\,\{\frac{\partial}{\partial y}|_\omega\}\ .$$

besides, we can observe that in this case $\Lambda \subsetneq \Delta^+ = \Delta$.

Definition 2.1. Let $X \in X(M)$ and θ a co-distribution on M. We say that X preserves θ if for each $x \in \text{dom}(X)$ and for each possible $t \in \mathbb{R}$

$$X_t^*\theta(X_t(x)) \subset \theta(x)$$

$\diamondsuit$

Remarks 2.2.

1. If X preserves θ then for each $x \in \text{dom}(X)$

$$X_t^*\theta(X_t(x)) = \theta$$

for each possible $t \in \mathbb{R}$.

2. Let F be a transitive family of vector fields on M, that is, the pseudo-group G_F built analogously to pseudo-group G_Σ for control systems satisfy

$$\exists x \in M : G_F(x) = M\ .$$

Let us suppose that F preserves θ, then, according to the remark above, the dimension of θ is independent from $x \in M$ and thus, θ is regular. It is important to note here the relevance of the orbit theorem, [9]. Indeed, given a control system Σ on M, the vector fields family $F = D = D(\Sigma)$ is transitive over each orbit of Σ. Particularly, each co-distribution θ preserved by D will be regular.

3. Note that $-\frac{\partial}{\partial x}$ does not preserve $\Lambda = \text{Span}\{f^{(k)}dx\}$ in example 2.0 (2).

4. It is well-known that the operator of exterior derivative d satisfies $d^2 = 0$. In particular, the exterior derivative of an exact 1 form df is null. The co-distributions introduced in Sec. 1 were defined by

$$\Lambda = \text{Span}\{d(L_{X^1} \circ L_{X^2} \circ \ldots \circ L_{X^k}(h_i)|X^j \in D,\ 1 \le i \le s\}$$

$$\Delta^+ = \text{Span}\{d(\rho \circ h_i)|\rho \in S_\Sigma, 1 \le i \le s\}$$

$$\Delta = \text{Span}\{d(\rho \circ h_i)|\rho \in G_\Sigma, 1 \le i \le s\}$$

consequently

$$d(\Lambda) = d(\Delta^+) = d(\Delta) = 0\ .$$

In particular, these co-distributions are involutive. So, we must try to find conditions under which this type of co-distributions is preserved by transitive families of vector fields.

The introduction of the concept that follows is thoroughly justified. let ω a 1-form in M, i.e.

$$\omega : x \in \text{dom}(\omega) \subset M \to \omega_x \in T_x^*M$$

and let $X \in X(M)$. Lie's derivative of ω in the direction of the vector field X in $x \in \text{dom}\,(X) \cap \text{dom}(\omega)$ is by definition

$$L_X(\omega)|_x(\cdot) = \frac{d}{dt}\Big|_{t=0} \omega_{X_t(x)}((X_t)_*(x)(\cdot))\ .$$

Now, let θ be a co-distribution in M parametrized by $\delta = (\omega^1, \omega^2, \ldots, \omega^r)$ over a "piece of trajectory of X in x", i.e.

$$\exists \varepsilon > 0 : \theta(X_t(x)) = \text{Span}\,\{\omega^1_{X_t(x)}, \omega^2_{X_t(x)}, \ldots, \omega^r_{X_t(x)}\}$$

for each $t \in (-\varepsilon, \varepsilon)$. Then, by definition, a vector field X preserves θ if

$$(X_t)^*\theta(X_t(x)) \subset \theta(x)$$

equivalently, if the curves

$$\psi_j(t) = X_t^*\omega^j_{X_t(x)} \in \theta(x), \quad 1 \le j \le r\ .$$

By derivation, we obtain for each $j = 1, 2, \ldots, r$,

$$\frac{d}{dt}\psi_j(t) = X_t^* L_X(\omega^j)(X_t(x))\ .$$

In particular, and since for each $j = 1, 2, \ldots, r$, $\omega^j_x \in \theta(x)$ we can conclude that a necessary condition for X to preserve θ is

$$\frac{d}{dt}\psi_j(t) \in \theta(x)$$

and then with $t = 0$, we obtain that Lie's derivative of ω^j in the direction of the vector field X at x must belong to θ, i.e.

$$L_X(\omega^j)\Big|_x \in \theta(x)$$

for each

$$x \in \text{dom}(X) \cap \text{dom}(\omega^j), \quad j = 1, 2, \ldots r\ .$$

However, as shown in the example 2.0 (2), this condition is not sufficient and therefore if we wish to study integrability via the above notion we need some additional hypothesis on these derivatives.

Definition 2.3. Let θ be a co-distribution

1. θ is said to be differentiable at x if there are 1-forms $\omega^1, \omega^2, \ldots, \omega^r$ differentiable in a neighborhood U in x satisfying

   i) $\omega \in \theta$ on U

   ii) $\theta(x) = \text{Span}\{\omega^1_x, \omega^2_x, \ldots, \omega^r_x\}$

2. Let θ be a differentiable at x, the map

$$y \in U \to \delta(y) : \mathbb{R}^r \to \theta(y) \subset T_y^*M$$

defined by

$$\delta(y)(t_1, t_2, \ldots, t_r) = \sum_{j=1}^r t_j \omega^j_y$$

is called a parametrization of θ centered in x. We denote $\delta = (\omega^1, \omega^2, \ldots, \omega^r)$.

3. Let $\delta = (\omega^1, \omega^2, \ldots, \omega^r)$ be a parametrization of θ centered in x. We say that a vector field X in M preserves θ infinitesimally at x along δ if:

i) there is an interval J containing the origin such that:

$$t \in J \Rightarrow \theta(X_t(x)) = \text{Span } \{\omega^1_{X_t(x)}, \omega^2_{X_t(x)}, \ldots, \omega^r_{X_t(x)}\}$$

ii) there are Lebesque-integrable functions $a_{ij} : J \to \mathbb{R}$ such that for each $t \in J$ and $j = 1, 2, \ldots, r$

$$L_X(\omega^j)(X_t(x)) = \sum_{j=1}^{r} a_{ij}(t)\omega^j_{X_t(x)}$$

$\diamond$

Motivated by the co-distributions defined in relation to the problem of the observability of control systems, each distribution will be assumed to be differentiable at x for each x in M.

Together with the concepts above the following lemma is obtained.

Lemma 2.4. If $X \in X(M)$ preserves a co-distribution θ infinitesimally at x in M along the parametrization $\delta = (\omega^1, \omega^2, \ldots, \omega^r)$ then, for each $t \in J$

$$X_t^*\theta(X_t(x)) = \theta(x)$$

Proof. It is sufficient to show that for each $t \in J$ and for each $j = 1, 2, \ldots, r$

$$\alpha_j(t) = X_t^*(x)\omega^j_{X_t(x)} \in \theta(x)$$

Consider an application $\lambda \in T_x^*M$ such that $\theta(x) \subset \text{Ker }(\lambda)$. Let us denote for $j = 1, 2, \ldots, r$

$$\gamma_j(t) = \lambda(\alpha_j(t)) \ .$$

After a derivation, we obtain

$$\begin{aligned} \frac{d}{dt}\alpha_j(t) &= X_t^* L_X(\omega^j)(X_t(x)) \\ &= X_t^*(\sum_{j=1}^{r} a_{ij}(t)\omega^j_{X_t(x)}) \\ &= \sum_{j=1}^{r} a_{ij}(t)\alpha_j(t)) \ . \end{aligned}$$

Then

$$\frac{d}{dt}\gamma_j(t) = \sum_{j=1}^{r} a_{ij}(t)\lambda(\alpha_j(t)) \ .$$

In particular if γ denotes $(\gamma_1, \gamma_2, \ldots, \gamma_r)$ as column vector, we obtain

$$\dot\gamma = (a_{ij}) \cdot \gamma$$

$$1 \le i, \ j \le r$$

then γ satisfies a linear differential equation with Lebesgue-integrable coefficients and thus the existence and uniqueness of the solutions are warranted. Now, for each $j = 1, 2, \ldots, r$

$$\alpha_j(0) = \omega^j_x \in \theta(x)$$

and by the construction of λ we obtain $\gamma(0) = 0$ and consequently $\gamma \equiv 0$. Otherwise

$$\lambda(\alpha_j(t)) = 0, \ j = 1, 2, \ldots, r$$

therefore $\alpha_j(t) \in \theta(x)$ for each $j \in \{1, 2, \ldots, n\}$. In fact, the reasoning above is true for every λ satisfying $\theta(x) \subset \text{Ker }(\lambda)$

$\diamond$

Remark 2.5. Lemma 2.4 establishes that if $X \in X(M)$ preserves θ infinitesimally at x along δ, then X preserves θ over the set

$$\{X_t(x) | t \in J\} \ .$$

This situation is still insufficient since the parametrization δ explains only a "piece of trajectory through x" and not on a neighbourhood of point x in M.

Definition 2.6. A parametrization $\delta = (\omega^1, \omega^2, \ldots, \omega^r)$ of θ is called exhaustive over a set $A \subset M$, if for each $y \in A$

$$\theta(y) = \text{Span}\{\omega^1_y, \omega^2_y, \ldots, \omega^r_y\}$$

$\diamond$

Now, we are really in position to make the best of definition 2.3.

Lemma 2.7. Let us suppose that $\delta = (\omega^1, \omega^2, \ldots, \omega^r)$ is an exhaustive parametrization of θ over an open V of M and that $X \in X(M)$ preserves θ infinitesimally along δ at all points of V. Then, X preserves θ in V.

Proof. We use a compactness argument. Let $T > 0$ and $y \in \bigcap_t \text{dom}(X_t)$. Let us define the set S by:

$$S = \{t \in [0, T] | X_\sigma^*(\theta(X_\sigma(y))) = \theta(y), \ \forall \sigma \in [0, T]\} \ .$$

Lemma 2.4 states that S is not empty. Let s be the suprem of S. Since X preserves θ infinitesimally in $X_s(y)$ along δ there exist $\varepsilon > 0$ such that

$$-\varepsilon < t < \varepsilon \Rightarrow X_t^*\theta(X_{t+s}(y)) = \theta(X_s(y)) \ .$$

Then, $s = T$ and X preserves θ. $\diamond$

Let F be a transitive family of vector fields on M and θ a co-distribution. The Lemmas in the paragraph above can be summarized as follows: if Lie's derivatives of exhaustive parametrization of θ in the direction of the elements of F can be written as linear combinations with Lebesgue-integrable coefficients, then F preserves θ. We must study, therefore, co-distributions having this type of parametrizations.

3 Integrability

We start a new concept.

Definition 3.0. Let $F \subset X(M)$ and θ a co-distribution in M. θ is said to be F-involutive if

$$L_X(W) \in \theta, \ \forall X \in F, \ \forall \omega \in \theta$$

◇

Remarks 3.1.

1. If F preserves θ then θ is F-involutive. In particular, Δ is D_Σ-involutive. In fact,

$$\Delta = \text{Span } \{d(h_i \circ \phi) | \phi \in G_\Sigma, \ i = 1, 2, \ldots, s\}$$

and, if $\beta \in G_\Sigma$ and $d(h_i \circ \phi) \in \Delta$ we have:

$$\beta^*(d(h_i \circ \phi)) = d(h_i \circ (\phi \circ \beta)) \ .$$

Since $\phi \circ \beta \in G_\Sigma$ it follows that $\beta^*(d(h_i \circ \phi)) \in \Delta$ for each $i = 1, 2, \ldots, s$. Consequently, $F = D_\Sigma$ preserves Δ. In fact, the elements generating Δ form parametrizations of this co-distribution over open sets of M.

2. From the definition of Λ itself we conclude that this co-distribution is D_Σ-involutive.

3. Δ^+ is D_Σ-involutive, however, as been before Δ^+ may not be preserved by D_Σ.

Theorem 3.2. Let F be a transitive family over M and θ a co-distribution in M. Then,

F preserves $\theta \Leftrightarrow \theta$ is regular and F-involutive.

Proof. Suppose that θ is regular and F-involutive and let $x \in M$ and $\delta = (\omega^1, \omega^2, \ldots, \omega^r)$ a parametrization of θ centered in x (in a neighbourhood of x). θ is regular and since the 1-forms ω^j, $j = 1, 2, \ldots, r$ are continuous, there must be a neighbourhood V_0 of x such that for each $y \in V_0$

$$\theta(y) = \text{Span } \{\omega_y^1, \omega_y^2, \ldots, \omega_y^r\}$$

(The argument used here is of the following type: if a continuous function is not null in a specific point then it is not null in a certain neighborhood of the point considered).

Consequently, δ is an exaustive parametrization of θ over V_0. By hypothesis θ is F-involutive and then

$$\forall w \in \theta, \forall X \in F, \ L_X(\omega) \in \theta \ .$$

Thus, for $y \in V_0$, $i = 1, 2, \ldots, r$ we have

$$L_X(\omega^i)\Big|_y = \sum_{j=1}^{r} a_{ij}\omega_y^j \ .$$

As $\{w^1, w^2, \ldots, w^r\}$ can be considered as a linearly independent set in V_0, we conclude that the functions $a_{ij} : V_0 \to \mathbb{R}$ are determined.

Now, if we prove that these functions are Lebesgue-integrable then since the reasoning above is independent of the point $y \in V_0$ and $X \in F$, Lemma 2.7 applies and hence F would preserve θ in V_0. As the exhaustive parametrizations of θ cover M thoroughly, the result would be proved. It is possible to prove something stronger: each a_{ij} is a differentiable function.

For each $y \in V_0$ we consider the linear mapping

$$\delta(y) : \mathbb{R}^r \to \theta(y) \subset T_y^* M$$

defined by

$$\delta(y)(t_1, t_2, \ldots, t_r) = \sum_{j=1}^{r} t_j \omega_y^j.$$

Then, δ is differentiable in V_0 and $\delta(y)$ has a range r. Le us also observe that

$$L_X(\omega^i)|_y = \delta(y)(a_{i1}(y), a_{i2}(y), \ldots, a_{ir}(y))$$

$$= \delta(y)(a_i(y))$$

is differentiable in V_0. Due to arguments on range, there are matrices $P(y) \in GL_r(\mathbb{R}), Q(y) \in M_{(m-r)\times r}(\mathbb{R})$, where m is of course the dimension of M and then the dimension of the co-tangent space $T_y^* M$, such that

$$\delta(y) = \begin{pmatrix} P(y) \\ Q(y) \end{pmatrix} \ .$$

The differentiability of $P(\cdot)$ is warranted by the differentiability of $\delta(\cdot)$. As for each $y \in V_0, P(y)^{-1}$ exist we conclude that $P(\cdot)^{-1}$ is also differentiable. Now

$$a_i(y) = (P(y)^{-1} 0) \begin{pmatrix} P(y) \\ Q(y) \end{pmatrix} a_i(y) = (P(y)^{-1} 0) L_X(\omega^i)\Big|_y \ .$$

So, the applications

$$a_i = (a_{i1}, a_{i2}, \ldots, a_{ir})$$

are differentiable in a neighbourhood of x.

Since the reciprocal was already proved, the proof of the theorem is finished. ◇

Extensions of Theorem 3.2.

1. The differentiability of the matrix (a_{ij}) in the above theorem is a condition quite stronger than the restriction Lebesgue-integrable. When explaining this situation, the concept below comes up naturally.

Definition 3.3. θ is said to be F-weakly involutive, if for each $x \in M$, there is a neighbourhood $\mathcal{U}$ of x and an exaustive parametrization δ in $\mathcal{U}$, such that, each $X \in F$ preserves θ infinitesimally in x along δ, with Lebesgue-integrable coefficients. ◇

Theorem 3.4. Let F be a transitive family of vector fields on M and θ an F-weakly-involutive co-distribution. Then F preserves θ

Proof. From the hypothesis we deduce that θ is a regular co-distribution and the rest of the proof is contained in the proof of the Theorem 3.2. ◇

2. In Lemma 2.4 the curves considered were

$$\alpha_j(t) = X_t^*(\omega^j)(X_t)(x)) \ .$$

The proof of this lemma results from the fact that $\beta = \lambda(\alpha)$ satisfies a linear differential equation with Lebesgue-integrable coefficients and with initial condition $\beta(0) = 0$. Let us consider again the definition 2.3 (3) but this time with restrictions on derivatives of highest order, i.e., condition (ii) is replaced by

$$L_X^{(s)}(\omega^i)(X_t(x)) = \sum_{j=1}^{r} a_{ij}(t)\omega^j_{X_t(x)}$$

with a_{ij} Lebesgue-integrable coefficients and $s \in \mathbb{N}$. Since

$$\alpha_i^{(s)}(t) = X_t^* L_X^{(s)}(\omega^i)X_t(x)$$

then

$$\beta_i^{(s)}(t) = \sum_{j=1}^{r} a_{ij}(t)\beta_j(t), \ \ 1 \le i \le r$$

however, this equation is equivalent to the system

$$\beta^{(1)}(t) = \gamma_1(t)$$

$$\gamma^{(1)}(t) = \gamma_2(t)$$

$$\gamma^{(1)}_{(s-1)})(t) = (a_{ij})\beta(t) \ .$$

If we assume that $L_X^{(p)}(\omega^i)(x) \in \theta(x)$, $s \ge p \ge 1$. i.e. if $\gamma_\ell = 0$ for $s - 1 \ge \ell \ge 1$ we will have:

$\beta(s)$ satisfies a linear differential equation with Lebesgue-integrable coefficients and $\beta^{(s)}(0) = 0$. Due to the uniqueness of the solution we conclude that $\beta^{(s)} \equiv 0$.

Inductively, we obtain

$$\beta^{(p)} \equiv 0 \text{ for } s \ge p \ge 1 \ .$$

Then $\beta \equiv 0$ and again the conclusion of the Lemma 2.4 will be valid.

Definition 3.5. An exhaustive parametrization $\delta = (\omega^1, \omega^2, \ldots, \omega^r)$ of θ in $\mathcal{U}$ is said to be F-weakly-involutive of order $n \in N$, if for each $X \in F$, there are integers $\ell_1, \ell_2, \ldots \ell_r \in \{1, 2, \ldots, n\}$ such that for each $p \in \{1, 2, \ldots, n\}$ and each $i = 1, 2, \ldots, r$

i) $L_X^{(p)}(\omega^i)(y) \in \theta(y), \ \forall y \in \mathcal{U}$

ii) The equation

$$\delta(y)a_i(y) = L_X^{(\ell_i)}(\omega^i)(y)$$

admits solution with $a_i : \mathcal{U} \to R$ Lebesgue-integrable.

◇

Theorem 3.6. Let F be a transitive family of vector fields on M, θ a co-distribution and $n \in \mathbb{N}$. If the domains of the parametrizations F-weakly-involutive of order n of θ over M, then F preserves θ.

Proof. It comes directly from the comment above and the Definition 3.5 itself.

◇

Let M be a manifold, X a vector field and ω a 1-form defined over M. We say that ω satisfies the Matsuda condition in relation to X in the point $x \in M$ if the series

$$\sum_{k=0}^{\infty} (-1)^k \cdot \frac{t^k}{k!} L_X^{(k)}(\omega)(x)$$

is of class C^1 in a neighbourhood $\mathcal{U}_0$ of $(0, x) \in \mathbb{R} \times M$ and the derivative may be computed by differentiating the series term by term, [6].

With this idea we get the following result:

Theorem 3.7. Let F be a transitive family of vector fields on M and θ a F-involutive co-distribution. If the domains of the exhaustive parametrizations of θ that satisfy Matsuda's condition in relation to F cover M, then F preserves θ.

Proof. let $\delta = (\omega^1, \omega^2, \ldots, \omega^r)$ be an exhaustive parametrization of θ in a neighbourhood $\mathcal{U}$ of x. Let us define for $j = 1, 2, \ldots, r$

$$\alpha_j(t, \tau) = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{t^k}{k!} L_X(\omega^j)(X_\tau(x))$$

$$\beta_j(t, \tau) = X_\tau^* \alpha_j(t, \tau)$$

Since θ is F-involutive, we can see that $\alpha_j(t, \tau) \in \theta(X_\tau(x))$. By taking a partial derivative, we get

$$\frac{\partial}{\partial t}\beta_j = \sum_{k=1}^{\infty} (-1)^k \frac{t^{k-1}}{(k-1)!} X_\tau^* L_X^{(k)}(\omega^j)(X_\tau(x))$$

$$\frac{\partial}{\partial \tau}\beta_j = \sum_{k=0}^{\infty} (-1)^k X_\tau^* L_X^{(k+1)}(\omega^j)(X_\tau(x)) \ .$$

In particular, for each $j = 1, 2, \ldots, r$

$$\frac{\partial}{\partial t}\beta_j + \frac{\partial}{\partial \tau}\beta_j = 0 \ .$$

Since $\beta_j(0,0) = \omega_x^j$ then $\beta_j(t,t) = \omega_x^j$, $\forall t$. Thus

$$X_{-\tau}^{*}\omega_x^j = \alpha_j(t,t) \in \theta(X_\tau(x))$$

and then F preserves θ. ◇

Analytic Distributions.

Let X and ω be analytic sections of the bundles TM and T^*M respectively. To study the curve

$$\alpha(t) = X_t^*\omega_{X_t(x)}$$

locally, it is sufficient to know the successive Lie's derivatives of the 1-form ω with respect to the vector field X at one point.

Theorem 3.8. Suppose that F is a transitive family of vector fields and θ is an analytic F-involutive co-distribution over M, such that the exhaustive analytic parametrizations of θ cover M. Then F preserves θ.

Proof. Let $x \in M$ and $\mathcal{U}$ be a neighbourhood of x and the domain of an analytic exhaustive parametrization $\delta = (\omega^1, \omega^2, \ldots, \omega^r)$. For each $X \in F$ and $i = 1, 2, \ldots, r$

$$\alpha_i(t) X_t^* \omega^i_{X_t(x)}$$

defines an analytic curve over T^*M. Since θ is F-involutive we get

$$\alpha_i^{(k)}(0) = L_X^{(k)}(\omega^i)(x) \in \theta(x) \ .$$

From the analyticity, we conclude the existence of $\varepsilon > 0$:

$$|t| < \varepsilon \Rightarrow \alpha_i(t) \in \theta(x)$$

δ is exhaustive over $\mathcal{U}$ and thus, by using the same type of argument utilized in Lemma 2.7 we see that X and then F preserves θ in $\mathcal{U}$. ◇

Theorem 3.8 is not valid in the C^∞ case. Indeed consider the system Σ defined by $M = \mathbb{R}^2$, $D = \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ and as in example 2.0 (2). Then

$$\Lambda = \text{Span } \{f^{(k)}dx\}$$

is D-involutive and $\delta = fdx$ is a global C^∞ parametrization of Λ. However, D cannot preserve this co-distribution because of the fact that Λ is not integrable in any point of the y axis. In fact, the dimension of Λ varies along any integral curve of the vector field $X = \frac{\partial}{\partial x}$.

This example also shows that in the C^∞ case Λ may not be integrable, even though the vector fields of D are analytic and complete and the system Σ is symmetric and controllable. Indeed, the conclusion of the non-integrability of Λ remains if we consider

$$D = \{\pm\frac{\partial}{\partial x}, \pm\frac{\partial}{\partial y}\}$$

instead of D as above.

Stefan's condition [8] is seen in the following concept.

Definition 3.9. Let $A \subset M$ be. A co-distribution θ is said to be locally analytic along A if: for each $x \in A$, $v \in \theta(x)$, there is an analytic 1-form ω defined in a neighbourhood of A, $\omega \in \theta$ and such that $\omega_x = v$. ◇

We observe that in the previous example there are exhaustive analytic parametrizations of Λ for $x > 0$ and for $x < 0$. However, it is not possible "to connect" the parametrizations via analytic continuation. In fact, Λ is not locally analytic along any piece of trajectory of the vector field $X = \frac{\partial}{\partial x}$ such that the set

$$A = \{X_t(z) | t \in J\} \ , \ z \in \mathbb{R}^2$$

contains points with positive and negative first coordinate. This is because for $v = dx$ and $(x_0, y_0) \in A$ with $x > 0$ there is no $\omega \in \Lambda$ such $\omega_{(x_0,y_0)} = v$. In fact, each

$$\omega = a(x,y)dx + b(x,y)dy \in \Lambda$$

is such that $a(x,y)$ is null over an open subset of R^2 so that $a \equiv 0$ in the whole plan, because of analyticity.

Theorem 3.10. Let F be transitive family of vector fields on M and θ an analytic F-involutive co-distribution. Then, F preserves θ if and only if for each $X \in F$, θ satisfies:

if $x \in \text{dom}(X)$ there exist $\varepsilon > 0$ such that θ is locally analytic along the set

$$A(X,x,\varepsilon) = \{X_t(x) | \ |t| < \varepsilon\} \ .$$

Proof. Let $X \in F$ and $x \in \text{dom}(X)$. There is an analytic parametrization $\delta = (w^1, w^2, \ldots, w^r)$ of θ centered in x and defined on $\mathcal{U}$. Let $\varepsilon > 0$ so that

$$-\varepsilon < t < \varepsilon \Rightarrow X_t(x) \in \mathcal{U} \ ,$$

and $y = X_{t_0}(x) \in A(X,x,\varepsilon)$. Consider $v \in \theta(y)$. Since X preserves θ, there are $u_1, u_2, \ldots, u_r$ such that

$$X_{t_0}^*(v) = \sum_{j=1}^{r} u_j \omega_x^j \ .$$

Then

$$w = X^*_{-t_0}(\sum_{j=1}^{r} u_j\omega^j)$$

is an analytic form defined in a neighbourhood of $A(X,x,\varepsilon)$ and $\omega_x = v$.

Conversely, let $X \in F$ and $x \in \mathrm{dom}(X)$. There exist $\varepsilon > 0$ such that θ is locally analytic along $A(X,x,\varepsilon)$. We must prove that

$$|t| < \varepsilon \Rightarrow X_t^*\theta(X_t(x)) \subset \theta(x) .$$

For each $v \in \theta(X_t(x))$ there exist $\omega \in \theta$ with $v = \omega_{X_t(x)}$. By deriving $\alpha(s) = X_s^*\omega_{X_s(x)}$ and evaluating at $s = 0$ we obtain:

$$\alpha(s) \in \theta(x) \text{ for each } s \in \mathrm{dom}(\alpha) .$$

Thus, $\alpha(t) \in \theta(x)$ and consequently due to the arbitrarity of v we conclude that

$$X_t^*\theta(X_t(x)) \subset \theta(k), \text{ for } |t| < \varepsilon .$$

The proof follows then in the same way as Lemma 2.7. ◇

This result generalizes theorem 3.8. In fact, in this case, only the exaustivity of θ along the trajectories of the vector fields of F is necessary.

4 Λ, Δ^+ and Δ

Some facts about Λ, Δ^+ and Δ are deduced. Other results may be found in [2].

Proposition 4.1. *Let $\Sigma = (M, D, \mathbf{R}^s, h)$ be a control system. If the dimension of Λ is constant then*

$$\Lambda = \Delta^+ = \Delta$$

.

Proof. Λ is regular and D_Σ-involutive by definition, so the Theorem 3.2 is applied and $D = D_\Sigma$ preserves Λ .

Let $i = 1, 2, \ldots, s$ and $X \in D$. For each possible $t \in \mathbf{R}$, the D-invariance of Λ ensures that

$$d(h_i \circ X_t)(x) = X_t^*(dh_i(x)) \in \Lambda(x), \ \forall x \in M .$$

The same reasoning is valid for any possible finite combination of elements in D. That is, if

$$\rho = X^1_{t_1} \circ X^2_{t_2} \circ \ldots \circ X^k_{t_k} \in G_\Sigma$$

then, for each $x \in M$

$$d(h_i \circ \rho) = \rho^*(dh_i(x)) \in \Lambda(x) .$$

Thus, $\Delta(x) \subset \Lambda(x)$ for each x in M and the result comes from the fact that $\Lambda \subset \Delta^+ \subset \Delta$ due to the construction itself of these objects. ◇

In [1] the following proposition was proved:
Let $\Sigma = (M, D, \mathbf{R}^s, h)$ be a transitive control system then

$$\mathrm{diom}(\Delta^+) \text{ constant} \Leftrightarrow \ \Delta^+ = \Delta$$

◇

In particular, we obtain:

Corollary 4.2. *Let Σ be a transitive control system, then*
a) $\Delta^+ = \Delta \Leftrightarrow \Delta^+$ is integrable.
b) Σ controllable $\Rightarrow \Delta^+ = \Delta$
c) Σ analytic $\Rightarrow \Lambda = \Delta^+ = \Delta$

Proof.
a) Since Σ is transitive, we deduce $\Delta = \Delta^+$ is integrable. Conversely, the integrability of Δ^+ implies that this co-distribution is regular and consequently $\Delta^+ = \Delta$ by the above result.

b) If Σ is controllable, then given $x, y \in M$, $\exists \rho \in S_\Sigma$ such that $y = \rho(x)$. Since Δ^+ is S_Σ-invariant, the dimension of Δ^+ is independent of the states in M, thus $\Delta^+ = \Delta$.

c) Under these conditions Λ is integrable and therefore regular. The result comes directly from Proposition 4.1. ◇

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