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Generalized NLS Hierarchies from Rational $\mathcal{W}$ Algebras

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Abstract

Finite rational $\mathcal{W}$ algebras are very natural structures appearing in coset constructions when a Kac-Moody subalgebra is factored out. In this letter we address the problem of relating these algebras to integrable hierarchies of equations, by showing how to associate to a rational $\mathcal{W}$ algebra its corresponding hierarchy. We work out two examples: the $sl(2)/U(1)$ coset, leading to the Non-Linear Schrödinger hierarchy, and the $U(1)$ coset of the Polyakov-Bershadsky $\mathcal{W}$ algebra, leading to a 3 field representation of the KP hierarchy already encountered in the literature. In such examples a rational algebra appears as algebra of constraints when reducing a KP hierarchy to a finite field representation. This fact arises the natural question whether rational algebras are always associated to such reductions and whether a classification of rational algebras can lead to a classification of the integrable hierarchies.

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1 Introduction

In a previous work [1] it has been shown that a natural structure of finite rational $\mathcal{W}_Q$ algebra (the index Q stands for "quotient") appears when looking to the subsector in the enveloping algebra of an affine Lie $\mathcal{G}$ or standard polynomial $\mathcal{W}$ algebra which commutes with respect to a given Kac-Moody subalgebra $\hat{\mathcal{G}} \subset \mathcal{G}, \mathcal{W}$. Such rational $\mathcal{W}_Q$ algebras can also be seen as specific realizations of non-linear W_∞ algebras of polynomial type.

In this letter we will point out that to each such rational $\mathcal{W}_Q$ algebra is associated a whole hierarchy of equations admitting an infinite number of hamiltonians in involution. These hierarchies could deserve the name of generalized Non-Linear Schrödinger (NLS) hierarchies. The simplest example of our framework is indeed provided by the Non-Linear Schrödinger hierarchy, which is obtained from the coset construction $sl(2)/U(1)$. Therefore we will refer to these generalized hierarchies also as coset hierarchies and to their associated finite $\mathcal{W}_Q$ algebras as rational coset $\mathcal{W}$ algebras[1].

Some comments are in order: the arising of a $\mathcal{W}_\infty$ algebra structure in the Witten's black hole [3] $sl(2)/U(1)$ coset model has been pointed out in many papers [4, 5] (for another coset construction leading to $\mathcal{W}_\infty$ algebra see [6]). In [5] a realization in terms of parafermions has been given. For a connection of such construction to the NLS hierarchy see e.g. [7].
The fact that an algebra with infinite elements is produced out of a finite number of parafermions is already an indication of the "degeneracy" of such algebra. Moreover, $\mathcal{W}_\infty$-algebra structures appear as algebras of constraints when reducing KP hierarchies to standard KdV-like hierarchies (see e.g. [8, 9, 10]): in this framework higher order constraints are just a consequence of a finite set of constraints. Here again the "degenerated" character of such $\mathcal{W}_\infty$ algebras appears. Rational $\mathcal{W}$ algebras are nothing else than $\mathcal{W}_\infty$ algebras which can be explicitly "solved", namely that can be expressed in closed form with a finite number of fields and a finite number of algebraic constraints among them which are consistent with the Poisson brackets structure of the underlining $\mathcal{W}_\infty$ algebra. In [1] a systematic way to produce rational algebras has been discussed. Here we will rely on the results of [1] to provide a framework for generalizing the works of [5, 7].

To be specific we will treat only cosets obtained by quotienting out an abelian $U(1)$ Kac Moody subalgebra, but the extension to non-abelian quotients is plainly straightforward. The whole discussion will be done for the classical case, so that the algebras involved are assumed to be Poisson brackets algebras. Results and conceptual points concerning quantum rational $\mathcal{W}$ algebras will be reported elsewhere.

A technical remark: we are interested in Poisson brackets of invariant quantities (with respect to the quotient subalgebra) defined in a manifestly covariant way. For that reason our framework is particularly suitable for computations because we do not need to take into account Dirac's brackets but only the original Poisson brackets structure.

It should be noticed that, despite striking analogies, namely two-bosons realization of the hierarchies, introduction of covariant derivatives, etc., our construction is basically different from the one proposed in [8]: the latter is based on currents realized through bosonic $\beta - \gamma$ systems, while our currents are standard Kac-Moody currents.

[1]For rational $\mathcal{W}$ algebras see also [2]

The plan of this paper is the following: in section 2 the basic features of rational coset $\mathcal{W}_Q$ algebras are recalled, the connection to non-linear W_∞ algebras is established, the formalism and conventions needed for later use are introduced. In section 3 the NLS hierarchy is revisited as a realization of the simplest possible example of $\mathcal{W}_Q$ algebra, based on the $sl(2)/U(1)$ coset; this is intended to illustrate the method which can be straightforwardly generalized to produce other coset hierarchies. In section 4 as an application of the above method, we will discuss the hierarchy associated to the simplest coset obtainible from a polynomial $\mathcal{W}$ algebra: this is the $U(1)$ coset of the Polyakov-Bershadsky [11] $\mathcal{W}$ algebra, which is of type $(1, \frac{3}{2}, \frac{3}{2}, 2)$. The hierarchy associated to such rational $\mathcal{W}$ algebra turns out to coincide with the first reduction of the four-field KP hierarchy of ref.[9].

2 Rational $\mathcal{W}$ algebras

In this section we will review the formulation of rational $\mathcal{W}_Q$ algebras as stated in [1]. For simplicity we will consider only abelian ($U(1)$) quotients.

Let $\mathcal{G}$ or $\mathcal{W}$ denote respectively a Kac-Moody or a $\mathcal{W}$ algebra admitting a subalgebra generated by a $U(1)$ Kac-Moody current $J(z)$:

$$\{J(z), J(w)\} = \gamma\delta'(z-w) \equiv \gamma\partial_w\delta(z-w) \tag{2.1}$$

(in the classical case the normalization factor γ can be fixed without loss of generality as will be done in the following). It is therefore possible to express any other element of the algebra in terms of a basis of fields W_{i,q_i} having a definite charge q_i with respect to $J(z)$, namely satisfying:

$$\{J(z), W_{i,q_i}(w)\} = q_i W_{i,q_i}\delta(z-w) \tag{2.2}$$

A derivative $\mathcal{D}$, covariant with respect to the above relations, can be introduced:

$$\mathcal{D}W_{i,q_i}(w) = (\partial - \tfrac{q_i}{\gamma}J(w))W_{i,q_i}(w) \tag{2.3}$$

The elements in $Com_J(\mathcal{G}, \mathcal{W})$, the subalgebra of the enveloping algebra commuting with $J(w)$, [2] are therefore spanned by the vanishing total charge monomials $(\mathcal{D}^{n_1}V_{1,q_1})(\mathcal{D}^{n_2}V_{2,q_2})...(\mathcal{D}^{n_j}V_{j,q_j})$, where the n_i's are non negative integers and the total charge is $q = q_1 + q_2 + ...q_j = 0$.

From now on we will concentrate only on invariants produced by bilinear combinations such as

$$\mathcal{D}^p V_+ \cdot \mathcal{D}^q V_- \tag{2.4}$$

(with $p, q \geq 0$ and $V_\pm$ have opposite charges), together with of course originally invariant fields. [3]

We summarize here the basic results of [1], with some extra comments:

[2] It is better conceptually to understand the derivative operator $\partial = \frac{d}{du}$ as an element of the original algebra, as this is the case for Kac-Moody algebras.

[3] This is only in order to avoid unnecessary technical complications in the following discussion, but hierarchies associated to multilinear invariants can be produced along the same lines as for the bilinear ones and are worth being studied

i) there exists a linear basis of fields, given by $V^{(p)} = \mathcal{D}^p V_+ \cdot V_-$ such that any bilinear invariants of the kind (2.4) is a linear combination of the $V^{(p)}$'s and the derivatives acting on them.
ii) the Poisson brackets algebra of the fields $V^{(p)}$'s among themselves is closed (possibly with the addition of other invariants, in the general case), but never in a finite way (the Poisson brackets of $V^{(p)}$ with $V^{(q)}$ necessarily generates on the right hand side terms depending on $V^{(p')}$, with $p' > p, q$). Moreover it can be explicitly checked that it is a non-linear algebra, so that it has the structure of a non-linear W_∞ algebra.
iii) due to the properties of the covariant derivative, the fields $V^{(p)}$, which are linearly independent, satisfy algebraic relations like the following quadratic ones

$$V^{(p+1)} \cdot V^{(0)} = V^{(0)} \cdot \partial V^{(p)} - V^{(p)} \cdot \partial V^{(0)} + V^{(p)} \cdot V^{(1)} \qquad (2.5)$$

Such relations allow to express algebraically the fields $V^{(p)}$, for $p \geq 2$ in terms of the fundamental fields $V^{(0)}$ and $V^{(1)}$. The above derived non-linear $\mathcal{W}_\infty$ algebra has therefore the structure of a rational $\mathcal{W}$ algebra. Notice that relations like (2.5) contain no informations if the fields $V_\pm$ are fermionics. (2.5) can still be applied to superalgebras if $V_\pm$ are bosonic superfields.
iv) if $Com_J(\mathcal{G}, \mathcal{W})$ contains a field $T(w)$ (the stress-energy tensor) whose Poisson brackets are the Virasoro algebra with non-vanishing central charge, and moreover $V^{(0)}$ is primary with conformal dimension h, then there exists a one-to-one correspondence between the fields $V^{(p)}$ of the basis, and an infinite tower of uniquely determined fields W_{h+p}, primary with respect to T, with conformal dimension $h+p$. This relation should be understood as follows: $V^{(p)}$ is the leading term in the associated primary field. The remaining terms are fixed without ambiguity, some of them just requiring W_{h+p} being primary, some others once a specific scheme to determine them is adopted (as an analogy, one should think to the choice of the renormalization scheme when dealing with renormalizable quantum field theories).
As we will see, the condition of having a non-vanishing central charge drops for the $sl(2)/U(1)$ coset model: therefore no infinite tower of primary fields associated to each $V^{(p)}$ can be generated (we have an infinite tower of "almost" primary fields associated to them). An infinite number of primary fields can still be produced, but they are of a trivial type, being just products of lower order primary fields. Anyway the structure of a rational algebra with two primary fields is mantained even in this case. The next simplest model admitting an infinite tower of invariant primary fields associated to $V^{(p)}$ is based on the coset $\frac{sl(2)\times U(1)}{U(1)}$, since now there exists another $U(1)$ current,commuting with $J(w)$, which allows to define an invariant stress-energy tensor with non-vanishing charge.

In the following, in order to illustrate how our procedure works we will treat explicitly two examples, so let us write down their algebra here.

2.1 case a: the $\frac{sl(2)}{U(1)}$ coset.

The classical $sl(2) - \mathcal{KM}$ algebra is given by the following Poisson brackets:

$$\begin{aligned} \{J_+(z), J_-(w)\} &= \delta'(z-w) - 2J_0(w)\delta(z-w) \equiv \mathcal{D}(w)\delta(z-w) \\ \{J_0(z), J_\pm(w)\} &= \pm J_\pm(w)\delta(z-w) \end{aligned}$$

$$\begin{aligned}\{J_0(z),J_0(w)\} &= -\tfrac{1}{2}\delta'(z-w)\\ \{J_\pm(z),J_\pm(w)\} &= 0\end{aligned} \tag{2.6}$$

$J_\pm$ play the role here of the fields $V_\pm$ in the previous subsection; they have conformal dimension 1 (here and in the following, the symbol $\delta'(z-w)$ is understood as $\partial_w\delta(z-w)$). The rational coset algebra of the commutant with respect to the J_0 current is given by the following Poisson brackets

$$\begin{aligned}\{W_2(z),W_2(w)\} &= 2W_2(w)\delta'(z-w)+\partial W_2(w)\delta(z-w)\\ \{W_2(z),W_3(w)\} &= 3W_3(w)\delta'(z-w)+\partial W_3(w)\delta(z-w)\\ \{W_3(z),W_3(w)\} &= 2W_2(w)\delta'''(z-w)+3\partial W_2(w)\delta(z-w)''+\\ &\quad [16V^{(2)}-8\partial W_3+8W_2{}^2-3\partial^2W_2](w)\delta'(z-w)+\\ &\quad \partial_w[8V^{(2)}-4\partial W_3+4W_2{}^2-2\partial^2W_2](w)\delta(z-w)\end{aligned} \tag{2.7}$$

We have preferred to express the above algebra in the basis of (uniquely determined) primary fields $W_2=J_+\cdot J_-$ and $W_3=\mathcal{D}J_+\cdot J_- - J_+\cdot\mathcal{D}J_-$. They have dimension 2, 3 respectively, while

$$V^{(2)} = \mathcal{D}^2J_+\cdot J_- = \frac{1}{4W_2}[W_3^2+2W_2\partial W_3+2W_2\partial^2W_2-\partial W_2\partial W_2]. \tag{2.8}$$

The second equality follows from the relation (2.5).
W_2 plays the role of a stress-energy tensor having no central charge. As already stated, in this simple example there exists no infinite tower of primary fields associated to the $V^{(p)}$'s fields, the only primary ones being $W_{2,3}$ and their products $W_2{}^mW_3{}^n$ for m,n non-negative integers.

The algebra of the fields $V^{(p)}=\mathcal{D}^pJ_+\cdot J_-$ is a non-linear $\mathcal{W}_\infty$ algebra: if we let from the very beginning identify $J_0\equiv 0$, then $J_\pm$ can be identified with the fields $\partial\beta$ and γ of a bosonic $\beta-\gamma$ system, the covariant derivative in $V^{(p)}$ must be replaced by the ordinary derivative and the non-linear $\mathcal{W}_\infty$ algebra is reduced to the standard linear w_∞ algebra.

2.2 case b: The $U(1)$ Polyakov-Bershadsky coset.

In this subsection we introduce the $U(1)$ commutant of the Polyakov-Bershadsky $\mathcal{W}$ algebra[11]. This algebra is associated to a non-abelian $sl(3)$ Toda model. For constructing general $\mathcal{W}$ algebras from Toda theories, both abelian and non-abelian, see e.g. [12, 13].

The Polyakov-Bershadsky $\mathcal{W}$ algebra is defined in terms of a stress-energy tensor $T(w)$, a $U(1)$ Kac-Moody current $J(w)$ and two bosonic charged fields of (covariantly conformal) dimension $\frac{3}{2}$. It is explicitly given by the following Poisson brackets:

$$\begin{aligned}\{J(z),J(w)\} &= \tfrac{3}{2}\delta'(z-w)\\ \{T(z),T(w)\} &= 2T(w)\delta'(z-w)+\partial T(w)\delta(z-w)-\tfrac{1}{2}\delta'''(z-w)\\ \{T(z),J(w)\} &= 0\end{aligned}$$

$$
\begin{aligned}
\{T(z), W_\pm(w)\} &= \tfrac{3}{2}W_\pm(w)\delta'(z-w) + (\mathcal{D}W_\pm)(w)\delta(z-w)\\
\{J(z), W_\pm(w)\} &= \pm\tfrac{3}{2}W_\pm(w)\delta(z-w)\\
\{W_+(z), W_-(w)\} &= (T-\mathcal{D}^2)(w)\delta(z-w)\\
\{W_+(z), W_+(w)\} &= \{W_-(z), W_-(w)\} = 0
\end{aligned}
\tag{2.9}
$$

The covariant derivative is now $\mathcal{D}W_\pm = (\partial \mp J)W_\pm$.

The rational coset $\mathcal{W}$ algebra of the $U(1)$ commutant of the above algebra has been written down in [1], the expression being given in terms of primary fields. Since, as already remarked, for the purpose of integrable hierarchies is not necessary to dispose of a basis of primary fields, here we prefer to express the coset algebra in terms of the fields $T(w)$ and $V^{(n)} = \mathcal{D}^n W_+ \cdot W_-$, this linear basis being more suitable for making computations (deriving equations of motions and so on). The algebra can be expressed in a closed form as follows

$$
\begin{aligned}
\{T(z), T(w)\} &= 2T(w)\delta'(z-w) + \partial T(w)\delta(z-w) - \tfrac{1}{2}\delta'''(z-w)\\
\{T(z), V^{(0)}(w)\} &= 3V^{(0)}(w)\delta'(z-w) + \partial V^{(0)}(w)\delta(z-w)\\
\{T(z), V^{(1)}(w)\} &= \tfrac{3}{2}V^{(0)}(w)\delta''(z-w) + V^{(1)}(w)\delta'(z-w) + \partial V^{(1)}(w)\delta(z-w)\\
\{V^{(0)}(z), V^{(0)}(w)\} &= (4V^{(1)} - 2\partial V^{(0)})(w)\delta'(z-w) + \partial(2V^{(1)} - \partial V^{(0)})(w)\delta(z-w)\\
\{V^{(0)}(z), V^{(1)}(w)\} &= V^{(0)}(w)\delta'''(z-w) + 2V^{(1)}\delta''(z-w) +\\
&\quad (V^{(2)} - 2\partial V^{(1)})(w)\delta'(z-w) +\\
&\quad (2\partial V^{(2)} - \partial^2 V^{(1)} - V^{(0)}\partial T)(w)\delta(z-w)\\
\{V^{(1)}(z), V^{(1)}(w)\} &= 2V^{(1)}(w)\delta'''(z-w) + 3\partial V^{(1)}(w)\delta''(z-w) +\\
&\quad (3V^{(3)} - 6\partial V^{(2)} + 3\partial^2 V^{(1)} - \tfrac{3}{2}V^{(0)}V^{(0)} - 2TV^{(1)})(w)\delta'(z-w) +\\
&\quad \partial(\partial^2 V^{(1)} - 3\partial V^{(2)} + 3V^{(3)} - TV^{(1)} - \tfrac{3}{4}V^{(0)}V^{(0)})(w)\delta(z-w)
\end{aligned}
\tag{2.10}
$$

The fields $T(w)$, $V^{(0,1)}$ are the fundamental ones. The fields $V^{(n)}$ for $n \geq 2$ being determined from (2.5).
Since in this case the invariant stress-energy tensor $T(w)$ admits a non-vanishing central charge, the commutant contains an infinite tower of primary fields and the rational algebra has an underlining structure of a non-linear $\mathcal{W}_\infty$ algebra of *primary* fields.

3 The NLS-hierarchy revisited.

In this section we will derive the hierarchy associated to the non-linear Schrödinger equation directly from the rational $\mathcal{W}$ algebra (2.7).

Before doing that, a few words need to be spent: one very remarkable feature of the rational algebras (2.7) and (2.10) is that, in both cases, it is possible to find out a basis of generating fields for the rational algebra such that for any Poisson brackets involving two fields of the basis, the term in the right hand side proportional to the delta-function turns out to be a total derivative. This property is of course immediately seen for the rational algebra (2.7) in the basis we have expressed it, but it is valid also for (2.10) as we will discuss in the next section. This very important property is shared by standard $\mathcal{W}$ algebras [14]. We will see that it is of crucial importance for making a connection with the hierarchies and is not unexpected.[4] Even if we do not have at present a general argument stating that this is always the case, it is indeed true that in any example of coset construction worked so far the above property is satisfied. Such examples include cosets realized from Kac-Moody and $\mathcal{W}$ algebras, with respect to abelian and non-abelian Kac-Moody quotient, cosets of superKac-Moody and super$\mathcal{W}$ algebras, even cosets realized for quantum algebras. The discussion which follows, done for the (2.7) algebra, can be generalized for any rational algebra satisfying the above property, which means at least a very large class of rational algebras.

Let us come back now to the rational algebra (2.7). Since we are not interested in its conformal property, it is more convenient to look at it as expressed in terms of the fields $V^{(n)} = \mathcal{D}^n J_+ \cdot J_-$, for n non-negative integer.

The above mentioned property tells us that the line integrals $H_1 = \int dw V^{(0)}(w)$, $H_2 = \int dw V^{(1)}(w)$ have vanishing Poisson brackets among themselves and can therefore be regarded as two compatible hamiltonians.
The equations of motion for the composite fields $V^{(n)}$ in the commutant are equivalent to the equations of motion for the original fields $J_{\pm,0}$. In one direction this statement is obvious; the converse is also true: at first let us remark that, by construction, J_0 commutes with any field in the commutant, therefore its equations of motion are always $\frac{d}{dt}J_0 = 0$, and is consistent to set $J_0 = 0$. Next, the following relations hold:

$$\frac{\partial J_+}{J_+} = \frac{V^{(1)}}{V^{(0)}}; \qquad \frac{\partial J_-}{J_-} = \frac{\partial V^{(0)} - V^{(1)}}{V^{(0)}} \tag{3.11}$$

they tell that the dynamics of $J_\pm$ can be reconstructed from the dynamics of $V^{(0,1)}$ through the non-local transformations

$$\begin{aligned} J_+(z) &= e^{\int^z dw\left(\frac{V^{(1)}}{V^{(0)}}\right)(w)} \\ J_-(z) &= e^{\int^z dw\left(\frac{\partial V^{(0)} - V^{(1)}}{V^{(1)}}\right)} \end{aligned} \tag{3.12}$$

There exists two compatible Poisson brackets structures which endorse our rational algebra of a bihamiltonian structure: the first Poisson bracket structure, the original one, is determined by the $sl(2) - \mathcal{KM}$ algebraic relations (2.6) for the fields $J_{\pm,0}$; the second one is again determined from (2.6), but after taking into account the field transformations $J_- \mapsto J_-$, $J_+ \mapsto \mathcal{D}J_+$. The compatibility of the two Poisson brackets means that, for any function f we have the equality

$$\frac{df}{dt} = \{H_1, f\}_2 = \{H_2, f\}_1 \tag{3.13}$$

The two Poisson brackets structures are the firsts of an infinite series relating the infinite number of hamiltonians in involution.

[4] as for the rational algebra (2.10), it is related to the fact that the Polyakov-Bershadsky algebra is associated to a fractional KdV hierarchy, see [15].

The equations of motion relative to the first hamiltonian imply that the $V^{(n)}$ fields are free fields: $\dot{V}^{(n)} = V^{(n)\prime}$ for any n (from now on we will use the standad conventions of dot and prime to denote time and spatial derivative respectively).
The equations of motions for the fields $J_\pm$ relative to the second hamiltonian are given by the following system (after setting $J_0 = 0$):

$$\dot{J}_\pm = \pm J_\pm'' \pm 2J_\pm(J_+J_-) \tag{3.14}$$

This is nothing else than the coupled system associated to the NLS equation; the standard NLS equation is recovered assuming the time imaginary and setting $J_-^* = J_+ = u$; such constraint is of course consistent with the above equation. We get for u the NLS equation in its standard form:[16]

$$i\dot{u} = u'' + 2u|u|^2 \tag{3.15}$$

In terms of the fields $V^{(n)}$ the equations of motion relative to the second hamiltonian read as follows:

$$\begin{aligned} \dot{V}^{(0)} &= 2\partial V^{(1)} - \partial^2 V^{(0)} \\ \dot{V}^{(1)} &= 2\partial V^{(2)} - \partial^2 V^{(1)} + 2V^{(0)}\partial V^{(0)} \end{aligned} \tag{3.16}$$

These two equations, together with the algebraic constraints (2.5) are sufficient to generate the whole tower of equations of motion

$$\dot{V}^{(n)} = 2\partial V^{(n+1)} - \partial^2 V^{(n)} + 2\sum_{k=1}^{n} \binom{n}{k} V^{(n-k)}\partial^k V^{(0)} \tag{3.17}$$

The NLS hierarchy is integrable, its integrability property being made manifest from the existence of a Lax pair representation in terms of pseudo-differential operators (PDO)(see [17] for more details and for the conventions here used).
Let $L = \partial + \sum_{n=0}^{\infty} u_n \partial^{-n-1}$ be the PDO of the KP hierarchy. The different flows are defined through the position

$$\frac{\partial L}{\partial t_k} = [L, L^k{}_+] \tag{3.18}$$

where k is a positive integer and $L^k{}_+$ denotes the purely differential part of the operator. The equations of motion of the NLS hierarchy relative to the second hamiltonian coincide with eq.(3.16,3.17) for $k = 2$ after reduction, namely after constraining the infinite number of fields u_n to be $u_n(x, t_2) = (-1)^n V^{(n)}$. Let us point out that, after having identified $u_0 \equiv V^{(0)}$, $u_1 \equiv -V^{(1)}$, the reduction of the remaining fields is uniquely determined by induction if we wish to reproduce from (3.18) the equations of motion (3.16,3.17): here again it is manifest the role of the above rational W algebra as underlining structure which allows to perform in a consistent way a reduction of the KP hierarchy having an infinite set of independent fields, to a KdV-like hierarchy involving only a finite number of independent fields.

The Lax pair version of (3.16,3.17) ensures the existence of an infinite number of first integrals of motion $K_r = < L^r >$, labelled by the integers $r \geq 1$. Here, as commonly used,

we have introduced the symbol $< A >= \int dw a_{-1}(w)$ where A is a generic pseudodifferential operator $A = ... + a_{-1}\partial^{-1} + ...$
Furthermore it can be explicitly checked that the first integrals are all in involution (this can also be seen as a consequence of the above mentioned bihamiltonian structure).
The next hamiltonian of the hierarchy, after $H_{1,2}$ is $H_3 = \int(V^{(2)} + V^{(0)}V^{(0)})$.
The infinite tower of hamiltonians can in principle be computed with an algorithmic procedure.

In this section we have basically rederived well known results concerning the NLS hierarchy. The important point however is that we were able to do so having, as only input, the existence of the rational algebra (2.7). Our framework can be generalized to produce other hierarchies from other rational algebras. In the next section we will furnish another example of that.

4 A 3-field hierarchy coset construction

In this section another example will be treated for convincing that the relation between rational algebras and hierarchies is not incidental. It is based on the rational algebra introduced in subsection 2.2. It is surely possible to derive the associated hierarchy directly from the fractional KdV hierarchy of the Polyakov-Bershadsky algebra:[15, 18] we plan to leave such derivation for an extended version of this paper. Here we will repeat the steps done in the previous section.

The basis of generating fields $T(w)$, $V^{(0,1)}$ for the rational algebra (2.10) does not satisfy the property that the coefficients of the delta-function terms are all total derivatives. However, there exists another basis, obtained from the previous one by simply replacing

$$V^{(1)} \mapsto \hat{V}^{(1)} = V^{(1)} - \tfrac{1}{4}T^2 \tag{4.19}$$

for which the property is satisfied.
It is clear that the two sets of fields are equivalent generating sets for the rational algebra (2.10). Notice that $\hat{V}^{(1)}$ is, up to total derivative contributions, the unique field of dimension 4 (the dimension of both $V^{(1)}$ and T^2), which satisfies the above property.
Therefore we have three hamiltonians, mutually in involution, which are the first ones of the infinite hierarchy. They are given by $H_1 = \int dw T(w)$, $H_2 = \int dw V^{(0)}(w)$ and $H_3 = \int dw \hat{V}^{(1)}(w)$.
As in the previous section, the dynamics for $T, V^{(n)}$ can be reconstructed from the dynamics of $T, W_\pm$ and conversely, with non-linear transformations relating $W_\pm$ to $V^{(0,1)}$ as eq.(3.12).

The equations of motion relative to the H_1 hamiltonian are the free field equations: $\dot{T} = T'$, $\dot{V}^{(n)} = V^{(n)\prime}$.
The system of equations of motion with respect to the H_2 hamiltonian is the following one:

$$\begin{aligned} \dot{T} &= 2V^{(0)\prime} \\ \dot{W}_\pm &= \pm W_\pm{}^{(2)} \pm T W_\pm \end{aligned} \tag{4.20}$$

(here we have denoted $W_\pm^{(k)} \equiv \mathcal{D}^k W_\pm$).
In terms of $V^{(0,1)}$ the above equations read:

$$\begin{aligned} \dot{V}^{(0)} &= 2\partial V^{(1)} - \partial^2 V^{(0)} \\ \dot{V}^{(1)} &= 2\partial V^{(2)} - \partial^2 V^{(1)} - V^{(0)}\partial T \end{aligned} \qquad (4.21)$$

For generic $V^{(n)}$, $n \geq 2$, the equations are determined from the above ones and are explicitly given by

$$\dot{V}^{(n)} = 2\partial V^{(n+1)} - \partial^2 V^{(n)} - \sum_{k=1}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) V^{(n-k)} \partial^k T \qquad (4.22)$$

Here again the integrability properties are made manifest by the existence of a Lax pair formalism. In this case the pseudodifferential operator L is given by $L = \partial^2 + \sum_{n=0}^{\infty} u_n \partial^{-n}$. The equations (4.21,4.22) are recovered from the flow

$$\frac{\partial L}{\partial t_1} = [L, L_+] \qquad (4.23)$$

once imposed the constraints $u_0(x,t_1) = T$ and $u_{n+1}(x,t_1) = (-1)^n V^{(n)}$ for $n \geq 0$.
The infinite tower of hamiltonians in involution can now be computed with the usual standard procedure. The first three hamiltonians of the hierarchy are the ones given above.

It is worth mentioning that the system (4.20) coincides, after field redefinitions, with the hierarchy eq.(30) of ref.[9]. The latter being a one-field reduction of the four-field representation of the KP hierarchy. It is indeed true that the above hierarchy admits three independent fields.

It is tempting from eq.(4.20) to set $T = 2V^{(0)}$ and to further reduce the hierarchy (4.20) to the NLS hierarchy studied in the previous section. Indeed the equations of motions for $W_\pm$ will look like eq. (3.14) (after a change in sign is taken into account: $W_- \mapsto -W_-$). This constraint plus the equations of motion make T and $V^{(0)}$ free fields. In order to be consistent however we have to require the compatibility of the dynamics of T with the dynamics of $V^{(0)}$: this imply that $V^{(0)}$ must satisfy a constraint given by a differential equation; this constraint itself must be compatible with the dynamics, a further constraints is generated and so on. An infinite tower of differential equations as constraints on the linearly independent fields $V^{(n)}$ is produced; the first two constraints of the series being given by

$$\begin{gathered} \partial(2V^{(1)} - V^{(0)} - \partial V^{(0)}) = 0 \\ \partial^2(4V^{(2)} - 4\partial V^{(1)} - 2V^{(0)}V^{(0)} - 2V^{(1)} + \partial V^{(0)} + \partial^2 V^{(0)}) = 0. \end{gathered} \qquad (4.24)$$

Conclusions

In this letter we have emphasized the role played by rational $\mathcal{W}$ algebras in studying hierachies of integrable equations. In particular we have shown how to produce such hierachies from known rational $\mathcal{W}$ algebras. A natural question arises: is the converse true? Any finite-field representation of KP hierarchies gives rise to an associated rational $\mathcal{W}$ algebra? It is tempting to answer in an adfirmative way. If this would be the case, then rational $\mathcal{W}$ algebras should be a fundamental tool for classifying hierarchies of integrable equations. This problem being transferred to the problem of classifing rational $\mathcal{W}$ algebras. It is far from being an easy problem to solve[5], but at least is a well-posed problem (finite structures like rational algebras seem more treatable objects than infinite ones, like non-linear $\mathcal{W}_\infty$ algebras). In forthcoming papers we plan to address these problems and to give at least a partial answer.
We should also mention here that the quantum version of this work is in preparation. It is amazing that rational algebras can be defined in the quantum case as well, and work just like the corresponding classical versions.
Last but not least we wish to point out that having at disposal the explicit rational algebra associated to an integrable hierarchy is very illuminating and can help to simplify proofs, especially when concerned induction. We have already exploited such property in this letter.

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[5] in the case of standard polynomial $\mathcal{W}$ algebras, all those arising as Toda reduction of a WZNW model are classified by the non-equivalent $sl(2)$ embeddings in a given Lie algebra [13]. For a discussion on the completeness of polynomial algebras so generated see [19]

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