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Statistical Mechanics of the Magnetized Pair Fermi Gas

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Abstract

Following our previous work on the magnetized pair Bose gas [1], we study in this paper the statistical mechanics of the charged relativistic Fermi gas with pair creation in d spatial dimensions. Initially, the gas in no external fields is studied, and we obtain expansions for the various thermodynamic functions in both the $\mu/m \to 0$ (neutrino) limit, and about the point $\mu/m = 1$, where μ is the chemical potential. We also present the thermodynamics of a gas of quantum-number conserving massless fermions. We then make a complete study of the pair Fermi gas in a homogeneous magnetic field, investigating the behavior of the magnetization over a wide range of field strengths. The inclusion of pairs leads to new results for the net magnetization due to the paramagnetic moment of the spins and the diamagnetic Landau orbits.

1 Introduction

There are many physical contexts in which the ideal Fermi gas is useful as a model for many-body finite-temperature systems. At low temperatures, where the single-particle energy spectrum can be taken as non-relativistic, the ideal Fermi gas [2] can describe weakly interacting systems such as He^3 and electrons in metals [3]. The thermodynamics of this system, both with [4] and without [5] an external magnetic field has been studied by Hore and Frankel in the region where $z \simeq 1$, where z is the fugacity, corresponding to temperatures near the Fermi temperature T_F.

Astrophysically, the electron gas in white dwarf stars [2] is at such a density and temperature that the energy spectrum must be given its full relativistic form, and this was also studied by Hore and Frankel [4]. However, at temperatures above pair threshold, as we have seen for the corresponding Bose gas case, the inclusion of antiparticles into the thermodynamics is essential. To this end, we give the full statistical mechanical treatment of the pair Fermi gas in no external fields in Section 2 of this paper.

The high-temperature asymptotic expansion for the thermodynamic potential in odd ($d = 2l + 1$) spatial dimensions for the field-free pair Fermi gas has been previously obtained by Actor [6] via the technique of "zeta-function regularization", and by Landsman and van Weert [7] using a Mellin transform technique, first developed by Haber and Weldon [8], described in full and used extensively in our study of the pair Bose gas. It is again our method of choice for the study of the pair Fermi system, and in fact much of the formulation for the Bose gas carries over readily to the analogous calculation for fermions. We recover the expansions in Refs. [6] and [7] readily, extend them to the case of even ($d = 2l$) dimensions, and then use them to find expansions for quantities such as the pressure, entropy, internal energy and specific heat in a region where $\overline{\mu} \simeq 1$. This corresponds to an intermediate quantum region, where the Fermi gas is neither degenerate ($T = 0$ and $\mu = \epsilon_F$, where ϵ_F is the Fermi energy), nor so hot that it behaves like a gas of neutrinos ($\overline{\mu} \to 0$). We will show that these expansions are about a temperature (T_0, say) which can be directly related to the Fermi temperature. We also find expansions for the specific heat in the $\overline{\beta} \to 0$, $\overline{\mu} \to 0$ neutrino limit.

The "neutrino limit" of the above discussion refers to an infinite-temperature limit of a gas of massive fermions, which has a conserved quantum number, such as electromagnetic $U(1)$ charge, or lepton number. In the case of fermions, it is also sensible to talk about a gas of massless quantum-number conserving particles (e.g. Dirac neutrinos), where particles and antiparticles remain distinguishable, and it is sensible to assign a finite chemical potential to the system. In contrast, there are no conserved quantum numbers for massless bosons, and so the chemical potential for a gas of them must be zero. Elze, Greiner and Rafelski [9] first studied a gas of massless fermions with quantum-number conservation in the context of hot quark matter in a giant MIT bag model for the nucleus. Here, the conserved quantity is baryon number. We review their work, and recover their results for the thermodynamic potential expansions. We point out, however, that their technique for including small-order mass corrections to the thermodynamic potential is flawed, and that the only systematic treatment is one where the full relativistic energy spectrum, containing the rest mass, is included. We also show that when this is done, in the case of fermions, the limit $m \to 0$ is a sensible one, and recovers the $m = 0$ results. This is entirely consistent with the boson case, where the lack of conserved quantum numbers for massless bosons means that an $m \to 0$ limit on the integrals or asymptotic expansions for the pair Bose gas with finite chemical potential is not sensible, and indeed gives results which are unphysical.

Our study then proceeds to the magnetized pair Fermi gas. Landau diamagnetism due to orbital motion in the homogeneous field, common to both bosons and fermions, must now in the case of fermions compete with the paramagnetism due to spin. Non-relativistically, it is possible to treat these two components of the magnetization separately, however relativistically we must calculate the net macroscopic magnetic moment of the system, in a manner following closely to that previously done for bosons.

Magnetized relativistic fermion systems are important astrophysically in the context of white dwarf and neutron stars. In the case of the latter, the fields on the surface are thought to be of the order of 10^{13} G (see Ref. [10], and references therein). Also, there may be large magnetic fields present when the quark gas forms during heavy-ion collisions at high temperature [11]. Furthermore, the cosmological plasmas present in the early universe, where fields were strong and temperatures high, and which we discussed previously in the context of the pair Bose gas, would have a fermionic component, so again the importance of studying both the magnetized pair Bose and Fermi plasmas in a cosmological scenario is evident.

The equation of state of electrons in strong magnetic fields has been calculated by Canuto and Chiu [12], as well as the thermodynamics and magnetization of this system. They showed that there is no permanent magnetic moment for an electron gas, and that at low temperatures it is weakly paramagnetic, by studying the "classical" properties of the magnetic moment. The magnetization at temperatures above pair threshold remains an open question, and is the topic of study of Section 3 of this paper. We show there that the net magnetization is positive, that is the pair Fermi gas is paramagnetic over a wide range of field strengths and temperatures.

Various authors have studied the magnetization of a gas of relativistic fermions with anomalous magnetic moment. Chiu, Canuto and Fassio-Canuto [13] did this for the electron gas in intense magnetic fields. Delsante and Frankel [10] have studied the statistical mechanics of a relativistic gas of spin-half magnetic moments in a strong magnetic field at $T = 0$, where there are no antifermions in equilibrium with the fermions. Chudnovsky [14] investigated a relativistic fermion gas with anomalous magnetic moments at finite temperature, but without including pair production. It is important to note that for real fermion systems, particularly in the presence of strong fields, that the anomalous moment should be included in the energy spectrum. However, in this paper, we have calculated the magnetization of a generalized pair fermion system without anomalous moments for weak, intermediate and strong fields, giving an indication of the interplay between Landau diamagnetism and the spin paramagnetism on the net magnetization of this system of "natural" fermion moments at finite temperature. The only other authors to study the magnetized pair Fermi gas as such, Miller and Ray [15], do so only in the case of strong fields, limiting their study to a display of the magnetization in its original integral form, and a numerical calculation of this for some region of the parameters of field and charge density. We emphasize that we will go significantly beyond these initial steps, as we did in the case of the magnetized pair Bose gas, to give asymptotic expansions for the magnetization over a wide range of field strengths.

Wherever possible, comparison between the results obtained for fermions and bosons will be made. Whilst the formulation of the statistical mechanics for both systems is remarkably similar, the differences therein reflect closely, sometimes remarkably, the quintessential contrasts between boson and fermion systems, and this we will highlight wherever possible.

2 The pair Fermi gas in no external fields

Obtaining the high-temperature expansions for the thermodynamic potential for the pair Fermi gas in no external fields can be readily done by adapting the Mellin transform method we previously described for free-field Bose gas. Having done this, we find a temperature expansion for the chemical potential around a characteristic temperature T_0, where $\overline{\mu}(T_0) = 1$, and give expansions for the various thermodynamic functions in this region. We then present the thermodynamics in the $T \to \infty$ neutrino limit.

2.1 High-temperature expansion

The thermodynamic potential for a pair Fermi gas is

$$\beta\Omega = -\sum_{\mathbf{p},\sigma} \log\left[1 + e^{-\beta(E(p,\sigma)-\mu)}\right] \quad + \quad \mu \to -\mu \ , \tag{1}$$

where we have explicitly indicated the sum over spin states with quantum number σ for spin-ς fermions; however $E(p) = (p^2 + m^2)^{1/2}$ for no external fields. Passing the momentum sums to integrals, and computing the sum over spins, we obtain

$$\frac{\Omega_d}{(2\varsigma + 1)V} = -\frac{2}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)\beta}\int_0^\infty dp\, p^{d-1} \log\left[1 + e^{-\beta(E(p)-\mu)}\right] \quad + \quad \mu \to -\mu \ . \tag{2}$$

The procedure for determining the $\overline{\beta} \ll 1$ expansions for Ω_d then follows precisely as in the Bose gas case; the logarithms are expanded, the momentum sums then computed to give a sum over modified Bessel functions, and in turn the sum given its Mellin integral representation. The Gauss hypergeometric functions appearing therein are then decomposed into components odd and even in $\overline{\mu}$, so that only the even terms remain, finally yielding:

$$\frac{\Omega_d}{(2\varsigma+1)V} = -\frac{m^{d+1}}{2^{d+1}\pi^{(d+1)/2}}$$
$$\times \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds \left(\frac{2}{\overline{\beta}}\right)^s \tau(s)\,\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s-d-1}{2}\right) {}_2F_1\left(\frac{s}{2},\frac{s-d-1}{2};\frac{1}{2};\overline{\mu}^2\right) \quad . \quad (3)$$

where Re $c > d+1$, and t'e function $\tau(z)$ is defined as

$$\tau(z) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^z} \; , \quad (4)$$

for Re $z \geq 1$, and more generally

$$\tau(z) = \left(1 - 2^{1-z}\right)\zeta(z) \quad . \quad (5)$$

In fact, apart from the sum over spins yielding the degeneracy factor of $2\varsigma + 1$, the only difference between (3) and the corresponding Bose gas form is the replacement of $\zeta(s)$ appearing in the latter with $\tau(s)$. This, however, changes the nature of the ensuing asymptotics significantly, as the tau function is analytic everywhere, in contrast to the $\zeta(z)$, which has a simple pole at $z = 1$.

Developing the $\overline{\beta} \ll 1$ asymptotic expansion requires calculating the residues of the integrand of (3). For odd dimensions ($d = 2l+1$), there is a double pole at $s = 0$, and simple poles at $s = 2l+2-2r$ for $r = 0,1,2,\ldots,l$, and at $s = -2q-2$ for $q = 0,1,2,\ldots$. The expansion for Ω_{2l+1} is then:

$$\frac{\Omega_{2l+1}}{(2\varsigma+1)V} = -\frac{m^{2l+2}}{2^{2l+1}\pi^{l+1}}\left\{\sum_{q=0}^{l}\left(\frac{2}{\overline{\beta}}\right)^{2l-2q+2}\frac{(-1)^q\,\Gamma(l-q+1)}{\Gamma(q+1)}\right.$$
$$\times\,\tau(2l-2q+2)\,{}_2F_1\left(l-q+1,-q;\frac{1}{2};\overline{\mu}^2\right)$$
$$+\frac{(-1)^l}{\Gamma(l+2)}\left[\log\left(\frac{\overline{\beta}}{\pi}\right)+\frac{1}{2}\left[\gamma-\psi(l+2)\right]+(l+1)\,\overline{\mu}^2\,{}_3F_2\left(1,1,-l;\frac{3}{2},2;\overline{\mu}^2\right)\right]$$

$$+(-1)^{l+1}\sum_{q=1}^{\infty}\left(\frac{\beta}{4\pi}\right)^{2q}\frac{(-1)^q\,\tau(2q+1)\,\Gamma(2q+1)}{\Gamma(q+1)\,\Gamma(l+q+2)}\,{}_2F_1\left(-q,-q-l-1;\frac{1}{2};\overline{\mu}^2\right)\Bigg\}. \tag{6}$$

This result has been previously obtained by Actor [6] and Landsman and van Weert [7].

In even ($d = 2l$) dimensions, the integrand of (3) has simple poles at $s = 0$ and at $s = 2l - 2q + 1$ for $q = 0, 1, 2, \ldots$. Calculating residues gives

$$\frac{\Omega_{2l}}{(2\varsigma+1)V} = -\frac{m^{2l+1}}{2^{2l}\pi^{l+1/2}}\Bigg\{\frac{(-1)^{l+1}\pi}{2\Gamma\left(l+\frac{3}{2}\right)}$$

$$+\sum_{q=0}^{\infty}\left(\frac{2}{\overline{\beta}}\right)^{2l-2q+1}\frac{(-1)^q\,\Gamma\left(\frac{2l+1-2q}{2}\right)}{\Gamma(q+1)}\tau(2l-2q+1)\,{}_2F_1\left(\frac{2l-2q+1}{2},-q;\frac{1}{2};\overline{\mu}^2\right)\Bigg\}. \tag{7}$$

The corresponding charge-density expansions are:

$$\frac{\rho_{2l+1}}{2\varsigma+1} = \frac{\overline{\mu}m^{2l+1}}{2^{2l-1}\pi^{l+1}}\Bigg\{\sum_{q=1}^{l}\left(\frac{2}{\overline{\beta}}\right)^{2l-2q+2}\frac{(-1)^{q+1}}{\Gamma(q)}\Gamma(l-q+2)$$

$$\times\tau(2l-2q+2)\,{}_2F_1\left(l-q+2,1-q;\frac{3}{2};\overline{\mu}^2\right)$$

$$+\frac{(-1)^l}{2\Gamma(l+1)}\left[{}_3F_2\left(1,1,-l;\frac{3}{2},2;\overline{\mu}^2\right)-\frac{l\overline{\mu}^2}{3}{}_3F_2\left(2,2,1-l;\frac{5}{2},3;\overline{\mu}^2\right)\right]$$

$$+\sum_{q=1}^{\infty}\left(\frac{\beta}{4\pi}\right)^{2q}\frac{(-1)^{q+l+1}\,\tau(2q+1)\,\Gamma(2q+1)}{\Gamma(q)\,\Gamma(l+q+1)}\,{}_3F_2\left(-q-l,1-q;\frac{3}{2};\overline{\mu}^2\right)\Bigg\}, \tag{8}$$

and

$$\frac{\rho_{2l}}{2\varsigma+1} = \frac{\overline{\mu}m^{2l}}{2^{2l-2}\pi^{l+1/2}}\Bigg\{\sum_{q=1}^{\infty}\left(\frac{2}{\overline{\beta}}\right)^{2l-2q+1}\frac{(-1)^{q+1}}{\Gamma(q)}\Gamma\left(\frac{2l-2q+3}{2}\right)$$

$$\times\tau(2l-2q+1)\,{}_2F_1\left(\frac{2l-2q+3}{2},1-q;\frac{3}{2};\overline{\mu}^2\right)\Bigg\}. \tag{9}$$

We note that the above asymptotic expansions are relatively much simpler forms than their boson counterparts. Mathematically, this is due to the absence of the Riemann zeta function pole. This in turn can be traced back to the original form for the fermionic integrals (2). In the Bose gas case, these were seen to have a branch-point at $\overline{\mu} = 1$. This determined the analytic behavior of the asymptotic expansions for the thermodynamic functions (such as the charge density), the branch-cut terms arising from the zeta function pole in the Mellin integral representation for the thermodynamic potential. Hence, the physical phenomenon of Bose-Einstein condensation was seen to be mathematically depicted by the analytic properties, in particular the singularity structure, of the Riemann zeta function. In the fermionic case, there is no branch-cut behavior in the original integrals, and the thermodynamic potential is analytic, due to the fact that the tau function appearing in the Mellin integral representation is entire. Again, this is the mathematical manifestation of the physical fact that fermions do not undergo a phase transition.

2.2 Thermodynamics in an intermediate quantum region

Whilst there is no phase transition for fermions, we are still able to give useful expansions for the thermodynamic functions in a region away from both the degenerate ($T = 0$) and neutrino ($T \to \infty$) limits. We choose to do expansions around a characteristic temperature T_0 (with the condition $T_0 \gg m$), which corresponds to $\overline{\mu} = 1$. We will show later how T_0 can be related to the Fermi temperature T_F. Whilst in the Bose case we were preoccupied with the nature of the phase transition, the absence of critical behavior for fermions means that we can pay some attention to the full gamut of thermodynamic functions, obtainable from Ω as demonstrated in Section 2 of [1].

2.2.1 $d = 1$

Using (8) for $l = 0$ to find an expansion for the chemical potential around T_0, we obtain

$$\overline{\mu} = 1 - 2\tau(3)\frac{\overline{\beta}_0}{(4\pi)^2}\left\{2t - 3t^2 + \mathcal{O}\left(t^3\right) + \ldots\right\} \quad . \tag{10}$$

This then allows us to give expansions for the various thermodynamic functions, such as pressure (P), entropy (S), internal energy (U) and specific heat (c_V):

$$\frac{P}{(2\varsigma+1)V} = \frac{m^2}{2\pi}\left\{\frac{\pi^2}{3\overline{\beta}_0^2}(1+2t) + \log\left(\frac{\overline{\beta}_0}{\pi}\right) - \gamma - \frac{1}{2} - t \right.$$
$$\left. + \tau(3)\left(\frac{\overline{\beta}_0}{4\pi}\right)^2 (5-2t) + \mathcal{O}\left(\overline{\beta}_0^4\right) + \ldots\right\} + \mathcal{O}\left(t^2\right) + \ldots \quad (11)$$

$$\frac{S}{(2\varsigma+1)V} = \frac{m}{2\pi}\left\{\frac{2}{3}\frac{\pi^2}{\overline{\beta}_0}(1+t) - \overline{\beta}_0(1-t)\right.$$
$$\left. - \frac{5\tau(3)}{8\pi^2}\overline{\beta}_0^3(1-3t) + \mathcal{O}\left(\overline{\beta}_0^5\right) + \ldots\right\} + \mathcal{O}\left(t^2\right) + \ldots \quad (12)$$

$$\frac{U}{(2\varsigma+1)V} = \frac{m^2}{2\pi}\left\{\frac{\pi^2}{3\overline{\beta}_0^2}(1+2t) + \log\left(\frac{\pi}{\overline{\beta}_0}\right) + \gamma + \frac{3}{2} - t\right.$$
$$\left. + 15\tau(3)\left(\frac{\overline{\beta}_0}{4\pi}\right)^2(1-2t) + \mathcal{O}\left(\overline{\beta}_0^4\right) + \ldots\right\} + \mathcal{O}\left(t^2\right) + \ldots \quad (13)$$

$$\frac{c_V}{2\varsigma+1} = \frac{m}{2\pi}\left\{\frac{2}{3}\frac{\pi^2}{\overline{\beta}_0}(1+t) + \overline{\beta}_0(1-t)\right.$$
$$\left. - \frac{15\tau(3)}{8\pi^2}\overline{\beta}_0^3(1-3t) + \mathcal{O}\left(\overline{\beta}_0^5\right) + \ldots\right\} + \mathcal{O}\left(t^2\right) + \ldots \quad (14)$$

2.2.2 $d \geq 2$

Expansions for the chemical potential and the thermodynamic functions can be readily generalized, to the first few terms, so that they are applicable for all $d \geq 2$. From (8) and (9) we find the $\overline{\mu}$-expansion

$$\overline{\mu} = 1 - (d-1)t + \frac{1}{2}d(d-1)t^2 + \mathcal{O}\left(t^3\right) + \ldots , \quad (15)$$

and the thermodynamic functions are given by

$$\frac{P}{(2\varsigma+1)V} = \frac{m^{d+1}}{2^d\pi^{(d+1)/2}}\left\{\tau(d+1)\Gamma\left(\frac{d+1}{2}\right)\left(\frac{2}{\overline{\beta}_0}\right)^{d+1}[1+(d+1)t]\right.$$
$$\left. + (d-2)\left(\frac{2}{\overline{\beta}_0}\right)^{d-1}\left[1 - \frac{d(d-1)}{d-2}t\right] + \mathcal{O}\left(\overline{\beta}_0^\alpha\right) + \ldots\right\} + \mathcal{O}\left(t^2\right) + \ldots \quad (16)$$

$$\frac{S}{(2\varsigma+1)V} = \frac{m^d}{2^{d-2}\pi^{(d+1)/2}} \left\{ \tau(d+1)\,\Gamma\!\left(\frac{d+3}{2}\right)\left(\frac{2}{\overline{\beta}_0}\right)^{d} [1+dt] \right.$$
$$\left. + (d-2)\,\tau(d-1)\,\Gamma\!\left(\frac{d+1}{2}\right)\left(\frac{2}{\overline{\beta}_0}\right)^{d-2}\left[1-\frac{d^2-2}{d-2}t\right] + \mathcal{O}\left(\overline{\beta}_0^{4-d}\right) + \ldots \right\}$$
$$+\mathcal{O}\left(t^2\right) + \ldots . \qquad (17)$$

$$\frac{U}{(2\varsigma+1)V} = \frac{m^{d+1}}{2^{d}\pi^{(d+1)/2}} \left\{ d\,\tau(d+1)\,\Gamma\!\left(\frac{d+1}{2}\right)\left(\frac{2}{\overline{\beta}_0}\right)^{d+1} [1+dt] \right.$$
$$\left. + \tau(d-1)\,\Gamma\!\left(\frac{d-1}{2}\right)\left(\frac{2}{\overline{\beta}_0}\right)^{d-1}\left[d^2-2d+2-(d-1)(d-2)^2 t\right] + \mathcal{O}\left(\overline{\beta}_0^{3-d}\right) \ldots \right\}$$
$$+\mathcal{O}\left(t^2\right) + \ldots . \qquad (18)$$

$$\frac{c_V}{2\varsigma+1} = \frac{m^d}{2^{d-1}\pi^{(d+1)/2}} \left\{ d^2\,\tau(d+1)\,\Gamma\!\left(\frac{d+1}{2}\right)\left(\frac{2}{\overline{\beta}_0}\right)^{d} [1+(d+1)t] \right.$$
$$\left. - 2\tau(d-1)\,\Gamma\!\left(\frac{d-1}{2}\right)\left(\frac{2}{\overline{\beta}_0}\right)^{d-2}\left[(d-2)^2-\left(d^3-2d^2+4d-5\right)t\right] + \mathcal{O}\left(\overline{\beta}_0^{4-d}\right) \ldots \right\}$$
$$+\mathcal{O}\left(t^2\right) + \ldots . \qquad (19)$$

where in (16) $\alpha = d-3$ for $d \geq 3$, and $\alpha = 0$ for $d = 2$.

We can relate the characteristic temperature T_0 to the Fermi temperature T_F. Given that there are no antifermions in equilibrium with the fermions at $T = 0$ (see, for example, the Appendix in Ref. [16]), we may write

$$\frac{\rho_d}{2\varsigma+1} = \frac{\pi^{d/2} p_F^d}{(2\pi)^d\,\Gamma\left(\frac{d}{2}+1\right)} \quad . \qquad (20)$$

where p_F is the Fermi momentum. Using $T_F = \epsilon_F = \left(p_F^2 + m^2\right)^{1/2}$ for a relativistic energy spectrum (as is necessary if the charge density of fermions at $T = 0$ is large enough for $\overline{\beta}_F \equiv m/T_F \lesssim 1$), and Eqns. (8) and (9), we obtain (for $d \geq 2$)

$$\left(\frac{T_0}{T_F}\right)^{d-1} = \frac{1}{2\tau(d-1)\,\Gamma(d+1)\,\overline{\beta}_F}\left\{1+\mathcal{O}\left(\overline{\beta}_F^2\right)+\ldots\right\} \quad . \qquad (21)$$

2.3 Neutrino limit

2.3.1 $d = 1$

Using Eqn. (8) for $l = 0$ we can find the behavior of the chemical potential in the $\bar{\mu} \to 0$, $\bar{\beta} \to 0$ neutrino limit:

$$\bar{\mu} = \frac{\pi \rho_1}{(2\varsigma + 1)\, m} \left\{ 1 - 4\tau(3) \left(\frac{\bar{\beta}}{4\pi} \right)^2 + \mathcal{O}\left(\bar{\beta}^4 \right) + \ldots \right\} \,. \tag{22}$$

which we may use to calculate the specific heat for a one-dimensional pair Fermi gas in the neutrino limit:

$$c_V = \frac{m}{2\pi} \left\{ \bar{\beta} - \frac{\tau(3)\, \rho_1^2 \bar{\beta}^3}{2 (2\varsigma + 1)^2 m^2} + \mathcal{O}\left(\bar{\beta}^5 \right) + \ldots \right\} \,. \tag{23}$$

2.3.2 $d \geq 2$

Again, results in the neutrino limit can be generalized to all $d \geq 2$. Using (8) and (9), we obtain for the chemical potential

$$\bar{\mu} = \frac{\pi^{(d+1)/2} \rho_d \bar{\beta}^{d-1}}{2 (2\varsigma + 1)\, \Gamma\left(\frac{d+1}{2} \right) m^d} \left\{ 1 + \mathcal{O}\left(\bar{\beta}^2 \right) + \ldots \right\} \,, \tag{24}$$

which, apart from the spin degeneracy factor, is precisely the same form as the equivalent boson result in the photon limit. Calculating the specific heat in the neutrino limit yields different expansions in odd and even dimensions. In odd dimensions, we have

$$\begin{aligned} \frac{c_V}{2\varsigma + 1} &= \frac{-m^{2l+1}}{2^{2l} \pi^{l+1}} \\ &\times \left\{ 2 \sum_{k=0}^{l} (-1)^{l-k+1} \frac{(2k+1)\, \Gamma(k+2)\, \tau(2k+2)}{\Gamma(l-k+1)} \left(\frac{2}{\bar{\beta}} \right)^{2k+1} + \frac{(-1)^{l+1} \bar{\beta}}{2\Gamma(l+2)} \right. \\ &+ 4\pi (-1)^{l+1} \sum_{k=1}^{l-1} (-1)^k \frac{\Gamma(2k+2)\, \tau(2k+1)}{\Gamma(k)\, \Gamma(k+l+2)} \left(\frac{\bar{\beta}}{4\pi} \right)^{2k+1} \\ &+ \left. \left(\frac{2^{2l-1} (2l+1)\, \pi^{2l+2} \rho_{2l+1}^2}{(2\varsigma + 1)^2\, \Gamma(l)\, \tau(2l)\, m^{4l+2}} - \frac{\tau(2l+1)}{2^{4l} \Gamma(l)\, \pi^{2l}} \right) \bar{\beta}^{2l+1} + \mathcal{O}\left(\bar{\beta}^{2l+3} \right) + \ldots \right\} \,, \end{aligned} \tag{25}$$

and in even dimensions

$$\frac{c_V}{2\varsigma+1} = \frac{-m^{2l}}{2^{2l-1}\pi^{l+1/2}}$$
$$\times \left\{ 2\sum_{q=0}^{2l-1} \frac{(-1)^q}{\Gamma(q+1)} \left(\frac{2}{\beta}\right)^{2l-2q} (2q-2l-2)\,\tau(2l-2q+1)\,\Gamma\left(\frac{2l-2q+3}{2}\right) \right.$$
$$\left. + \left(\frac{2^{2l-1} l \pi^{2l+1} \rho_{2l}^2}{(2\varsigma+1)^2\,\Gamma(2l)\,\tau(2l-1)\, m^{4l}} + \frac{(l-1)\,\tau(1-2l)}{2^{2l-2}\Gamma\left(\frac{2l+3}{2}\right)\Gamma(2l+1)} \right) \beta^{2l} + \mathcal{O}\left(\beta^{2l+2}\right) + \dots \right\}. \tag{26}$$

The result in three dimensions is

$$c_V = \frac{7}{30}\pi^2 T^3 - \frac{m^2}{12}T - \frac{m^4}{16T} - \left(\frac{18\rho_3^2}{(2\varsigma+1)^2 m^3} - \frac{\tau(3)\, m^6}{64\pi^4} \right) \frac{1}{T^3} + \mathcal{O}\left(\frac{1}{T^5}\right) + \dots \; . \tag{27}$$

The leading-order term should be the specific heat of a neutrino gas. However, even if we set $\varsigma = 0$ for singly-polarized neutrinos, the leading-order term in (27) is twice the expected value. This is because of the degeneracy of particles and antiparticles in the neutrino limit, so that we have effectively counted twice. This problem did not arise in the boson case, as the pair degeneracy serendipitously accounted for the fact that photons have two polarization states.

2.4 Gas of massless quantum-number conserving fermions

In the case of fermions, it is meaningful to take an $m \to 0$ limit of the thermodynamic potential asymptotic expansions (6) and (7), because conserved quantum numbers exist. For example, in the case of Dirac neutrinos, the conserved quantum number is lepton number, or in the case of a quark bag model for the nucleus, as studied by Elze *et al.* [9], it is baryon number. Charge densities for these quantum numbers, and their conjugate chemical potentials, can therefore be non-zero. Contrast this to the boson case, where of course there are no conserved quantum numbers for massless particles, so that the charge density and chemical potential must vanish. This is reflected by the fact that the $m \to 0$ limit on the asymptotic forms for the boson thermodynamic functions for finite μ is ill defined.

Elze *et al.* [9] were able to calculate the integrals for the massless pair Fermi gas directly for all odd dimensions; they found that in this case the thermodynamic potential is, remarkably, a finite polynomial. We recover this result by taking the $m \to 0$ limit of (6):

$$\frac{\Omega_{2l+1}(m=0)}{(2\varsigma+1)V} = \frac{-1}{2^{2l+2}\pi^{l+1/2}}\left\{2\Gamma(l+1)\sum_{q=0}^{l}(2T)^{2l-2q+2}\mu^{2q}\frac{\tau(2l-2q+2)}{\Gamma(q+1)\Gamma\left(q+\frac{1}{2}\right)} + \frac{\mu^{2l+2}}{(l+1)\Gamma\left(l+\frac{3}{2}\right)}\right\} . \quad (28)$$

The result in even dimensions is the infinite series

$$\frac{\Omega_{2l}(m=0)}{(2\varsigma+1)V} = -\frac{\Gamma\left(l+\frac{1}{2}\right)}{2^{2l}\pi^{l}}\sum_{q=0}^{\infty}(2T)^{2l-2q+1}\mu^{2q}\frac{\tau(2l-2q+1)}{\Gamma(q+1)\Gamma\left(q+\frac{1}{2}\right)} . \quad (29)$$

We now note that in Ref. [9], Elze *et al.* also attempt to find "finite mass corrections" to the above expansions, by returning to the original integrals for the thermodynamic potential, expanding the integrand in the parameter $\overline{\mu}^{-1}$, and then evaluating the integrals appearing in this expansion. This procedure misses contributions to Ω, such as the logarithmic term in (6). The only consistent method is to develop expansions after the integral over momentum states is evaluated, as the Mellin transform method we have exhibited successfully does.

3 The magnetized pair Fermi gas

3.1 Mellin integral representation for the thermodynamic potential

Having completed our study of the field-free pair Fermi gas, we now move on to the magnetized gas. The corresponding study for the magnetized pair Bose gas provided many interesting, new results for the magnetic properties of the system. In the case of fermions, the statistics and the inclusion of spin will significantly affect the magnetic properties of the gas.

The thermodynamic potential for a gas of magnetized fermion pairs is obtained by passing the sum of Eqn. (1) to integrals, using the appropriate boundary conditions and density of states [2]:

$$\Omega_d = \frac{-eB}{2^{d-2}\pi^{d/2}\Gamma\left(\frac{d}{2}-1\right)\beta}\sum_{n=0}^{\infty}\sum_{\sigma=-1,1}\int_0^{\infty} dp\, p^{d-3}\log\left[1+e^{-\beta(E_{n,\sigma}(p)-\mu)}\right] + \mu \to -\mu \ , \quad (30)$$

where the single-particle energy spectrum for spin-half fermions is given by

$$E_{n,\sigma}^2(p) = p^2 + (2n-\sigma+1)eB + m^2 \ . \quad (31)$$

Here, σ has the values -1 and 1 for spin-down and spin-up states respectively. Computing the momentum integrals gives

$$\frac{\Omega_d}{V} = -\frac{\overline{B}m^{d+1}}{2^{(d-1)/2}\pi^{(d+1)/2}}\sum_{n=0}^{\infty}\sum_{\sigma=-1,1}\sum_{j=1}^{\infty}\left[(2n-\sigma+1)\overline{B}+1\right]^{(d-1)/4}\frac{(-1)^{j+1}e^{j\overline{\beta}\overline{\mu}}}{\left(j\overline{\beta}\right)^{(d-1)/2}}$$
$$\times K_{(d-1)/2}\left(j\overline{\beta}\left[(2n-\sigma+1)\overline{B}+1\right]^{1/2}\right) + \overline{\mu}\to-\overline{\mu} \ . \quad (32)$$

The procedure for finding the Mellin integral representation for the thermodynamic potential follows closely the procedure described for the magnetized pair Bose gas, so that we obtain

$$\frac{\Omega_d}{V} = -\frac{\overline{B}m^{d+1}}{2^d\pi^{(d+1)/2}}\sum_{n=0}^{\infty}\sum_{\sigma=-1,1}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\left(\frac{2}{\overline{\beta}}\right)^s\frac{\tau(s)\,\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s-d+1}{2}\right)}{\left[(2n-\sigma+1)\overline{B}+1\right]^{(s-d+1)/2}}$$
$$\times {}_2F_1\left(\frac{s}{2},\frac{s-d+1}{2};\frac{1}{2};\frac{\overline{\mu}^2}{(2n-\sigma+1)\overline{B}+1}\right) \ . \quad (33)$$

The hypergeometric function is given its sum representation, and the sums over n and σ computed first to give the Mellin integral representation in the form

$$\frac{\Omega_d}{V} = -\frac{m^{d+1}}{\pi^{(d+1)/2}}\sum_{k=0}^{\infty}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\left(\frac{2}{\overline{B}}\right)^{(s-d-1)/2+k}\frac{\zeta(s)\,\Gamma\left(\frac{s}{2}+k\right)\Gamma\left(\frac{s-d+1}{2}+k\right)}{\overline{\beta}^s\,\Gamma(2k+1)}\overline{\mu}^{2k}$$
$$\times\left[2\zeta\left(\frac{s-d+1}{2}+k,\frac{1}{2\overline{B}}\right)-\left(2\overline{B}\right)^{(s-d+1)/2+k}\right] \ , \quad (34)$$

where $\mathrm{Re}\, c > d+1$.

As in the boson case, closing the contour to the left and evaluating residues gives the $\overline{B}\overline{\beta}^2 \ll 1$ expansions for the thermodynamic potential. However, because the upper bound on μ for fermions is ϵ_F, which is determined by the number density of particles at $T = 0$, Eqn. (34) indicates that care must be taken with the magnitude of the chemical potential when using these asymptotic expansions, as they may be divergent for $\overline{\mu} \gg 1$. To evaluate the $\overline{B}\overline{\beta}^2 \gg 1$ expansions we must return to the original integrals, which we do later.

In odd dimensions, the pole structure of the integrand of (34) is:

(i). Simple poles at $s = 2l - 2q\,(q = 0, 1, 2, \ldots l-1)$ when $k = 0$, due to $\Gamma\left(\frac{s}{2} - l\right)$.

(ii). A double pole at $s = 0$ when $k = 0$, due to $\Gamma\left(\frac{s}{2} - l\right)\Gamma\left(\frac{s}{2}\right)$.

(iii). Simple poles at $s = -2r-2\,(r = 0, 1, 2, \ldots)$ when $k = 0$, due to $\Gamma\left(\frac{s}{2} - l\right)\Gamma\left(\frac{s}{2}\right)\tau(s)$.

(iv). Simple poles at $s = 2l - 2k + 2$ when $0 \le k \le l+1$, due to $\zeta\left(\frac{s}{2} - l + k, \frac{1}{2\overline{B}}\right)$.

(v). Simple poles at $s = 2l - 2k - 2q\,(q = 0, 1, 2, \ldots l-k)$ when $1 \le k \le l$, due to $\Gamma\left(\frac{s}{2} - l + k\right)$.

(vi). Simple poles at $s = -2k - 2r\,(r = 0, 1, 2, \ldots)$ when $k \ge 1$, due to $\Gamma\left(\frac{s}{2} - l + k\right)\Gamma\left(\frac{s}{2} + k\right)\tau(s)$.

The expansion for the thermodynamic potential in odd dimensions is then given by

$$
\begin{aligned}
\frac{\Omega_{2l+1}}{V} = &-\frac{m^{2l+2}}{2^{2l+1}\pi^{l+1}}\left\{\sum_{k=0}^{l+1}\frac{2^{2l+3}}{\overline{\beta}^{2l-2k+2}}\tau(2l-2k+2)\frac{\Gamma(l+1)}{\Gamma(2k+1)}\overline{\mu}^{2k}\right.\\
&+\sum_{q=0}^{l-1}(-1)^q\left(\frac{2}{\overline{\beta}}\right)^{2l-2q}\frac{\tau(2l-2q)\,\Gamma(l-q)}{\Gamma(q+1)}\left(2\overline{B}\right)^{q+1}\left[2\zeta\left(-q, \frac{1}{2\overline{B}}\right) - \left(\frac{1}{2\overline{B}}\right)^q\right]\\
&-\frac{\left(-2\overline{B}\right)^{l+1}}{\Gamma(l+1)}\left(\left[\log\left(\frac{\pi^2}{2\overline{B}\,\overline{\beta}^2}\right) - \gamma + \psi(l+1)\right]\left[\zeta\left(-l, \frac{1}{2\overline{B}}\right) - \frac{1}{2\left(2\overline{B}\right)^l}\right]\right.\\
&\left.+2\,\frac{d}{ds}\zeta\left(\frac{s}{2} - l, \frac{1}{2\overline{B}}\right)\bigg|_{s=0} - \frac{1}{2\left(2\overline{B}\right)^l}\log\left(2\overline{B}\right)\right)\\
&+\sum_{k=1}^{l}\sum_{q=0}^{l-k}(-1)^q\frac{2^{2l-2q}}{\overline{\beta}^{2l-2k-2q}}\frac{\tau(2l-2k-2q)\,\Gamma(l-q)}{\Gamma(q+1)\,\Gamma(2k+1)}\overline{\mu}^{2k}\left(2\overline{B}\right)^{q+1}
\end{aligned}
$$

$$\times\left[2\zeta\left(-q,\frac{1}{2\overline{B}}\right)-\left(\frac{1}{2\overline{B}}\right)^{q}\right]$$

$$+\left(2\overline{B}\right)^{l+1}\sum_{q=1}^{\infty}(-1)^{q+l}\left(\frac{\overline{B}\,\overline{\beta}^{2}}{8\pi^{2}}\right)^{q}\frac{\tau(2q+1)\,\Gamma(2q+1)}{\Gamma(q+1)\,\Gamma(q+l+1)}$$

$$\times\left[2\zeta\left(-q-l,\frac{1}{2\overline{B}}\right)-\left(\frac{1}{2\overline{B}}\right)^{q+l}\right]$$

$$+4\left(2\overline{B}\right)^{l}\sum_{k=1}^{\infty}\frac{\overline{\mu}^{2k}}{\Gamma(2k+1)}\sum_{q=1}^{\infty}(-1)^{k+q+l-1}\left(\frac{\overline{\beta}}{2\pi}\right)^{2k+2q-2}\left(\frac{\overline{B}}{2}\right)^{q}$$

$$\times\frac{\tau(2k+2q-1)\,\Gamma(2k+2q-1)}{\Gamma(q)\,\Gamma(q+l)}\left[2\zeta\left(-q-l+1,\frac{1}{2\overline{B}}\right)-\left(\frac{1}{2\overline{B}}\right)^{q+l-1}\right]\Bigg\}\ . \quad (35)$$

In even dimensions, we have

(i). Simple poles at $s=2l-2k+1$ for all k, due to $\zeta\left(\frac{s+1}{2}-l+k,\frac{1}{2\overline{B}}\right)$.

(ii). Simple poles at $s=2l-2k-2q-1$ for all k and $q=0,1,2,\ldots$, due to $\Gamma\left(\frac{s+1}{2}-l+k\right)$.

(iii). A simple pole at $s=0$ for $k=0$, due to $\Gamma\left(\frac{s}{2}+k\right)$.

We then have

$$\frac{\Omega_{2l}}{V}=-\frac{m^{2l+1}}{2^{2l}\pi^{l+1/2}}\left\{\frac{1}{2}\Gamma\left(\frac{1}{2}-l\right)\left(2\overline{B}\right)^{l+1/2}\left[2\zeta\left(\frac{1}{2}-l,\frac{1}{2\overline{B}}\right)-\left(\frac{1}{2\overline{B}}\right)^{l-1/2}\right]\right.$$

$$+\sum_{k=0}^{\infty}\frac{2^{2l+2}}{\overline{\beta}^{2l-2k+1}}\tau\left(2l-2k+1\right)\frac{\Gamma\left(l+\frac{1}{2}\right)}{\Gamma(2k+1)}\overline{\mu}^{2k}$$

$$+\sum_{k=0}^{\infty}\frac{\overline{\mu}^{2k}}{\Gamma(2k+1)}\sum_{q=0}^{\infty}\frac{(-1)^{q}\,\tau\left(2l-2k-2q-1\right)\Gamma\left(l-q-\frac{1}{2}\right)}{\Gamma(q+1)}\overline{\beta}^{2k+2q-2l+1}2^{2l-2q-1}$$

$$\left.\times\left(2\overline{B}\right)^{q+1}\left[2\zeta\left(-q,\frac{1}{2\overline{B}}\right)-\left(\frac{1}{2\overline{B}}\right)^{q}\right]\right\}\ . \quad (36)$$

The corresponding charge densities are given by

$$\rho_{2l+1}=\frac{m^{2l+1}}{2^{2l+1}\pi^{l+1}}\left\{\sum_{k=1}^{l+1}\frac{2^{2l+3}}{\overline{\beta}^{2l-2k+2}}\tau\left(2l-2k+2\right)\frac{\Gamma(l+1)}{\Gamma(2k)}\overline{\mu}^{2k-1}\right.$$

$$+\sum_{k=1}^{l}\sum_{q=0}^{l-k}(-1)^{q}\frac{2^{2l-2q}}{\overline{\beta}^{2l-2k-2q}}\frac{\tau\left(2l-2k-2q\right)\Gamma\left(l-q\right)}{\Gamma(q+1)\,\Gamma(2k)}\overline{\mu}^{2k-1}\left(2\overline{B}\right)^{q+1}$$

$$\times\left[2\zeta\left(-q,\frac{1}{2\overline{B}}\right)-\left(\frac{1}{2\overline{B}}\right)^{q}\right]$$

$$+4\left(2\overline{B}\right)^{l}\sum_{k=1}^{\infty}\frac{\overline{\mu}^{2k-1}}{\Gamma(2k)}\sum_{q=1}^{\infty}(-1)^{k+q+l-1}\left(\frac{\overline{\beta}}{2\pi}\right)^{2k+2q-2}\left(\frac{\overline{B}}{2}\right)^{q}$$

$$\times\frac{\tau(2k+2q-1)\,\Gamma(2k+2q-1)}{\Gamma(q)\,\Gamma(q+l)}\left[2\zeta\left(-q-l+1,\frac{1}{2\overline{B}}\right)-\left(\frac{1}{2\overline{B}}\right)^{q+l-1}\right]\Bigg\}\ . \quad (37)$$

and

$$\rho_{2l}=\frac{m^{2l}}{2^{2l+2}\pi^{l+1/2}}\left\{\sum_{k=0}^{\infty}\frac{2^{2l}}{\overline{\beta}^{2l-2k+1}}\tau\left(2l-2k+1\right)\frac{\Gamma\left(l+\frac{1}{2}\right)}{\Gamma(2k)}\overline{\mu}^{2k-1}\right.$$

$$+\sum_{k=1}^{\infty}\frac{\overline{\mu}^{2k-1}}{\Gamma(2k)}\sum_{q=0}^{\infty}\frac{(-1)^{q}\,\tau\left(2l-2k-2q-1\right)\Gamma\left(l-q-\frac{1}{2}\right)}{\Gamma(q+1)}\overline{\beta}^{2k+2q-2l+1}2^{2l-2q-1}$$

$$\left.\times\left(2\overline{B}\right)^{q+1}\left[2\zeta\left(-q,\frac{1}{2\overline{B}}\right)-\left(\frac{1}{2\overline{B}}\right)^{q}\right]\right\}\ . \quad (38)$$

It is clear that the above expansions are rapidly convergent under the conditions $\overline{B}\,\overline{\beta}^{2}\ll 1$ and $\overline{\mu}\,\overline{\beta}\ll 1$. Both of these constraints indicate a high-temperature limit, as expected.

3.2 Weak fields

The weak-field limit $\overline{B}\ll 1$ corresponds, in the case of electrons in three dimensions, to fields less than 4×10^{13} G. Hence, this limit covers a wide range of field strengths; almost certainly most astrophysical fields in the present era. (For heavier leptons, the bound on the field strength increases as m^2).

With this in mind, we now investigate the magnetization of the pair Fermi system in these weak fields. To do this, we must use the large-parameter asymptotic expansions presented in Appendix A of [1] for the Hurwitz zeta function, and its derivative, in a similar fashion to the evaluation of the same limit in the boson case. We then obtain for the magnetization

$$M_{2l+1}=-\frac{em^{2l}\overline{B}}{2^{2l-1}\pi^{l+1}}\left\{\sum_{q=1}^{l-1}\left(\frac{2}{\overline{\beta}}\right)^{2l-2q}\frac{(-1)^{q}}{\Gamma(q)}\tau(2l-2q)\,\Gamma(l-q)\right.$$

$$+\frac{(-1)^{l}}{2\Gamma(l)}\left[\log\left(\frac{\pi^{2}}{\overline{\beta}^{2}}\right)-\gamma+\psi(l)\right]$$

$$+\sum_{k=1}^{l}\sum_{q=1}^{l-k}2^{2l-2q}(-1)^{q}\,\tau(2l-2k-2q)\frac{\Gamma(l-q)}{\Gamma(q)\,\Gamma(2k+1)}\frac{\overline{\mu}^{2k}}{\overline{\beta}^{2l-2k-2q}}$$

$$+\sum_{q=1}^{\infty}(-1)^{l+q}\left(\frac{\overline{\beta}}{4\pi}\right)^{2q}\frac{\Gamma(2q+1)\,\tau(2q+1)}{\Gamma(q+1)\,\Gamma(q+l)}$$

$$+4\sum_{k=1}^{\infty}\frac{\overline{\mu}^{2k}}{\Gamma(2k+1)}\sum_{q=1}^{\infty}(-1)^{l+q+k+1}\left(\frac{\overline{\beta}}{2\pi}\right)^{2k+2q-2}\frac{\Gamma(2q+2k-1)\,\tau(2q+2k-1)}{\Gamma(q)\,\Gamma(q+l-1)}\Bigg\}$$

$$\times\left\{\frac{1}{3}\overline{B}+\mathcal{O}\left(\overline{B}^{3}\right)+\ldots\right\}\ , \tag{39}$$

and

$$M_{2l}=-\frac{\epsilon m^{2l-1}\overline{B}}{2^{2l-2}\pi^{l+1/2}}\Bigg\{\frac{1}{2}\Gamma\left(\frac{3}{2}-l\right)$$

$$+\sum_{k=0}^{\infty}\frac{\overline{\mu}^{2k}}{\Gamma(2k+1)}\sum_{q=1}^{\infty}(-1)^{q}\,2^{2l-2q-1}\frac{\Gamma\left(l-q-\frac{1}{2}\right)}{\Gamma(q)}\tau(2l-2k-2q-1)\,\overline{\beta}^{2k+2q-2l+1}\Bigg\}$$

$$\times\left\{\frac{1}{3}\overline{B}+\mathcal{O}\left(\overline{B}^{3}\right)+\ldots\right\}\ , \tag{40}$$

for odd and even dimensions respectively.

3.2.1 $d=3$

There are two natural regions in which to find temperature expansions for the magnetization: the intermediate quantum region $\overline{\mu}\simeq 1$, and the neutrino limit $\overline{\mu}\rightarrow 0$. To obtain the former, we again have $\overline{\mu}(T_0)=1$, and using (37) and (38), we have

$$\overline{\mu}=1-(d-1)t+\mathcal{O}\left(t^{2}\right)+\ldots\ . \tag{41}$$

The expansion for the magnetization in three dimensions is then

$$M_3 = \frac{\mu_0 \rho_3 \overline{\beta}_0^2}{\pi^2} \left[\log\left(\frac{\pi}{\overline{\beta}_0}\right) - \gamma + t\left\{1 + \mathcal{O}(t) + \ldots\right\} + \mathcal{O}\left(\overline{\beta}_0^2\right) + \ldots \right] \left[\overline{B} + \mathcal{O}\left(\overline{B}^3\right) + \ldots\right] . \tag{42}$$

We may also find the magnetization in the neutrino limit, by firstly using the charge-density equations (37) and (38) to find the behavior of the chemical potential in d-dimensions,

$$\overline{\mu} = \frac{\pi^{(d+1)/2} \rho_d \overline{\beta}^{d-1}}{\tau\,(d-1)\,\Gamma\left(\frac{d+1}{2}\right) m^d} + \mathcal{O}\left(\overline{\beta}^d\right) + \ldots , \tag{43}$$

so that for $d = 3$, the leading-order magnetization in the neutrino limit is

$$M_3 = \frac{\epsilon m^2}{6\pi^2} \left[\log\left(\frac{\pi}{\overline{\beta}}\right) - \gamma + \mathcal{O}\left(\overline{\beta}^2\right) + \ldots \right] \left[\overline{B} + \mathcal{O}\left(\overline{B}^3\right) + \ldots\right] . \tag{44}$$

Clearly, the magnetization law indicates paramagnetism (as the susceptibility $\chi = \partial M/\partial B$ is positive), and that the magnitude of the magnetization increases logarithmically with temperature. We saw in the case of the pair Bose gas how pair production caused a macroscopic diamagnetic moment which under certain conditions was large enough to expel the external field from the medium. With fermions, the low-temperature paramagnetic behavior [2], due essentially to the alignment of spins with the field, is enhanced by the freedom to produce pairs in the high-temperature region, the net sum of diamagnetic (Landau moment) and paramagnetic (intrinsic spin moment) contributions resulting in overall paramagnetism. In one sense, this result is surprising. Chiu and Canuto [12] suggested in their study of the magnetized Fermi gas that increasing the temperature from $T = 0$, where the gas is degenerate and paramagnetic, so that the fermions become thermally distributed through all Landau levels, will ultimately result in Landau diamagnetism dominating the paramagnetic spin contribution. As we have seen, pair production allows the paramagnetic contribution to remain dominant (even though, in the relativistic regime, the para- and diamagnetic contributions cannot be explicitly evaluated, as they can be at low temperatures). Finally, the absence of critical behavior and branch-cuts in the Fermi integrals means that the leading-order magnetization law for fermions in three dimensions applies universally in the intermediate quantum region

(42) through to the neutrino limit (44), which is seen more clearly if one substitutes the leading-order form for the charge density into (42).

3.2.2 $d \geq 4$

The leading-order magnetization law in the intermediate quantum region can be sufficiently generalized to all dimensions $d \geq 4$:

$$M_d = \frac{em^{d-1}}{2^{d-2}\pi^{(d+1)/2}} \left[\tau (d-3) \Gamma\left(\frac{d-3}{2}\right) \left(\frac{2}{\overline{\beta}_0}\right)^{d-3} \{1 + (d-3) t + \dots\} + \mathcal{O}\left(\varepsilon\left(\overline{\beta}_0\right)\right) + \dots \right] \left[\frac{1}{3}\overline{B} + \mathcal{O}\left(\overline{B}^3\right) + \dots\right], \quad (45)$$

where $\varepsilon\left(\overline{\beta}_0\right) = 1$ for $d = 4$, $\varepsilon\left(\overline{\beta}_0\right) = \log \overline{\beta}_0$ for $d = 5$, and $\varepsilon\left(\overline{\beta}_0\right) = \overline{\beta}_0^{d-5}$ for $d \geq 6$.

The generalization to all $d \geq 4$ can also be done in the neutrino limit, and we obtain a similar form for the the magnetization:

$$M_d = \frac{em^{d-1}}{2^{d-2}\pi^{(d+1)/2}} \left[\tau (d-3) \Gamma\left(\frac{d-3}{2}\right) \left(\frac{2}{\overline{\beta}}\right)^{d-3} + \mathcal{O}\left(\varepsilon\left(\overline{\beta}\right)\right) + \dots \right] \times \left[\frac{1}{3}\overline{B} + \mathcal{O}\left(\overline{B}^3\right) + \dots\right], \quad (46)$$

where ε is as defined previously.

We note again that the leading-order behavior of the magnetization has the same form from the intermediate quantum region, through to the neutrino limit. Also, the leading-order temperature behavior for all $d \geq 4$ is algebraic, and only in three dimensions do we see the appearance of a non-algebraic law. For fermions in weak fields, the leading-order field dependence of the magnetization is always $M \sim B$, in contrast to the pair Bose gas in three dimensions, where the branch-cuts in the Bose integrals led to, for example, a weak-field $M \sim -B^{1/2}$ law near the zero-field condensation temperature. To summarize, in all dimensions, pair creation at high temperatures causes the Fermi gas to become increasingly paramagnetic.

3.3 Intermediate fields

As we discussed in the case of bosons, the intermediate-field region $1 \ll \overline{B} \ll \overline{\beta}^{-2}$, whilst corresponding to very large fields ($\gg \mathcal{O}(10^{13})$ G for electrons), are physically relevant in the context of cosmological scenarios, such as the electroweak phase transition. Thus, we expand the Hurwitz zeta functions and derivatives appearing in the thermodynamic potential of Eqns. (35) and (36) in the limit $\overline{B} \gg 1$. This corresponds to a small a expansion for $\zeta(z,a)$ and its z-derivative, given by us in Appendix A of [1]. Then, evaluating the magnetization gives

$$
\begin{aligned}
M_{2l+1} = \frac{\epsilon}{\pi^{l+1}}\left(\frac{m}{2\overline{\beta}}\right)^{2l} \Bigg\{ & \sum_{q=1}^{l-1}(-1)^q\, 2^{2l-q+1}\frac{(q+1)\,\Gamma(l-q)}{\Gamma(q+1)}\zeta(-q)\,\tau(2l-2q) \\
& \times\left(\overline{B}\,\overline{\beta}^2\right)^q\left\{1+\mathcal{O}\left(\overline{B}^{-1}\right)+\ldots\right\} \\
+\frac{2^l(-1)^{l+1}}{\Gamma(l+2)} & \left[\zeta(-l)\left(\log\left(\frac{2\overline{B}\,\overline{\beta}^2}{\pi^2}\right)+\gamma-\psi(l+1)+\frac{1}{(l+1)}\right)\right. \\
& \left. -(l+1)^2\zeta'(-l)\right]\left(\overline{B}\,\overline{\beta}^2\right)^l\left\{1+\mathcal{O}\left(\overline{B}^{-1}\right)+\ldots\right\} \\
+\sum_{k=1}^{l}\frac{\left(\overline{\mu}\,\overline{\beta}\right)^{2k}}{\Gamma(2k+1)} & \sum_{q=1}^{l-k}(-1)^q\, 2^{2l-q+1}\frac{(q+1)\,\Gamma(l-q)}{\Gamma(q+1)}\zeta(-q)\,\tau(2l-2k-2q) \\
& \times\left(\overline{B}\,\overline{\beta}^2\right)^q\left\{1+\mathcal{O}\left(\overline{B}^{-1}\right)+\ldots\right\} \\
+\sum_{q=1}^{\infty}\frac{(-1)^{q+l}}{(4\pi)^{2q}} & 2^{q+l+1}\frac{(q+l+1)\,\Gamma(2q+1)}{\Gamma(q+1)\,\Gamma(q+l+1)}\tau(2q+1)\,\zeta(-q-l) \\
& \times\left(\overline{B}\,\overline{\beta}^2\right)^{q+l}\left\{1+\mathcal{O}\left(\overline{B}^{-1}\right)+\ldots\right\} \\
+8\sum_{k=1}^{\infty}\frac{1}{\Gamma(2k+1)} & \left(\frac{\overline{\mu}\,\overline{\beta}}{2\pi}\right)^{2k}\sum_{q=1}^{\infty}\frac{(-1)^{k+q}\,2^{q+l}}{(4\pi)^{2q}}\frac{(q+l)\,\Gamma(2q+2k-1)}{\Gamma(q)\,\Gamma(q+l)} \\
& \times\tau(2q+2k-1)\,\zeta(-q-l+1)\left(\overline{B}\,\overline{\beta}^2\right)^{q+l-1}\left\{1+\mathcal{O}\left(\overline{B}^{-1}\right)+\ldots\right\}\Bigg\},
\end{aligned}
\tag{47}
$$

and

$$M_{2l} = \frac{e}{\pi^{l-1/2}} \left(\frac{m}{2\overline{\beta}}\right)^{2l-1} \left\{ \frac{1}{2}\left(l+\frac{1}{2}\right)\zeta\left(\frac{1}{2}-l\right)\left(2\overline{B}\,\overline{\beta}^2\right)^{l-1/2}\left\{1+\mathcal{O}\left(\overline{B}^{-1}\right)+\ldots\right\}\right.$$

$$+\sum_{k=0}^{\infty}\frac{\left(\overline{\mu}\,\overline{\beta}\right)^{2k}}{\Gamma(2k+1)}\sum_{q=1}^{\infty}(-1)^q\,2^{2l-q}\frac{(q+1)\,\Gamma\left(l-q-\frac{1}{2}\right)}{\Gamma(q+1)}$$

$$\left.\times\,\tau(2l-2q-2k-1)\,\zeta(-q)\left(\overline{B}\,\overline{\beta}^2\right)^q\left\{1+\mathcal{O}\left(\overline{B}^{-1}\right)+\ldots\right\}\right\}. \qquad (48)$$

As we have already seen, the absence of critical behavior for fermions meant that in weak fields, the leading-order magnetization was the same in both the $\overline{\mu} \simeq 1$ and $\overline{\mu} \to 0$ region in all dimensions. From (47) and (48), it can be seen that this smooth behavior also manifests itself in the transition from weak to intermediate fields; in fact to leading order the weak- and intermediate-field magnetizations in most cases are the same, the only difference being that the field corrections to these leading-order magnetizations are in descending powers of the field (i.e. of the form $\overline{B}^{-1}$, etc.), rather than small-field corrections (of $\mathcal{O}\left(\overline{B}^3\right)$, say). The two cases where the magnetization does change for $\overline{B} \gg 1$ is when $d = 3$, which warrants some discussion.

3.3.1 $d = 3$

The leading-order temperature behavior of the charge-density equations (37) and (38) remains unchanged from the weak-field forms. Thus, the leading order behavior for the chemical potential in these regions (Eqns. (41) and (43) respectively), also are unaltered. Therefore, in both the intermediate-field region, and the neutrino limit, the leading-order magnetization is given by

$$M_3 = -\frac{em^2}{48\pi^2}\overline{B}\left[\log\overline{B}\,\overline{\beta}^2+\mathcal{O}(1)+\ldots\right]. \qquad (49)$$

This new magnetization law for intermediate fields indicates that the gas remains paramagnetic, as of course $\overline{B}\,\overline{\beta}^2 \ll 1$ in this region, and so M_3 is still positive. This is not unexpected, as a stronger field will demand an even larger paramagnetic moment from spin alignment. The fact that $\overline{B}$ premultiplies the logarithm in (49), where it now also appears in the argument, shows this clearly.

3.3.2 $d \geq 4$

The leading-order magnetization for all $d \geq 4$ in the intermediate-field region, in both the intermediate quantum region and in the neutrino limit, is simply the weak field result. Explicitly,

$$M_d = \frac{em^{d-1}}{2^{d-2}\pi^{(d+1)/2}} \left[\tau (d-3) \Gamma\left(\frac{d-3}{2}\right) \left(\frac{2}{\overline{\beta}}\right)^{d-3} + \mathcal{O}\left(\delta\left(\overline{B}\,\overline{\beta}^2\right)\right) + \ldots \right] \frac{1}{3}\overline{B} \qquad (50)$$

where $\delta\left(\overline{B}\,\overline{\beta}^2\right) = \left(\overline{B}\,\overline{\beta}^2\right)^{3/2}$ for $d = 4$, $\delta\left(\overline{B}\,\overline{\beta}^2\right) = \left(\overline{B}\,\overline{\beta}^2\right)^2 \log \overline{B}\,\overline{\beta}^2$ for $d = 5$, and $\delta\left(\overline{B}\,\overline{\beta}^2\right) = \left(\overline{B}\,\overline{\beta}^2\right)^2$ for $d \geq 6$.

Again, paramagnetism is enhanced in all dimensions with the larger field strength, as the leading-order form for the magnetization remains unaltered. It remains now to be seen what happens when $\overline{B}\,\overline{\beta}^2 \gg 1$, the strong-field limit.

3.4 Strong fields

In our study of the magnetized pair Bose gas, the strong-field condition $\overline{B}\,\overline{\beta}^2 \gg 1$ necessitated a return to the original integral form for the thermodynamic potential, as the Mellin transform method for generating the weak- and intermediate-field expansions did not readily yield the asymptotics of this strong-field region. This also applies to fermions, and so returning to the sum form for Ω_d of Eqn. (32), and separating out the $n = 0$, $\sigma = 1$ term gives

$$\begin{aligned} \frac{\Omega_d}{V} = -\frac{\overline{B}m^{d+1}}{2^{(d-1)/2}\pi^{(d+1)/2}} \Bigg\{ & \sum_{j=1}^{\infty} \frac{(-1)^{j+1} e^{j\overline{\beta}\overline{\mu}}}{\left(j\overline{\beta}\right)^{(d-1)/2}} K_{(d-1)/2}\left(j\overline{\beta}\right) \\ & + 2\sum_{n=0}^{\infty}\sum_{j=1}^{\infty} \left[(2n+2)\overline{B}+1\right]^{(d-1)/4} \frac{(-1)^{j+1} e^{j\overline{\beta}\overline{\mu}}}{\left(j\overline{\beta}\right)^{(d-1)/2}} \\ & \times K_{(d-1)/2}\left(j\overline{\beta}\left[(2n+2)\overline{B}+1\right]^{1/2}\right) \Bigg\} + \overline{\mu} \to -\overline{\mu}\,. \end{aligned} \qquad (51)$$

Clearly, aside from the prefactor of $\overline{B}$, the first piece is essentially the sum representing the thermodynamic potential of a $(d-2)$-dimensional *field-free* pair Fermi gas. A term of this nature did not appear in the case of spinless bosons, and again its presence here is due to the interplay between spin paramagnetism and Landau diamagnetism. The nature of the asymptotic expansions for this term will be determined now by the size of $\overline{\beta}$, and so immediately there is a contrast with the Bose gas in strong fields, which we showed to have a leading-order magnetization law independent of temperature. As we are interested in the region $\overline{\beta} \ll 1$, we shall restrict our expansions of this part of the Fermi thermodynamic potential to this only; a non-relativistic treatment would require $\overline{\beta} \gg 1$ expansions, which are not of interest to us in this paper.

Writing the first piece of (51) as Ω'_d, and the field-free thermodynamic potential of Eqns. (35) and (36) as $\Omega_d\left(\overline{B}=0\right)$, then

$$\Omega'_d = \frac{m^2\overline{B}}{2\pi}\,\Omega_{d-2}\left(\overline{B}=0\right) \; . \tag{52}$$

The other pieces of Ω_d of (51) above can be expanded in the $\overline{B}\,\overline{\beta}^2 \gg 1$ limit by using the large parameter expansion for the modified Bessel functions given in Section 3.5 of [1]. As in the case of the pair Bose gas in strong fields, all terms in this piece with $n \geq 1$ will be exponentially smaller than the $n = 0$ term, so that we have to leading order

$$\frac{\Omega_d}{V} = \frac{\Omega'_d}{V} - \frac{2m^{d+1}\overline{B}\left(2\overline{B}+1\right)^{(d-2)/4}}{\left(2\pi\overline{\beta}\right)^{d/2}}\left\{f_{d/2}\left(\exp\left[\overline{\mu}-\left(2\overline{B}+1\right)^{1/2}\right]\right)+\overline{\mu}\rightarrow-\overline{\mu}\right\} \; . \tag{53}$$

where $f_\nu(z)$ are the Fermi functions, defined for $\mathrm{Re}\,\nu \geq 1$ by

$$f_\nu(z) = \sum_{j=1}^{\infty}(-1)^{j+1}\frac{z^j}{j^\nu} \; . \tag{54}$$

If we wish to find expansions for the magnetization around $\overline{\mu} \simeq 1$, then clearly the Fermi-function components of Ω_d in (53) will be exponentially smaller than any contribution from Ω'_d. Furthermore, it is then easy to relate the magnetization to the pressure of the field-free gas via (52), as

$$M_d = -\frac{e}{m^2V}\frac{\partial\Omega'_d}{\partial\overline{B}} = -\frac{e}{2\pi}\frac{\partial}{\partial V}\Omega_{d-2}\left(\overline{B}=0\right) = \frac{e}{2\pi}P_{d-2}\left(\overline{B}=0\right) \; . \tag{55}$$

The chemical potential around the characteristic temperature T_0 is now given by

$$\overline{\mu} = 1 - (d-3)\,t + \mathcal{O}\left(t^2\right) + \dots . \tag{56}$$

and our expansions for the pressure in Section 2 (11) and (16) will therefore allow us to find the magnetization expansions in the $\overline{\mu} \simeq 1$ region for the gas in strong fields. As an example, we give the magnetization in three dimensions (for which we use the one-dimensional pressure expansion (11)):

$$M_3 = \frac{em^2}{4\pi^2}\left\{\frac{\pi^2}{3\overline{\beta}_0^2}(1+2t) + \log\left(\frac{\overline{\beta}_0}{\pi}\right) - \gamma - \frac{1}{2} - t \right.$$
$$\left. + \tau(3)\left(\frac{\overline{\beta}_0}{4\pi}\right)^2(5-2t) + \mathcal{O}\left(\overline{\beta}_0^4\right) + \dots\right\} + \mathcal{O}\left(t^2\right) + \dots . \tag{57}$$

Interestingly, the field-independence of this magnetization law seems to have been caused by the strength of the field; it must be restated that this magnetization is due essentially to the fermions in the lowest Landau level with spin aligned with the field. Once the field is so high that the occupation of other spin states and Landau orbits is exponentially damped, the net paramagnetic moment is due only to these $n = 0$, $\sigma = 1$ fermions. However, the exclusion principle forces these fermions to occupy linear momentum states above the ground state, and hence the sensitivity of the magnetization to the temperature. Contrast this to the Bose gas in strong fields of Section 3.5 of [1]. Bose statistics allow all of the bosons to occupy the ground state, and hence the gas is able to maintain its ground-state, zero-temperature behavior up to $T^2 \lesssim eB$.

We should also also mention that it may be tempting to investigate expansions in other regions of $\overline{\mu}$-space, particularly $\overline{\mu} \to 0$ and $\overline{\mu} \simeq 2\overline{B}$. However, the former will take us to a region which violates $\overline{B}\,\overline{\beta}^2 \gg 1$, and the latter will result in divergence problems for the expansions of Ω_d', and so seems to be a low-temperature region which is not of interest to us here. Finally, it is important to note that in this strong-field region the role of the vacuum, and interactions, cannot be overlooked. This is a matter which we discussed in [1].

4 Discussion and concluding remarks

Our study of the pair Fermi gas has taken us from consideration of the thermodynamic function expansions for no external fields, through to a consideration of the magnetized gas over a wide range of field strengths. It was readily apparent that the gas is paramagnetic, due essentially to the contribution of the spins, which overcome the diamagnetic Landau orbit component; however for a relativistic energy spectrum an explicit calculation of the contribution of each component is not possible. Importantly, for each field-strength region, a different leading-order magnetization law was found (in our three dimensions, at least), indicating that the roles of field strength and temperature, and there relative magnitudes, are crucial to the behavior of this non-interacting gas. Curiously, but explicably, the magnetization in the strongest fields was found to be field-independent.

The role of pair production in the statistical mechanics of the magnetized pair Fermi gas was a more subtle one than in the corresponding Bose system. There, it provided the mechanism for total field expulsion; in the case of fermions, it allowed the paramagnetic moment of the spins to maintain its dominance over Landau diamagnetism.

The applicability of the magnetized pair Fermi gas as a model for various physical systems, such as exotic stellar objects and in cosmology, has recently led to considerable interest in the study of many-fermion systems at finite temperature and density in external magnetic fields. Chodos *et al.* [17] have presented a formulation for the electron propagator in a magnetic field with finite charge density and chemical potential. More recently, Elmfors *et al.* [18] have also investigated this system. They essentially present (in Lagrangian language) the effective action of a system of non-interacting fermions, and also include the vacuum contribution, which they renormalize. The particle part of their Lagrangian is exactly the thermodynamic potential of the magnetized pair Fermi gas, which we have considered in detail in Section 3 of this paper. Elmfors *et al.* are able to glean some leading-order information from a Poisson-summation form for Ω_d, for example they find the leading-order magnetization in the weak-field limit in three dimensions, our Eqn (44). However, the full nature of the field and temperature expansions, which we have given (where applicable) in the intermediate quantum region and neutrino limit for weak, intermediate and strong fields, is readily obtained from the Mellin-transform technique which we have employed. Also, Elmfors *et al.* have investigated the low-temperature behavior of the pair Fermi system, when the system is degenerate. Whilst we have not considered this here, we point out that a study of the ground state of the relativistic Fermi gas has been given by Delsante and Frankel [19]. Finally, Chiu [20] discusses various aspects of the ground state of pair fermion systems in intense magnetic fields.

At this point, we conclude our study of the magnetized pair Fermi gas. The next great challenge, for both pair Fermi and Bose systems, is to include interactions, such as interparticle electromagnetic couplings, or particle self-interactions. One would hope that a successful perturbative treatment of such systems would provide even greater insight into the behavior of exotic stellar objects, and cosmological events such as the electroweak phase transition, where the roles of temperature, finite charge densities, fields, and interactions are all important.

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