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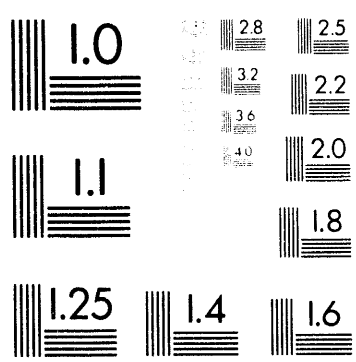

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FINITE BOSON MAPPINGS OF FERMION SYSTEMS

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ABSTRACT

We discuss a general mapping of fermion pairs to bosons that preserves Hermitian conjugation, with an eye towards producing finite and usable boson Hamiltonians that approximate well the low-energy dynamics of a fermion Hamiltonian.

1. Introduction

The dynamics of strongly interacting many-fermion systems is at the heart of nuclear physics, and gives rise to both the riches and the difficulties therein. A basic goal is simple to state: given an interaction (which itself is a difficult and separate problem) and the corresponding fermion Hamiltonian $\hat{H}_F$, solve the stationary Schrödinger equation and find the (low-lying) eigenstates,

$$\hat{H}_F \left|\Psi_\lambda\right\rangle = E_\lambda \left|\Psi_\lambda\right\rangle , \tag{1}$$

and transition matrix elements between the eigenstates $T_{\lambda'\lambda} = \langle \Psi_{\lambda'} | \hat{T} | \Psi_\lambda \rangle$. For more than three or four particles integration of the Schrödinger or related equations is numerically intractable. Instead what one often does is devise a many-body basis which can be truncated to a computationally managable size, and then solve (1) as a *matrix diagonalization problem*. In the standard fermion shell model the many-body basis, and its truncation, is built from single-particle configurations which in turn arise from, or so at least we pretend, a mean-field or Hartree-Fock calculation. This picture encompasses many basic properties of nuclei, such as deformation, but for detailed correlations the number of configurations needed becomes simply enormous. For example for ful-shell calculations of rare earth nuclides, the size of the fermion Fock space, i.e. the dimension of the matrix to be diagonalized, is of the order[1,2] 10^{15-21}! While some very clever techniques[2,3] are being employed to attack this problem, it is sensible to ask about truncations other than in the single-particle picture.

One alternative approach is to build/truncate the Fock space based on two-particle degrees of freedom. We know pairwise correlations are important from the BCS theory of superconductivity[4], where the wavefunction is a condensate of boson-like Cooper pairs of electrons coupled to zero linear momentum, from the binding-energy systematics of even and odd nuclei, and of course from the success of the phenomenological Interacting Boson Model[5] (IBM). In the latter many states and transition amplitudes are successfully described using only s- and d- (angular momentum $J = 0, 2$) bosons, which are widely thought to represent coherent nucleon pairs[1,6]. In both cases an enormous number of fermion degrees of freedom are well modeled by only a handful of boson degrees of freedom. Note that although the algebraic limits of the IBM—SU(3), SU(5), O(6), and so on—illuminate collective behavior, in the most general case it, like the fermion shell model, also reduces to a matrix diagonalization problem, with the critical advantage of a computationally tractable model space.

Our goal then is to find a much smaller, and thus managable, boson Fock space, and a set of boson images of the Hamiltonian ($\hat{h}_B$) and transition operators ($\hat{t}_B$) in that boson space that reproduce or at least approximate well the low-energy dynamics of the original fermion system,

$$\hat{h}_B \left|\Phi_\lambda\right) = E_\lambda \left|\Phi_\lambda\right), \quad \left(\Phi_{\lambda'}\right| \hat{t}_B \left|\Phi_\lambda\right) = T_{\lambda'\lambda}. \tag{2}$$

The mapping of fermion systems to bosons has a long history[7], starting at least with Holstein and Primakoff[8] in 1940 and continuing through Dyson[9] and Belyaev and Zelevinskii[10] to name just a few. After the introduction of the IBM Otsuka, Arima, Iachello[6](OAI) (along with Talmi[1]) investigated its microscopic foundations by mapping fermion shell model states to bosons.

Because of the importance and controversy of this topic we are revisiting boson mappings. Most boson expansion techniques concentrate on mapping operators and algebras[9,10]. Instead we follow Marumori[11] and OAI[6] by mapping matrix elements.

2. Fermion pairs, matrix elements, and truncation

The first step is to define many-body states built from pairs of fermions and then calculated matrix elements. We work in a fermion space with 2Ω single-particle states. For the fermion shell-model basis states one often uses Slater determinants, antisymmeterized products of single-fermion wavefunctions which we can write using Fock creation operators: $a_j^\dagger, j = 1, \cdots, 2\Omega$ on the vacuum $a_{i_1}^\dagger \cdots a_{i_n}^\dagger \left|0\right\rangle$ for n fermions. For an even number of fermions we instead construct basis states from $N = n/2$ fermion pair creation operators,

$$\left|\psi_\beta\right\rangle = \prod_{m=1}^{N} \hat{A}_{\beta_m}^\dagger \left|0\right\rangle, \quad \hat{A}_\beta^\dagger \equiv \frac{1}{\sqrt{2}} \sum_{ij} \left(\mathbf{A}_\beta^\dagger\right)_{ij} a_i^\dagger a_j^\dagger. \tag{3}$$

We always choose the $\Omega(2\Omega-1)$ matrices $\mathbf{A}_\beta$ to be antisymmetric to preserve the underlying fermion statistics, thus eliminating the need later on distinguish between 'ideal' and 'physical' bosons. Generic one- and two-body fermion operators we represent by $\hat{T} \equiv \sum_{ij} T_{ij} a_i^\dagger a_j$, $\hat{V} \equiv \sum_{\mu\nu} \langle \mu | V | \nu \rangle \hat{A}_\mu^\dagger \hat{A}_\nu$, where $T_{ij} = \langle i | \hat{T} | j \rangle$; from such operators one can construct a fermion Hamiltonian H_F.

Now one needs matrix elements of these states, including the overlap: $\langle \Psi_\alpha | \Psi_\beta \rangle$, $\left\langle \Psi_\alpha \left| \hat{H}_F \right| \Psi_\beta \right\rangle$, $\left\langle \Psi_\alpha \left| \hat{T} \right| \Psi_\beta \right\rangle$, and so on. These matrix elements are much more difficult top compute than the corresponding matrix elements between Slater determinants. Silvestre-Brac and Piepenbring[12], laboriously using commutation relations, found general expressions for the matrix elements. Rowe, Song and Chen[13] using 'vector coherent states' (we would say fermion-pair coherent states) found matrix elements between pair-condensate wavefunctions, that is states of the form $\left(\hat{A}^\dagger\right)^N |0\rangle$. Using a theorem by Lang et al.[3], we have generalized[14] the method of Rowe, Song and Chen and recovered (actually discovered independently) the general expressions of Silvestre-Brac and Piepenbring. One could now solve the Schrödinger equation (1) numerically, after truncating the fermion Fock space by restricting the set of pairs, denoted by $\{\bar{\alpha}\}$, used to construct the many-body states.

Before moving on we make two comments. The first regards the choice of truncation. Rowe, Song and Chen[13] give a variational principle that seems obviously useful in this regards. Otsuka and Yoshinaga[15] start from HFB states; the two approaches can probably be related in some approximation. The second comment is that naive truncations motivated by the IBM may not be successful. Halse, Jaqua and Barrett[17] find that $J = 0, 2$ pairs do not describe well the low-lying spectra of a $Q \cdot Q$ Hamiltonian in a single $j = 17/2$ shell, and that while inclusion of $J = 4$ pairs improves the situation considerably even that model space is somewhat lacking. That is, while phenomenology does not require $J = 4$ pairs microscopy does. We address this situation through effective interaction theory in section 4.

3. Boson representations of matrix elements

We now want to translate the fermion matrix elements into boson space. We take the simple mapping of fermion states into boson states

$$|\psi_\beta\rangle \to |\phi_\beta) = \prod_{m=1}^{N} b_{\beta_m}^\dagger |0), \tag{4}$$

where the $b^\dagger$ are boson creation operators. We construct boson operators that preserve matrix elements, introducing boson operators $\hat{\mathcal{T}}_B$, $\hat{\mathcal{V}}_B$, and most important importantly the *norm operator* $\hat{\mathcal{N}}_B$ such that $(\phi_\alpha|\,\hat{\mathcal{T}}_B\,|\phi_\beta) = \langle\psi_\alpha|\,\hat{T}\,|\psi_\beta\rangle$, $(\phi_\alpha|\,\hat{\mathcal{V}}_B\,|\phi_\beta) = \langle\psi_\alpha|\,\hat{V}\,|\psi_\beta\rangle$. and $(\phi_\alpha|\,\hat{\mathcal{N}}_B\,|\phi_\beta) = \langle\psi_\alpha\,|\psi_\beta\rangle$. We term $\hat{\mathcal{T}}_B, \hat{\mathcal{V}}_B$ the boson *representations* of the fermion operators $\hat{T}$, $\hat{V}$. The details of the construction is given in Reference 14, and one finds the 'linked-cluster' (a la Kishimoto and Tamura[18] although with differences) expansion of the representations to be of the form

$$\hat{\mathcal{N}}_B = 1 + \sum_{\ell=2}^{\infty} \sum_{\{\sigma,\tau\}} w_\ell^0(\sigma_1,\ldots,\sigma_l;\tau_1,\ldots,\tau_l) \prod_{i=1}^{\ell} b_{\sigma_i}^\dagger \prod_{j=1}^{\ell} b_{\tau_j}. \tag{5}$$

and similarly for $\hat{\mathcal{V}}_B, \hat{\mathcal{T}}_B$. In the norm operator the ℓ-body terms embody the fact that the fermion-pair operators do not have exactly bosonic commutation relations, and act to enforce the Pauli principle.

The norm operator [19] can be conveniently and compactly expressed in terms of the kth order Casimir operators of the unitary group SU(2Ω), $\hat{C}_k = 2^k \,\mathrm{tr}\,(\mathbf{P})^k$, $\mathbf{P} = \sum_{\sigma\tau} b_\sigma^\dagger b_\tau \mathbf{A}_\sigma \mathbf{A}_\tau^\dagger$ (and so is both a matrix and a boson operator; the trace is over the matrix indices and not the boson Fock space)

$$\hat{\mathcal{N}}_B = :\exp\left(-\frac{1}{2}\sum_{k=2}^{\infty}\frac{(-1)^k}{k}\hat{C}_k\right): \tag{6}$$

where the colons ':' refer to normal-ordering of the boson operators. Similarly — and this is a new result we have not seen elsewhere in the literature — the representations $\hat{\mathcal{T}}_B, \hat{\mathcal{V}}_B$ can also be written in compact form[20,14]:

$$\hat{\mathcal{T}}_B = 2\sum_{\sigma,\tau} :\mathrm{tr}\left[\mathbf{A}_\sigma \mathbf{T} \mathbf{A}_\tau^\dagger \mathbf{G}\right] b_\sigma^\dagger b_\tau \hat{\mathcal{N}}_B:, \tag{7}$$

$$\begin{aligned}\hat{\mathcal{V}}_B = \sum_{\mu,\nu} \langle\mu|V|\nu\rangle \sum_{\sigma,\tau} :\Big\{&\mathrm{tr}\left[\mathbf{A}_\sigma \mathbf{A}_\mu^\dagger \mathbf{G}\right] \mathrm{tr}\left[\mathbf{A}_\nu \mathbf{A}_\tau^\dagger \mathbf{G}\right] \\ &+4\,\mathrm{tr}\left[\mathbf{A}_\sigma \mathbf{A}_\mu^\dagger \mathbf{P} \mathbf{G} \mathbf{A}_\mu \mathbf{A}_\tau^\dagger \mathbf{G}\right]\Big\} b_\sigma^\dagger b_\tau \hat{\mathcal{N}}_B:,\end{aligned} \tag{8}$$

where $\mathbf{G} = (\mathbf{1} + 2\mathbf{P})^{-1}$. These representations have the powerful property of exactly expressing the fermion matrix elements under *any* truncation, a fact not previously appreciated in the literature[19]. Their compact form also makes them useful for formal manipulation[14,20].

With the boson representations of fermion operators in hand one can express the fermion Schrödinger equation as a generalized boson eigenvalue equation,

$$\hat{\mathcal{H}}_B\,|\Phi_\lambda) = E_\lambda \hat{\mathcal{N}}_B\,|\Phi_\lambda)\,. \tag{9}$$

Here $\hat{\mathcal{H}}_B$ is the boson representation of the fermion Hamiltonian Every physical fermion eigenstate in (1) has a corresponding eigenstate, with the same eigenvalue, in (9). Because the space of states constructed from pairs of fermions is overcomplete, there also exist spurious boson states that do not correspond to unique physical fermion states. The overcompleteness also means that (9) is harder to solve exactly than (1). So one truncates the model space. As mentioned in the previous section such truncations can seriously distort the spectrum, and so one resorts to *effective representations.*

3a. Effective representations, part one

We follow the usual Feshbach derivation and partition the *boson* Fock space using P to project out the allowed space and Q its compliment, with $P + Q = 1$, $P^2 = P$, $Q^2 = Q$ and $PQ = QP = 0$. Then the truncated representations are simply

$$[\mathcal{H}_B]_T = P\mathcal{H}_B P, \tag{10}$$

$$[\mathcal{N}_B]_T = P\mathcal{N}_B P, \tag{11}$$

and $|\Phi)_T = P|\Phi)$. Then the generalized eigenvalue equation in the full space (9) becomes

$$\left[\hat{\mathcal{H}}_B\right]_T^{\text{eff}}(E_\lambda)\,|\Phi_\lambda)_T = E_\lambda \left[\hat{\mathcal{N}}_B\right]_T^{\text{eff}}(E_\lambda)\,|\Phi_\lambda)_T\,, \tag{12}$$

with

$$\begin{aligned}\left[\hat{\mathcal{H}}_B\right]_T^{\text{eff}}(E) &= P\mathcal{H}_B P + P\mathcal{H}_B Q\frac{1}{Q(\mathcal{H}_B - E\mathcal{N}_B)Q}Q\mathcal{H}_B P \\ -P\mathcal{H}_B Q\frac{E}{Q(\mathcal{H}_B - E\mathcal{N}_B)Q}Q\mathcal{N}_B P &- P\mathcal{N}_B Q\frac{E}{Q(\mathcal{H}_B - E\mathcal{N}_B)Q}Q\mathcal{H}_B P \\ &+P\mathcal{H}_B Q\frac{E^2}{Q(\mathcal{H}_B - E\mathcal{N}_B)Q}Q\mathcal{H}_B P + E\mathcal{A}(E), \qquad (13) \\ \left[\hat{\mathcal{N}}_B\right]_T^{\text{eff}}(E) &= P\mathcal{N}_B P + \mathcal{A}(E).\end{aligned}$$

There is some ambiguity in definition of these effective representations as embodied by the operator $\mathcal{A}(E)$. One can also in principle construct energy-independent, but non-Hermitian, effective representations[21]. In subsection 4c however we'll argue that $\mathcal{A}(E)$ might be useful in this regard.

4. Boson images

In general the boson representations given in (6), (7) and (8) do not have good convergence properties, so that simple termination of the series such as (5) in ℓ-body terms is impossible and use of the generalized eigenvalue equation (9), as written, is problematic. Instead we "divide out" the norm operator to obtain the *boson image*:

$$h_B \sim \text{"}\mathcal{H}_B/\mathcal{N}_B.\text{"} \qquad (14)$$

That this is reasonable is suggested by the explicit forms of (7) and (8). The hope of course is that h_B is finite or nearly so, so that a 1+2-body fermion Hamiltonian is mapped to an image

$$\bar{h}_B \sim \theta_1 b^\dagger b + \theta_2 b^\dagger b^\dagger bb + \theta_3 b^\dagger b^\dagger b^\dagger bbb + \theta_4 b^\dagger b^\dagger b^\dagger b^\dagger bbbb + \ldots \qquad (15)$$

with the ℓ-body terms, $\ell > 2$, zero or greatly suppressed.

4a. Exact results

It turns out that for a number of cases the image of the Hamiltonian is exactly finite. In particular, for the full boson Fock space the representations factor in a simple way: $\hat{\mathcal{T}}_B = \hat{\mathcal{N}}_B \hat{T}_B = \hat{T}_B \hat{\mathcal{N}}_B$ and $\hat{\mathcal{V}}_B = \hat{\mathcal{N}}_B \hat{V}_B = \hat{V}_B \hat{\mathcal{N}}_B$, where the factored operators $\hat{T}_B$, $\hat{V}_B$, which we term the boson images of $\hat{T}$, $\hat{V}$, have simple forms:

$$\hat{T}_B = 2 \sum_{\sigma\tau} \mathrm{tr}\left(\mathbf{A}_\sigma \mathbf{T} \mathbf{A}_\tau^\dagger\right) b_\sigma^\dagger b_\tau, \qquad (16)$$

$$\hat{V}_B = \sum_{\mu\nu} \langle \mu | V | \nu \rangle \left[b_\mu^\dagger b_\nu + 2 \sum_{\sigma\sigma'} \sum_{\tau\tau'} \mathrm{tr}\left(\mathbf{A}_\sigma \mathbf{A}_\mu^\dagger \mathbf{A}_{\sigma'} \mathbf{A}_\tau^\dagger \mathbf{A}_\nu \mathbf{A}_{\tau'}^\dagger\right) b_\sigma^\dagger b_{\sigma'}^\dagger b_\tau b_{\tau'} \right] \qquad (17)$$

In general one can find a image Hamiltonian $\hat{H}_B = \hat{T}_B + \hat{V}_B$. This result, and its relation to other mappings such as the non-Hermitian Dyson mapping, is found in Marshalek[22].

Thus any boson representation of a Hamiltonian factorizes: $\hat{\mathcal{H}}_B = \hat{\mathcal{N}}_B \hat{H}_B$. Since the norm operator is a function of the SU(2Ω) Casimir operators it commutes with the boson images of fermion operators[14], and one can simultaneously diagonalize both $\hat{\mathcal{H}}_B$ and $\hat{\mathcal{N}}_B$. Then Eqn. (9) becomes

$$\hat{H}_B \left|\Phi_\lambda\right) = E'_\lambda \left|\Phi_\lambda\right). \qquad (18)$$

where $E'_\lambda = E_\lambda$ for the physical states, but E'_λ for the spurious states is no longer necessarily zero. The boson Hamiltonian $\hat{H}_B$ is by construction Hermitian and, if one starts with at most only two-body interactions between fermions, has at most two-body boson interactions. All physical eigenstates of the original fermion Hamiltonian will have counterparts in (18). It should be clear that transition amplitudes between physical eigenstates will be preserved. Spurious states will also exist but, since the norm operator $\hat{\mathcal{N}}_B$ commutes with the boson image Hamiltonian $\hat{H}_B$, the physical eigenstates and the spurious states will not admix.

The boson Schrödinger equation (18), though finite, is not much use as the boson Fock space is still much larger than the original fermion Fock space, and we still must truncate the boson Fock space. Although the representations remain exact under truncation, the factorization into the image does not persist in general: $\left[\hat{\mathcal{H}}_B\right]_T \neq \left[\hat{\mathcal{N}}_B\right]_T \left[\hat{H}_B\right]_T$. This was recognized by Marshalek[22]. (An alternate formulation[22] does not require the complete Fock space, but mixes physical and spurious states and so always requires a projection operator.)

We can however find sufficient conditions such that a factorization

$$\left[\hat{\mathcal{H}}_B\right]_T = \left[\hat{\mathcal{N}}_B\right]_T H_D \tag{19}$$

does exist, in particular, if the subset $\{\bar{\alpha}\}$ of fermion-pair creation and annihilation operators form a closed subalgebra, that is,

$$\left[\hat{A}_{\bar{\alpha}}, \left[\hat{A}_{\bar{\beta}}, \hat{A}^\dagger_{\bar{\gamma}}\right]\right] = \sum_{\bar{\delta}} C^{\bar{\delta}}_{\bar{\alpha}\bar{\beta}\bar{\gamma}} \hat{A}_{\bar{\delta}}. \tag{20}$$

Favorite examples are the SO(8) and Sp(6) models[23] which specify how to choose the pair operators. In general H_D, though finite, is not Hermitian; we then call it a *Dyson* image[9].

That one can find a Dyson image for a closed subalgebra is no surprise. A more intriguing result is that under certain conditions given elsewhere [14,20] even in the truncated space the Dyson image of some interactions is in fact Hermitian. For example, the $Q \cdot Q$ and other multipole-multipole interactions in the SO(8) and Sp(6) models have Hermitian Dyson images. Other interactions do not have Hermitian Dyson images, such as pairing in any model and the SO(7) interaction in the SO(8) model. It so happens that these particular cases nonetheless can be brought into finite, Hermitian form as discussed in the next section.

4b. Approximate or numerical images

The most general image Hamiltonian one can define is

$$h_B \equiv U \left[\tilde{\mathcal{N}}_B\right]_T^{-1/2} \left[\mathcal{H}_B\right]_T \left[\tilde{\mathcal{N}}_B\right]_T^{-1/2} U^\dagger, \tag{21}$$

which is manifestly Hermitian for any truncation scheme and any interaction, with U a unitary operator. (Because the norm is a singular operator it cannot be inverted. Instead $\left[\tilde{\mathcal{N}}_B\right]_T^{-1/2}$ is calculated from the norm only in the physical subspace, with the zero eigenvalues which annihilate the spurious states retained. Then h_B does not mix physical and spurious states.) This Hermitian image h_B is related to the Dyson image H_D by a similarity transformation $\mathcal{S} = U\left[\mathcal{N}_B\right]_T^{1/2}$,

$$h_B = \mathcal{S} H_D \mathcal{S}^{-1}. \tag{22}$$

How can we understand this similarity transformation? The original fermion states $|\Psi_{\tilde{\alpha}}\rangle$ are not orthogonal as reflected by the norm operator. The similarity transformation $\mathcal{S}$ orthogonalizes this states inasmuch $\mathcal{S}\mathcal{N}_B\mathcal{S}^{-1} = 1$ in the physical space and $= 0$ in the spurious space. This is akin to Gram-Schmidt orthogonalization and the freedom to choose U, and $\mathcal{S}$, corresponds to the freedom one has in ordering the vectors to be Gram-Schmidt orthgonalized. For example, OAI[6] orthogonalized their basis in effect by choosing $\mathcal{S}$ such that the seniority-zero state is mapped to itself,

$$\mathcal{S}(s^\dagger)^n\,|0) \rightarrow (s^\dagger)^n\,|0)\,, \tag{23}$$

and in general if $|\nu)$ is some state of seniority ν, then

$$\mathcal{S}\,|\nu) \rightarrow |\nu) + |\nu-2) + |\nu-4) + \ldots \tag{24}$$

While h_B is manifestly Hermitian it is not necessarily finite, as sketched in (15). Our success in finding finite images in the previous section gives us hope that the high-order many-body terms may be small. Furthermore we can use the freedom in the choice of U to our advantage. Consider the SO(8) model[23] and its three algebraic limits: the pure pairing interaction, the $Q\cdot Q$ interaction, which can be written in terms of SO(6) Casimir operators, and the linear combination of pairing and $Q\cdot Q$ which can be written in terms of SO(7) Casimirs. The Dyson image of $Q\cdot Q$ is Hermitian and $H_D = h_B$ with $U = 1$. The Dyson images of the pairing and SO(7) interactions are finite but non-Hermitian. We have found U's $\neq 1$ for both these cases (but not the same U) such that their respective Hermitian images h_B are finite; the one for pairing is exactly the OAI prescription.

When OAI did their boson mapping they found a many-body dependence in the coefficients for their boson interaction. It is possible that this many-body dependence was induced at least in part by their orthogonalization scheme, that is effectively their choice of U. We can illustrate this in the SO(8) model with the SO(7) interaction, whose spectrum is exactly known and for which we can find a U such that its image is finite with no many-body dependence; this is the left-hand spectrum in Figure 1.

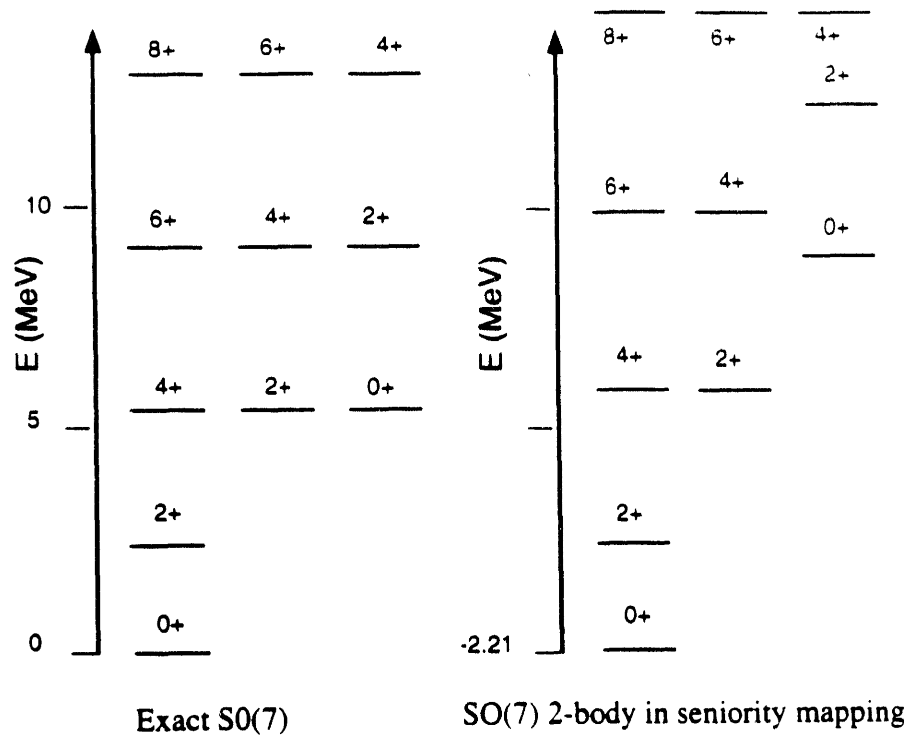

Fig. 1: Spectrum of SO(7) interaction in SO(8) model with exact (left) and approximate (right) two-body boson Hamiltonians.

For the right-hand side we took the U appropriate for pairing, that is the OAI prescription, and calculated the spectrum keeping only the strict two-body terms. The distortion in the spectrum from the exact result, such as the overall energy shift and the large perturbation in the third band, indicates missing many-body terms. That is, if one mapped the SO(7) interaction using the canonical OAI procedure one would find of necessity a many-body dependence in the interaction coefficients. By orthogonalizing the basis in the a different way, however, as expressed by a different choice of U, the many-body dependent vanishes. We are currently exploring how to exploit this freedom to best effect.

4c. Effective representations, part two

Now that we have discussed boson images we observe that effective operator theory is more efficient when applied to representations rather than images, by which we mean that the corrections are smaller. Suppose one started with the image Hamiltonian in the full space, H_B as defined in section 4a, and from that constructed an effective image in the usual way,

$$H_B^{\text{eff}} = PH_BP + PH_BQ\frac{1}{Q(E-H_B)Q}QH_BP = [H_B]_T + \Delta H_B(E). \qquad (25)$$

Now compare that with the effective image constructed from effective representations,

$$h_B^{\text{eff}}(E) \equiv \left(\left[\tilde{\mathcal{N}}_B\right]_T^{\text{eff}}(E)\right)^{-1/2} [\mathcal{H}_B]_T^{\text{eff}} \left(\left[\tilde{\mathcal{N}}_B\right]_T^{\text{eff}}(E)\right)^{-1/2} = h_B + \Delta h_B(E) \qquad (26)$$

(leaving aside the issue of the choice of an overall unitary transformation U). Now $h_B \neq [H_B]_T$, and consider those cases, such as SO(8) and Sp(6), in which the P-space decouples completely from the Q space. In those cases $\Delta h_B = 0$ but ΔH_B cannot be zero. Hence in general, we believe, the corrections from using effective representations Δh_B will be smaller than those from directly performing effective operator theory on the image, ΔH_B.

We noted previously that there is some ambiguity in the definition of the effective representations as denoted in (13) by $\mathcal{A}(E)$. We now would like to speculate on its possible use. In effective operator theory the eigenstates are no longer orthogonal because of the truncation of the model space; this is embodied by the fact that one uses either energy-dependent or non-Hermitian effective interactions. We propose that this non-orthogonality could also be embedded in the choice of $\mathcal{A}(E)$, so that the similarity transform on the basis is now $U\sqrt{\mathcal{N}_B + \mathcal{A}(E)}\,|\Phi_{\tilde{\alpha}})$. The ambiguity operator $\mathcal{A}(E)$ could be chosen so as to minimize the energy dependence of the final boson image. Although the similarity transformation is now energy-dependent, this would not show up in the calculation of the spectrum, but only in the calculation of effective transition operators. These speculations need to be explored in greater detail.

5. Summary

In order to investigate rigorous foundations for the phenomenological Interacting Boson Model, we have presented a rigorous microscopic mapping of fermion pairs to bosons that can preserve Hermicity. We have also briefly discussed the application of effective operator theory to account for that part of the model space excluded through truncation. In several analytic cases the resulting boson image Hamiltonian is finite. In the most general case the image Hamiltonian may not be finite but we have demonstrated there is some freedom in the mapping that one can possibly exploit to minimize the many-body terms. Several open questions remain: What in general is the best way to find U? If the Dyson image is finite though non-Hermitian, does there always exist a U such that the Hermitian image h_B is also finite? Can one exploit ambiguity in the effective representations to minimize the energy dependence?

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