



**INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS**

**ANALYTIC FORMULAE
FOR THE HARTREE-FOCK ORDER PARAMETER
AT ARBITRARY p/q FILLING FACTORS
FOR THE 2DEG IN A MAGNETIC FIELD**

Alejandro Cabo Montes de Oca

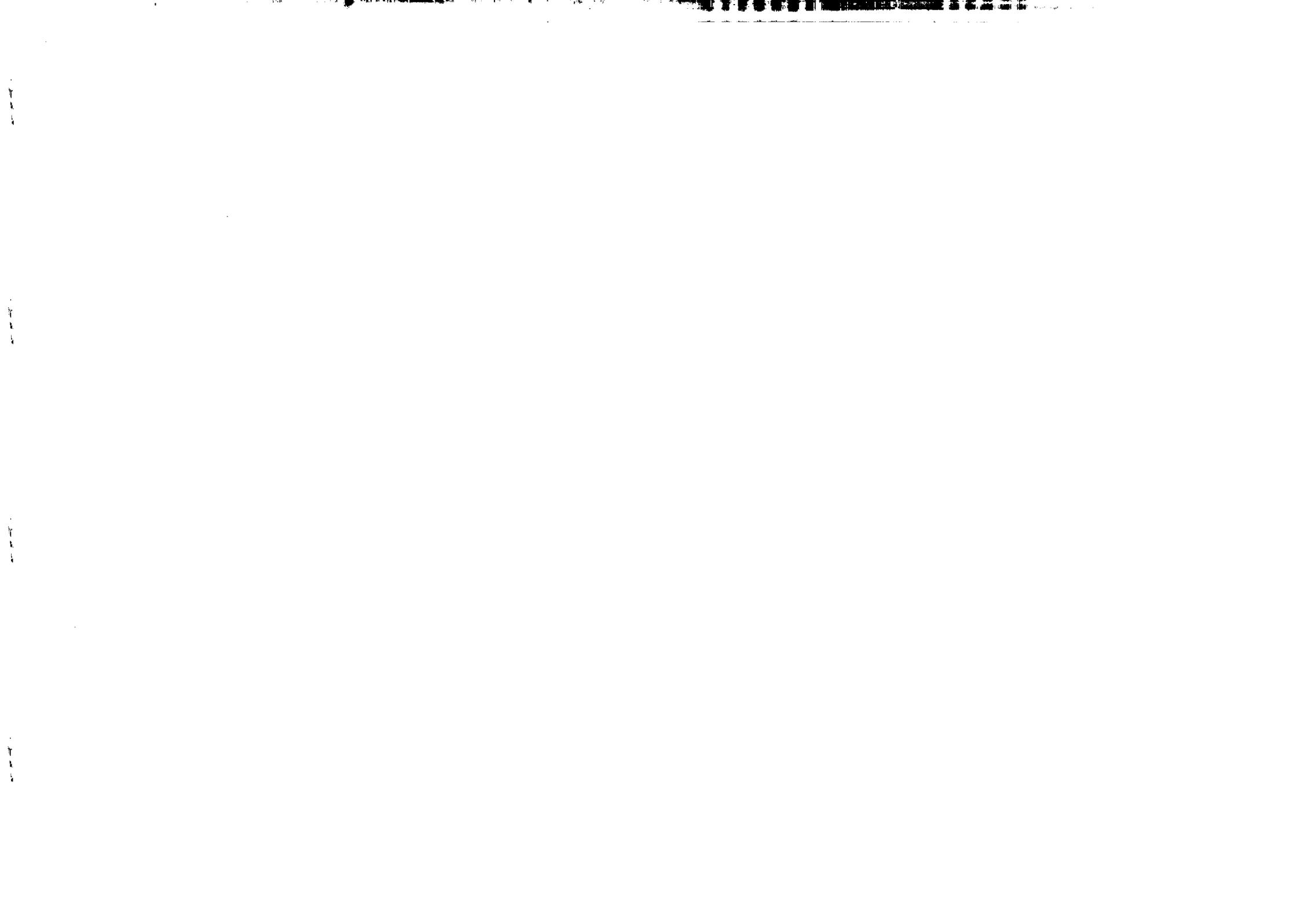


**INTERNATIONAL
ATOMIC ENERGY
AGENCY**



**UNITED NATIONS
EDUCATIONAL,
SCIENTIFIC
AND CULTURAL
ORGANIZATION**

MIRAMARE-TRIESTE



International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

**ANALYTIC FORMULAE FOR THE HARTREE-FOCK ORDER
PARAMETER AT ARBITRARY p/q FILLING FACTORS
FOR THE 2DEG IN A MAGNETIC FIELD**

Alejandro Cabo Montes de Oca ¹
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

Analytic expressions for order parameters are given for the previously introduced general class of Hartree Fock states at arbitrary filling factors $\nu = p/q$ for odd q values. The order parameters are expressed as sums of magnetic translations eigenvalues over the filled single electron states. Simple summation formulae for the band spectra in terms of the same eigenvalues are also presented. The energy per particle at $\nu = 1/3$ is calculated for various states differing in the way of filling of the $1/3$ of the orbitals. The calculated energies are not competing with the usual CDW results. However the high degree of electron overlapping allows for the next corrections to modify this situation. The discussion suggests these Hartree-Fock Slater determinants as interesting alternatives for the Tao-Thouless parent states which may correct their anomalous symmetry and correlation functions properties.

MIRAMARE - TRIESTE

July 1994

¹Permanent address: Group of Theoretical Physics, Instituto de Cibernética Matemática y Física, Calle, No. 309, esq.15, Vedado, La Habana, Cuba.

1 Introduction

In a previous paper a class of exact solutions of the Hartree-Fock problem associated to the 2DEG in a magnetic field was presented [1]. In this work further results related with these explicit solutions are given. In particular simple analytical expressions for the order parameters are obtained for any of those states at arbitrary filling factors $\nu = p/q$ (for odd q). These relations for the order parameter show them as simple summations of eigenvalues of the magnetic translations operators. The results would further simplify an analysis of quantum field theories (QFT) which can be constructed with those solutions as perturbative ground states. Those theories seem to be worth considering because it looks feasible that they can describe physically realizable Hall crystal states [2,3] or cooperative rings of exchange phases [4,5]. Up to now difficulties in checking these possibilities seem to be connected with the lack of detailed information about the almost approximately and numerically known Hartree-Fock solutions [6-16]. There are few indications in the above sense. Firstly, the solutions given in [1] show the q fold degeneracy which should be expected in the FQHE ground state [17]. Additionally, these perturbative ground states would lack the rotational symmetry and correlation functions difficulties of the interesting Tao-Thouless parent states [18-22]. Moreover, they show an enhanced degree of electron overlapping in comparison with the one existing in the lowest energy CDW states. Therefore, the second order (or higher) corrections to the Hartree-Fock result, which in the lowest energy CDW state are negligible [9], could alter the Hartree-Fock result for the energy per particle appreciably.

The detailed information on the Hartree-Fock states given here and in [1] should be useful in simplifying higher order perturbative corrections. The possibility that crystal like states could be relevant for the description of the FQHE ground state was also supported by the results of Chui and collaborators [23-25]. In ref. [25] crystal like states lowering the energy per particle of the Laughlin wave function were investigated and in ref. [23] signals of crystal like correlations in few particle calculations were discussed.

A point which attracts the attention in the Tao-Thouless approach is the energy per particle evaluation [20]. As in ref. [25], the result was lower in value than the corresponding one in the Laughlin state. The here presented wavefunctions, as considered for the interaction free ground states of a modified Tao-Thouless treatment, seem to offer an opportunity to repair the above mentioned shortcomings of that approach: the high rotational asymmetry and the unphysical density-density correlation functions [21,22]. The comments above can rise the question about the possibility for the construction of a consistent description of the FQHE. Further calculations of the higher contributions to the energy per particle will be considered elsewhere.

The paper is organized as follows. In section 2 a basis of functions diagonalizing the Hartree-Fock hamiltonian is introduced. The quantum numbers of the basis wavefunctions in it are given in a slightly modified way as it was done in [1]. Concretely, the single particle states are required to be eigenfunctions of the magnetic translations shifts in a fraction q of the diagonal of the unit cell which is associated with the vectors \vec{a}_1 and \vec{a}_2 . This procedure allows for a discussion being symmetric with respect to the orientation of these vectors. We must stress here that a set of basis functions constructed in the same spirit than the one in [1] was independently discussed by Ferrari [26,27]. Unfortunately, we had no information on that work at the time of sending our paper for publication. The focus of ref. [1] however, was centered on the different question of the construction of a wide

class of exact Hartree-Fock solutions.

Section 3 is devoted to the evaluation of the explicit formulae for the order parameter (or equivalently the Fourier coefficients of the charge density) at any filling factor value $\nu = p/q$ for an odd q value. The order parameter appears expressed as a summation over the eigenvalues of the magnetic translations associated to the filled single electron states. Similar relations are expected to hold for the more conventional CDW states. The discussion applies for arbitrary form of the lattice structure whenever the flux per unit cell is a flux quantum. An interesting property of this result is that, at variance with the case of the CDW which show gaussian like behaviour, the order parameter can have an inverse power law asymptotic at high momenta. The sum rules for the order parameter [9] are shown to be satisfied by the explicit expressions in the general case.

In Section 4 the summation formulae for the selfenergies are calculated. They are expressed as sums of the same sort as the ones appearing in the usual relations for the energy per particle in the Hartree-Fock approximation [6]. The spectrum, in a special way of filling, consists of exactly q energy branches, each one of them being fourfold degenerate due to the special way of construction of the magnetic Bloch states [1].

Section 5 considers in more detail the important $q=3$ particular case at the $\nu = 1/3$ value of the filling factor. The energy per particle in the Hartree-Fock approximation is calculated for various states differing in the way of filling of $1/3$ of the electron orbits. The calculated value clearly depends on the ways of filling which are infinite in number. The results for any of the cases calculated here are not better than the one corresponding to lowest energy CDW states which is $-0.388 e^2/r_0$. However, as it was also stressed before, due to the enhanced electron overlapping, the next perturbative corrections should be evaluated in order to have a good approximation for the energy per particle which is a relevant parameter for a FQHE ground state candidate. In the lowest energy CDW solutions these corrections were estimated in [9] to be insufficient for lowering the energy down to the $-0.41 e^2/r_0$ result in the Laughlin state.

Finally, in the summary the results are commented again and perspectives for the further continuation of the work are advanced.

2 Diagonalizing basis

The purpose of this section is to present a slight modification for the parametrization of the elements of the set of functions diagonalizing the Hartree-Fock hamiltonian given in [1]. In the final step of the construction in that reference, it was imposed on the elements of the basis the condition of to be eigenfunctions of a magnetic translation in a fraction q of one of the two unit cell vectors introduced, specifically \vec{a}_1 . This is a somewhat asymmetrical parametrization with respect to the directions of the two unit cell vectors. It was selected in [1] as a matter of simplicity. Here we impose a more symmetrical condition which is implemented technically in a very similar way. Concretely, the functions are required to be eigenvectors of the magnetic translations but now in a fraction q of the vector $\vec{a}_1 + \vec{a}_2$ which is the diagonal of the unit cell.

In the same way as in [1] the diagonalizing functions become

$$\chi_{\vec{k}}^{(\sigma, \nu, r)}(\vec{x}) = \sum_{s=1}^q \sum_{\{\vec{R}^\alpha\}} \sum_{\alpha=0}^{s-1} C(s, \sigma, \nu, \vec{k}) \sigma^{\delta_1^\alpha} \nu^{\delta_2^\alpha} (-1)^{\delta_1^\alpha \delta_2^\alpha} \exp\{i\vec{k} \cdot \vec{R}^\alpha\} T_{\vec{r}_s} T_{\vec{R}^\alpha} \phi(\vec{x}) \quad (1)$$

where

$$C(s, \sigma, \nu, \vec{k}) = \frac{(-\sigma\nu)^{s-1}}{2\sqrt{q}} \exp\left[\frac{i(s-1)\vec{k} \cdot \vec{a}_3}{q} - \frac{2\pi i(r-1)(s-1)}{q}\right] \quad (2)$$

in which $\sigma, \nu = \pm 1, r, s = 1, 2, \dots, q$ and all the definitions and nomenclature are the same as in [1].

The functions (1) are now invariant up to a phase under any translation on a vector of the lattice generated by the unit cell vectors \vec{a}_1 (or \vec{a}_2) and $\vec{a}_1 + \vec{a}_2$. Then, it becomes possible to introduce new unit cell vectors on that lattice in the following symmetrical way

$$\vec{b}_1 = \frac{(1+q)}{2q} \vec{a}_1 + \frac{(1-q)}{2q} \vec{a}_2, \quad (3)$$

$$\vec{b}_2 = \frac{(1-q)}{2q} \vec{a}_1 + \frac{(1+q)}{2q} \vec{a}_2. \quad (4)$$

Inversely, it follows

$$\vec{a}_1 = \frac{(1+q)}{2} \vec{b}_1 + \frac{(q-1)}{2} \vec{b}_2, \quad (5)$$

$$\vec{a}_2 = \frac{(q-1)}{2} \vec{b}_1 + \frac{(q+1)}{2} \vec{b}_2. \quad (6)$$

The above definitions allow for the vectors of the lattice generated by \vec{b}_1 and \vec{b}_2 to be written in the alternative ways

$$\begin{aligned} \vec{l} &= l_1 \vec{b}_1 + l_2 \vec{b}_2, \quad l_1, l_2 = -\infty \dots \infty, \\ &= \vec{R}^\alpha + \vec{l}_s, \quad \vec{l}_s = \frac{(s-1)}{q} (\vec{a}_1 + \vec{a}_2). \end{aligned} \quad (7)$$

It should be remarked again that we will use systematically the symbols and lattice variables already given in the complementary paper [1]. In (7), for example, \vec{R}^α is defined in (3) of reference [1].

From (7), it is possible to obtain for l_1 and l_2

$$l_1 = \frac{q}{2} [2(n_1 + n_2) + \delta_1^\alpha + \delta_2^\alpha + 2(s-1)/q] + \frac{1}{2} [2(n_1 - n_2) + \delta_1^\alpha - \delta_2^\alpha], \quad (8)$$

$$l_2 = \frac{q}{2} [2(n_1 + n_2) + \delta_1^\alpha + \delta_2^\alpha + 2(s-1)/q] - \frac{1}{2} [2(n_1 - n_2) + \delta_1^\alpha - \delta_2^\alpha], \quad (9)$$

which express the change of variables between the set of parameters (n_1, n_2, α, s) and the new components (l_1, l_2) .

The inverse mapping can be written as

$$n_1 = \text{Int}[\tau_1/2], \quad (10)$$

$$n_2 = \text{Int}[\tau_2/2], \quad (11)$$

$$\delta_1^\alpha = \text{Int}[\tau_1 - 2\text{Int}[\tau_1/2]], \quad (12)$$

$$\delta_2^\alpha = \text{Int}[\tau_2 - 2\text{Int}[\tau_2/2]], \quad (13)$$

$$\frac{(s-1)}{q} = \tau_1 - 2\text{Int}[\tau_1/2] - \text{Int}[\tau_1 - 2\text{Int}[\tau_1/2]] \quad (14)$$

$$= \tau_2 - 2\text{Int}[\tau_2/2] - \text{Int}[\tau_2 - 2\text{Int}[\tau_2/2]], \quad (15)$$

where

$$\tau_1 = \frac{1}{2q} [(q+1)l_1 + (1-q)l_2], \quad (16)$$

$$\tau_2 = \frac{1}{2q} [(1-q)l_1 + (q+1)l_2], \quad (17)$$

and $\text{Int}[A]$ is the integer part of A .

By means of the new set of variables (l_1, l_2) the functions (1) can be represented in the alternative way

$$\chi_{\vec{k}}^{(\sigma, \nu, r)}(\vec{x}) = \sum_{\{\vec{l}\}} \exp[i\vec{k} \cdot \vec{l}] T_l \phi(\vec{x}) \frac{1}{2\sqrt{q}} \sigma^{\delta_1^r} \nu^{\delta_2^r} (-\sigma\nu)^{(s-1)} (-1)^{\delta_1^r \delta_2^r} \exp\left[\frac{ie}{\hbar c} \vec{A}(\vec{r}_s) \cdot \vec{P}^{\sigma} - \frac{2\pi}{q} (r-1)(s-1)i\right], \quad (18)$$

where $\{\vec{l}\}$ represents the set of lattice vectors \vec{l} .

Expression (18) can be fully expressed in terms of the new variables (l_1, l_2) by using the following identities

$$(-1)^{(s-1)} (-1)^{\delta_1^r \delta_2^r} \exp\left[\frac{ie}{\hbar c} \vec{A}(\vec{l}_s) \cdot \vec{P}^{\sigma}\right] = (-1)^{l_1 l_2}, \quad (19)$$

$$\sigma^{\delta_1^r} \nu^{\delta_2^r} (\sigma\nu)^{(s-1)} \exp[i\vec{k} \cdot \vec{l} - 2\pi i(r-1)(s-1)/q] = \exp[i\vec{P}_{\vec{k}}^{(\sigma, \nu, r)} \cdot \vec{l}], \quad (20)$$

in which the vector \vec{P} is defined by

$$\vec{P}_{\vec{k}}^{(\sigma, \nu, r)} = \vec{k} + \pi [-(\sigma-1) - (q-1)(\nu-1) + 2(q-1)(r-1)] \vec{\tau}_1 + \pi [-(\nu-1) - (q-1)(\sigma-1) + 2(q-1)(r-1)] \vec{\tau}_2. \quad (21)$$

The unit cell vectors in the reciprocal lattice $\vec{\tau}_i$ are given in [1] (where they are called \vec{b}_i as satisfying

$$\vec{\tau}_i \cdot \vec{a}_j = \frac{1}{2} \delta_{ij} \quad i, j = 1, 2.$$

After substituting (19) and (20) in (18) the basis functions can be written as summations over the lattice variables \vec{l} in the following simplified way

$$\chi_{\vec{k}}^{(\sigma, \nu, r)}(\vec{x}) = \sum_{\{\vec{l}\}} \frac{1}{2\sqrt{q}} (-1)^{l_1 l_2} \exp[i\vec{P}_{\vec{k}}^{(\sigma, \nu, r)} \cdot \vec{l}] T_l \phi(\vec{x}), \quad (22)$$

for all the values of the indices

$$\sigma, \nu = \pm 1, \quad r = 1, 2, \dots, q,$$

$$\vec{k} = \frac{4\pi}{N} (q_1 \vec{\tau}_1 + q_2 \vec{\tau}_2),$$

$$-\frac{N}{4} < q_i \leq \frac{N}{4}, \quad i = 1, 2.$$

The functions χ satisfy the eigenvalue equations

$$T_{\vec{l}} \chi_{\vec{k}}^{(\sigma, \nu, r)}(\vec{x}) = \lambda_{\vec{k}}^{(\sigma, \nu, r)}(\vec{l}) \chi_{\vec{k}}^{(\sigma, \nu, r)}(\vec{x}), \quad (23)$$

$$\lambda_{\vec{k}}^{(\sigma, \nu, r)}(\vec{l}) = (-1)^{l_1 l_2} \sigma^{1/2(q+1)l_1 + (q-1)l_2} \nu^{1/2[(q+1)l_2 + (q-1)l_1]} \exp\left[-i\vec{k} \cdot \vec{l} + \frac{(1-q)\pi(r-1)(l_1 + l_2)i}{q}\right]. \quad (24)$$

The norm of the states χ in the large area V_N with sides $N\vec{a}_1$ and $N\vec{a}_2$, can be calculated to be

$$\begin{aligned} \mathcal{N}_{\vec{k}}^{(\sigma, \nu, r)} &= \int_{V_N} d\vec{x} [\chi_{\vec{k}}^{(\sigma, \nu, r)}(\vec{x})]^* \chi_{\vec{k}}^{(\sigma, \nu, r)}(\vec{x}) \\ &= \frac{N^2}{4} \sum_{\{\vec{l}\}} (-1)^{l_1 l_2} \exp\left[-\frac{1}{4\tau_0^2} \vec{l}^2 + i\vec{P}_{\vec{k}}^{(\sigma, \nu, r)} \cdot \vec{l}\right]. \end{aligned} \quad (25)$$

Relation (25) shows that the norm is proportional to the wave function evaluated at the origin of coordinates. Moreover, the functions (22), as well as the norms (25), can be expressed as Theta functions of rational characteristics with an argument proportional to the vector $\vec{P}_{\vec{k}}^{(\sigma, \nu, r)}$. This fact could allow to derive additional relations being of interest for the considered problem.

The set of functions defined here can be associated to any form of the lattice whenever the unit cell defined by \vec{a}_1 and \vec{a}_2 has q flux quanta passing through it. In addition, by construction, the flux through the cell defined by \vec{b}_1 and \vec{b}_2 is equal to one flux quantum.

In what follows these functions are used for making an explicit calculation of the order parameter or equivalently the Fourier coefficients of the charge density.

Before continuing, let us define a set of orthonormalized functions as follows

$$\phi_{\vec{k}}^{(\sigma, \nu, r)}(\vec{x}) = \frac{1}{\sqrt{\mathcal{N}_{\vec{k}}^{(\sigma, \nu, r)}}} \chi_{\vec{k}}^{(\sigma, \nu, r)}(\vec{x}). \quad (26)$$

Also a compact index for the one electron states (26) can be defined introducing the vector

$$\vec{p} = p_1 \vec{\tau}_1 + p_2 \vec{\tau}_2. \quad (27)$$

where the components are expressed in terms of the previous indices in the following way

$$p_1 = \frac{4\pi q_1}{N} - \pi(\sigma-1) - \pi(q-1)(\nu-1) + 2\pi(q-1)(r-1), \quad (28)$$

$$p_2 = \frac{4\pi q_2}{N} - \pi(\nu-1) - \pi(q-1)(\sigma-1) + 2\pi(q-1)(r-1). \quad (29)$$

Then, the wavefunctions (26) can be written as follows

$$\phi_{\vec{p}}(\vec{x}) = \frac{1}{2\sqrt{qN_{\vec{p}}}} \sum_{\vec{r}} (-1)^{i_1 i_2} \exp[i\vec{p} \cdot \vec{l}] T_{\vec{r}} \phi(\vec{x}), \quad (30)$$

$$N_{\vec{p}} = \frac{N^2}{4} \sum_{\vec{r}} (-1)^{i_1 i_2} \exp[-\frac{1}{4r_0^2} \vec{l}^2 + i\vec{p} \cdot \vec{l}]. \quad (31)$$

The representation (30) makes clear that the states are invariant under a shift in \vec{p} given by the lattice vectors

$$\delta\vec{p} = 4\pi(n_1\vec{\sigma}_1 + n_2\vec{\sigma}_2), \quad (32)$$

where the unit cell vectors here satisfy

$$\vec{\sigma}_i \cdot \vec{b}_j = \frac{1}{2} \delta_{ij}, \quad i = 1, 2. \quad (33)$$

The eigenvalues (24) can also be rewritten as follows

$$\lambda^{\vec{p}}(\vec{l}) = (-1)^{i_1 i_2} \exp(-i\vec{p} \cdot \vec{l}). \quad (34)$$

3 Explicit determination of the order parameter

In this section the calculation in terms of sums of eigenvalues of the order parameter will be considered.

Since the density is periodic in a lattice generated by \vec{b}_1 and \vec{b}_2 it will have a Fourier expanded expression like

$$\rho(\vec{x}) = \sum_{\vec{Q}} \rho(\vec{Q}) \exp(i\vec{Q} \cdot \vec{x}) \quad (35)$$

where the spectrum is concentrated in the momentum values of the form

$$\vec{Q} = 4\pi(n_1\vec{\sigma}_1 + n_2\vec{\sigma}_2) \quad (36)$$

In (36) the unit cell vector of the reciprocal lattice are defined through

$$\vec{\sigma}_1 = -\frac{\vec{n} \times \vec{b}_2}{4\pi r_0^2}, \quad \vec{\sigma}_2 = \frac{\vec{n} \times \vec{b}_1}{4\pi r_0^2}, \quad (37)$$

or equivalently

$$\vec{b}_1 = -4\pi r_0^2 \vec{n} \times \vec{\sigma}_2, \quad \vec{b}_2 = 4\pi r_0^2 \vec{n} \times \vec{\sigma}_1. \quad (38)$$

The Fourier component of the density can be calculated through the inverse Fourier transform in the way

$$\begin{aligned} \rho(\vec{Q}) &= \frac{1}{V_N} \int_{V_N} d\vec{x} \rho(\vec{x}) \exp(-i\vec{Q} \cdot \vec{x}) \\ &= \frac{1}{V_N} \sum_{\{c\}} \int_{V_N} d\vec{x} [\phi^c(\vec{x})]^* \phi^c(\vec{x}) \exp(-i\vec{Q} \cdot \vec{x}), \end{aligned} \quad (39)$$

where $V_N = 2\pi r_0^2 q N^2$, as defined above, is the area of the region of N^2 cells on which periodicity boundary conditions were imposed in ref.[1], and the quantum numbers of the basis functions (26) are designed by c . The summation over c in (39) runs over all the values corresponding to the filled one electron states in the considered Hartree-Fock solution. Of special interest in some part of the discussion below will be the many body states in which all the one particle states for p specified values of r , let say $r_i, i = 1, \dots, p$, are filled for all values of σ, ν and \vec{k} . We will call by the band r_i the set of all states for arbitrary (σ, ν, \vec{k}) at fixed r_i . There are q of such bands. After substituting (26) for the functions appearing in (39), calculating the gaussian integral which appear and doing some algebraic calculations, the following formula for $\rho(\vec{Q})$ is received

$$\begin{aligned} \rho(\vec{Q}) &= \frac{\exp(-\frac{1}{2} r_0^2 \vec{Q}^2)}{V_N} \sum_{\{(\sigma, \nu, \vec{k}, r)\}_f} \\ &= \frac{\sum_{\{l\}} (-1)^{i_1 i_2} \exp\{-\frac{p}{4r_0^2} + i\vec{l} \cdot [\vec{P}_{\vec{k}}^{(\sigma, \nu, r)} - \frac{1}{2}(\vec{Q} + i\vec{n} \times \vec{Q})]\}}}{\sum_{\{l\}} (-1)^{i_1 i_2} \exp\{-\frac{p}{4r_0^2} + i\vec{l} \cdot \vec{P}_{\vec{k}}^{(\sigma, \nu, r)}\}}, \end{aligned} \quad (40)$$

where $\{(\sigma, \nu, \vec{k}, r)\}_f$ is the set of filled one electron states $\chi_{\vec{k}}^{(\sigma, \nu, r)}$.

The order parameter Δ is defined in the usual way

$$\Delta(\vec{Q}) = 2\pi r_0^2 \exp(\frac{r_0^2 \vec{Q}^2}{4}) \rho(\vec{Q}). \quad (41)$$

Relation (40) can be further transformed by expressing \vec{Q} in terms of a new variable \vec{x}^* according to

$$\begin{aligned} \vec{x}^* &= r_0^2 \vec{n} \times \vec{Q} = -n_2 \vec{b}_1 + n_1 \vec{b}_2, \\ \vec{Q} &= \frac{1}{r_0^2} \vec{x}^* \times \vec{n} = 4\pi(n_1 \vec{\sigma}_1 + n_2 \vec{\sigma}_2). \end{aligned} \quad (42)$$

In terms of \vec{x}^* the numerator in (40) can be expressed as a magnetic translation with a displacement given by \vec{x}^* of a wavefunction pertaining to the basis, being evaluated at the origin of coordinates. But, the denominator is also proportional to the same wavefunction evaluated at the origin. Moreover, since the functions of the basis are eigenfunctions of the magnetic translations in the lattice vectors \vec{l} and noticing that \vec{x}^* in (42) is one of such vectors, the following expression for Δ arises

$$\begin{aligned} \Delta(\vec{Q}) &= \frac{2\pi r_0^2}{V_N} \sum_{\{(\sigma, \nu, \vec{k}, r)\}_f} \frac{[T_{\vec{x}^*} \chi_{\vec{k}}^{(\sigma, \nu, r)}(\vec{x})]_{\vec{x}=0}}{\chi_{\vec{k}}^{(\sigma, \nu, r)}(\vec{0})} \\ &= \frac{2\pi r_0^2}{V_N} \sum_{\{(\sigma, \nu, \vec{k}, r)\}_f} \chi_{\vec{k}}^{(\sigma, \nu, r)}(\vec{x}^*) \\ &= \frac{2\pi r_0^2}{V_N} (-1)^{n_1 n_2} \sum_{\{\vec{p}\}_f} \exp[-i\vec{p} \cdot \vec{x}^*]. \end{aligned} \quad (43)$$

Expression (43) satisfies the required relation

$$\Delta(0) = \frac{1}{qN^2} \sum_{\{\vec{p}\}_f} 1 = \nu \quad (44)$$

where ν is the filling factor. The sum rule derived in [9] for any Hartree-Fock solution also follows in the way described below

$$\begin{aligned}\sum_{\vec{Q}} |\Delta(\vec{Q})|^2 &= \frac{1}{(qN^2)^2} \sum_{\{\vec{p}\}_f} \sum_{\{\vec{p}'\}_f} \sum_{\vec{Q}} \exp[-i(\vec{p}-\vec{p}') \cdot \vec{x}^2] \\ &= \frac{1}{(qN^2)^2} \sum_{\{\vec{p}\}_f} \sum_{\{\vec{p}'\}_f} \delta_{\vec{p},\vec{p}'} (qN^2) \\ &= \frac{1}{qN^2} \sum_{\{\vec{p}\}_f} 1 \\ &= \nu.\end{aligned}\quad (45)$$

4 Order parameters for special states

Let us consider in this section some particular states which differs by the way of selecting the filled electron orbitals.

a. States with p of the q bands filled.

The order parameter of the special Hartree-Fock states with p of the above defined q bands filled, can be calculated through (43).

The summation in (43) can be explicitly performed by using the identities

$$\sum_{\sigma=\pm 1} \sum_{\nu=\pm 1} \sigma^{-\frac{1}{2}(q+1)n_2 + \frac{1}{2}(q-1)n_1} \nu^{\frac{1}{2}(q+1)n_1 - \frac{1}{2}(q-1)n_2} = 4 \delta_{[n_2],0} \delta_{[n_1],0}, \quad (46)$$

$$\sum_{k=0}^n \exp[i(kx+a)] = \text{cosec}(x/2) \sin((n+1)/2) \exp[(nx/2+a)i], \quad [28], \quad (47)$$

where

$$[N] = \begin{cases} 1 & \text{for odd } N \\ 0 & \text{for even } N. \end{cases} \quad (48)$$

The final result for the order parameter is

$$\begin{aligned}\Delta(\vec{Q}) &= \frac{1}{q} \sum_{i=1}^p \exp\left[\frac{1}{q}(1-q)\pi(r_i-1)(n_1-n_2)i\right] \delta_{[n_2],0} \delta_{[n_1],0} \\ &\quad \frac{\sin\left(\frac{\pi}{4q}[-(1-q)n_2 + (1+q)n_1]\right)}{\frac{\pi}{4q}[-(1-q)n_2 + (1+q)n_1]} \cdot \frac{\sin\left(\frac{\pi}{4q}[-(1+q)n_2 + (1-q)n_1]\right)}{\frac{\pi}{4q}[-(1+q)n_2 + (1-q)n_1]},\end{aligned}\quad (49)$$

in which it must be considered that the $\sin(x)/x$ functions appearing are defined as equal to one at $x=0$.

Relation (49) shows a decaying of Δ in a power like fashion for high values of the Fourier harmonics \vec{Q} . This result is at sharp contrast with the outcome in the gaussian approximation in which the behaviour is exponentially diminishing [9].

At $\vec{Q}=0$ from (49) follows

$$\begin{aligned}\Delta(0) &= \left(\sum_{i=1}^p 1\right) \frac{1}{q} \\ &= \frac{p}{q} = \nu.\end{aligned}\quad (50)$$

That is, the value of the order parameter as required by (39) is equal to the filling factor. The filling factor is equal to p/q because there are p of the q bands filled.

Again, the sum rule obeyed by $\Delta(\vec{Q})$ for this particular state [9] can be checked to follow in a way given below

$$\begin{aligned}\sum_{\vec{Q}} |\Delta(\vec{Q})|^2 &= \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \frac{1}{q^2} \left[\frac{\sin\left(\frac{\pi}{q}[(1-q)m_2/2 + (1+q)m_1/2]\right)}{\frac{\pi}{q}[(1-q)m_2/2 + (1+q)m_1/2]} \right]^2 \\ &\quad \left[\frac{\sin\left(\frac{\pi}{q}[(1+q)m_2/2 + (1-q)m_1/2]\right)}{\frac{\pi}{q}[(1+q)m_2/2 + (1-q)m_1/2]} \right]^2 \\ &\quad \sum_{r_i, r_j} \exp\left[\frac{2\pi(1-q)(r_i-r_j)(m_1-m_2)}{q}\right].\end{aligned}\quad (51)$$

But defining new variables N_1, N_2, s according with

$$N_1 + \frac{(s-1)}{q} = \frac{(q+1)m_1}{2q} + \frac{(1-q)m_2}{2q}, \quad (52)$$

$$N_2 + \frac{(s-1)}{q} = \frac{(1-q)m_1}{2q} + \frac{(1+q)m_2}{2q}, \quad (53)$$

where N_1, N_2 are integers and s takes the values $s=1, 2, \dots, q$, the summation (51) can be written as follows

$$\begin{aligned}\sum_{\vec{Q}} |\Delta(\vec{Q})|^2 &= \sum_{N_1, N_2, s} \frac{1}{q^2} \frac{[\sin\left(\frac{(s-1)\pi}{q}\right)]^2}{\pi^2 \left[N_1 + \frac{(s-1)}{q}\right]^2} \frac{[\sin\left(\frac{(s-1)\pi}{q}\right)]^2}{\pi^2 \left[N_2 + \frac{(s-1)}{q}\right]^2} \\ &\quad \sum_{j=1}^p \sum_{i=1}^p \exp\left[\frac{4(1-q)\pi(r_i-r_j)(s-1)i}{q}\right].\end{aligned}\quad (54)$$

Now the use of the formula ([28])

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k+a)^2} = \frac{\pi^2}{[\sin(\pi a)]^2}, \quad (55)$$

leads to

$$\sum_{\vec{Q}} |\Delta(\vec{Q})|^2 = \sum_{j=1}^p \sum_{i=1}^p \sum_{s=1}^q \frac{1}{q^2} \exp\left[\frac{4\pi(r_i-r_j)(s-1)i}{q}\right]. \quad (56)$$

Since $r_i - r_j < q$ the argument $4\pi(r_i - r_j)/q$ can't be a multiple of 2π if $r_i \neq r_j$ because q is odd. Therefore, if $r_i \neq r_j$, the number $\exp[4\pi(r_i - r_j)/q]$ is one of the roots of the

identity of order q which differs of the identity. Thus, the sum over s in (56) vanish when $r_i \neq r_j$. Hence, the necessary property follows again

$$\begin{aligned} \sum_{\vec{q}} |\Delta(\vec{Q})|^2 &= \sum_{j=1}^p \sum_{i=1}^p \frac{1}{q} \delta_{ij} \\ &= \frac{p}{q} = \nu. \end{aligned} \quad (57)$$

b. $\nu = 1/3$ state for all the $|\vec{p}| \leq \sqrt{2/3}/r_0$ electron orbitals filled.

The previously considered states correspond to disconnected zones of filled orbitals in the space of indices \vec{p} . As an opposite case, let us consider here (for the special conditions $\nu = 1/3$ and an hexagonal lattice) the state obtained by filling the one electron orbits in a circular region in the \vec{p} space containing one third of the total number of states, that is \vec{p} is satisfying

$$|\vec{p}| \leq \sqrt{2/3} \frac{1}{r_0}. \quad (58)$$

The spherical symmetry of the selected region simplifies the calculation of the order parameter through the formula (43) in the continuum limit. The result becomes

$$\begin{aligned} \Delta(\vec{Q}) &= 2\pi r_0^2 (-1)^{n_1 n_2} \int_{|\vec{p}| \leq \sqrt{2/3}/r_0} \frac{d\vec{p}}{(2\pi)^2} \exp(-i\vec{q} \cdot \vec{x}^{\vec{p}}) \\ &= \frac{2}{3} (-1)^{n_1 n_2} \frac{J_1(\sqrt{\frac{8\pi}{3\sqrt{3}}} \sqrt{n_1^2 + n_2^2 + n_1 n_2})}{\sqrt{\frac{8\pi}{3\sqrt{3}} \sqrt{n_1^2 + n_2^2 + n_1 n_2}}}, \end{aligned} \quad (59)$$

which also satisfies

$$\Delta(0) = \frac{1}{3}. \quad (60)$$

c. Tao-Thouless like states at $\nu = \frac{1}{3}$.

Finally, in this section we will consider states at $\nu = \frac{1}{3}$ and hexagonal lattice symmetry for which in the thermodynamical limit the fraction of the filled one electron states in the \vec{p} space is given by

$$f(\vec{p}) = \frac{1}{3} + .2222 \sum_{i=0}^5 \exp(\vec{p} \cdot \vec{x}_i^{\vec{p}}), \quad (61)$$

in which

$$\vec{x}_i^{\vec{p}} = \sqrt{\frac{4\pi r_0^2}{\sqrt{3}}} \left(\cos\left[\left(i - \frac{1}{2}\right) \frac{\pi}{3}\right], \sin\left[\left(i - \frac{1}{2}\right) \frac{\pi}{3}\right] \right), \quad (62)$$

where i ranges from 0 to 5.

The coefficient in the harmonic term in (61) has been selected as the one having the highest value but maintaining $f(\vec{p})$ positive for all the values of \vec{p} . It can be noticed that $f(\vec{p})$ is also periodic. For a vanishing coefficient of the harmonic term the $f(\vec{p}) = 1/3$ corresponds to a Tao-Thouless parent state but constructed in the new basis of the considered Hartree-Fock solutions. Because in this case the only (low momenta) order parameter being different from zero is $\Delta(0)$, it follows that the energy per particle becomes

$$\epsilon = -\sqrt{\frac{\pi}{8}} \frac{e^2}{r_0}. \quad (63)$$

In the original Tao-Thouless discussion only after calculating the correlation energy in the screening approximation the result (63) for the energy per particle was reduced. The purpose of considering the modified parent states defined by (61) is to clearly illustrate those states as particular extremals of the Hartree-Fock equations.

For the order parameter associated to $f(\vec{p})$ follows

$$\begin{aligned} \Delta(\vec{Q}) &= \frac{r_0^2}{2\pi} (-1)^{n_1 n_2} \int d\vec{p} f(\vec{p}) \exp(-i\vec{q} \cdot \vec{x}^{\vec{p}}) \\ &= \frac{1}{3} \delta_{\vec{Q},0} + 0.2222 (-1)^{n_1 n_2} \sum_{i=0}^5 \delta_{\vec{Q},\vec{Q}_i}. \end{aligned} \quad (64)$$

The introduction of the harmonic form for the \vec{p} dependence of the fraction of filled states $f(\vec{p})$ was also motivated by the aim of upmost increasing the order parameters associated to the smaller nonvanishing momenta values \vec{Q} in order to help for a reduction of the energy per particle value.

In conclusion, the high degree of symmetry of the considered Hartree-Fock states allows to obtain analytical expressions in terms of elementary functions for their order parameters. This results avoid limitations for further applications associated to the numerical character of various discussions of the Hartree-Fock problem [7-16]. Specifically, the availability of analytical results can help to investigate the role of the next order corrections in these states. Such contributions could be more relevant in these states than what they are in the lowest energy CDW state discussed in [9].

5 Selfenergy summation formula

The aim of this section is to present formulae giving the wavevector dependence of the selfenergies of the here considered solutions as summations of the similar sort to those defining the energy per particle. Since $\Delta(\vec{Q})$ is explicitly known, the evaluation of the band spectrum for the class of Hartree-Fock states is greatly simplified.

The selfenergies can be calculated by the formula given in [1] in the form

$$\begin{aligned} \epsilon^c &= e^2 \int d\vec{x} d\vec{x}' [\phi^c(\vec{x})]^* \phi^c(\vec{x}') \sum_{\{a\}_f} \frac{[\phi^a(\vec{x}')]^* \phi^a(\vec{x}') - n_0}{|\vec{x} - \vec{x}'|} \\ &\quad - e^2 \int d\vec{x} d\vec{x}' [\phi^c(\vec{x})]^* \phi^c(\vec{x}') \sum_{\{a\}_f} \frac{[\phi^a(\vec{x}')]^* \phi^a(\vec{x})}{|\vec{x} - \vec{x}'|}. \end{aligned} \quad (65)$$

in which n_0 is the jellium compensating background density and the indices for the basis functions are shortly written (similarly as it was done before) as $c, a = (\sigma, \nu, \vec{k}, r)$, where $\{a\}_f$ is the set of all the filled one electron states associated to the Slater determinant.

The selfenergy (65) is fully determined by the one-electron magnitude

$$\begin{aligned} \rho^c(\vec{x}, \vec{z}) &= [\phi^c(\vec{x})]^* \phi^c(\vec{z}) \\ &= [\phi^c(\vec{x})]^* \phi^c(\vec{x} - \vec{z}), \end{aligned} \quad (66)$$

in which $\vec{z} = \vec{x} - \vec{x}'$. After using the eigenvalue relation

$$T_1 \phi^c(x) = \lambda^c(\vec{l}) \phi^c(\vec{x}) \quad (67)$$

and also the one obtained from it by substituting $\vec{x} \rightarrow \vec{x} - \vec{z}$, relation (66) becomes

$$\rho^c(\vec{x} - \vec{l}, \vec{z}) = \exp\left[\frac{i\vec{e}}{\hbar c} \vec{A}(\vec{l}) \cdot \vec{z}\right] \rho^c(\vec{x}, \vec{z}). \quad (68)$$

Formula (68) implies that the Fourier transform of $\rho(\vec{x}, \vec{z})$ over the \vec{x} variable has the following structure

$$\rho^c(\vec{x}, \vec{z}) = \sum_{\vec{Q}} \exp[i(\vec{Q} + \frac{e}{\hbar c} \vec{A}(\vec{z})) \cdot \vec{x}] \rho^c(\vec{Q}, \vec{z}) \quad (69)$$

with

$$\rho^c(\vec{Q}, \vec{z}) = \frac{1}{V_N} \int_{V_N} \exp[-i(\vec{Q} + \frac{e}{\hbar c} \vec{A}(\vec{z})) \cdot \vec{x}] \rho^c(\vec{x}, \vec{z}) d\vec{x}, \quad (70)$$

where \vec{Q} are the lattice vectors associated to the Fourier transform of the density which are given in (36).

The explicit calculation of $\rho(\vec{Q}, \vec{z})$ can be done after substituting the form for ϕ^c given by (26) in (70) and performing the gaussian integration over \vec{x} . The calculation is essentially the same that was performed for arriving to (40) for the Fourier components of the density. There is only the additional dependence on the shift $\vec{z} = \vec{x} - \vec{x}'$. The result is

$$\rho^c(\vec{Q}, \vec{z}) = \rho^c(\vec{Q}) \exp\left[-\frac{\vec{z}^2}{4r_0^2} - i\frac{\vec{Q}}{2} \cdot (\vec{z} + i\vec{n} \times \vec{z})\right], \quad (71)$$

where

$$\rho^c(\vec{Q}) = \frac{1}{2\pi r_0^2} \Delta^c(\vec{Q}) \exp\left[-\frac{r_0^2 \vec{Q}^2}{4}\right], \quad (72)$$

$$\begin{aligned} \Delta^c(\vec{Q}) &= \frac{2\pi r_0^2}{V_N} \lambda^{(\sigma, \nu, \vec{k}, r)}(\vec{x}^{\vec{Q}}) \\ &= \frac{2\pi r_0^2}{V_N} \exp[-i\vec{k} \cdot \vec{x}^{\vec{Q}}] (-1)^{n_1 n_2} \sigma^{-\frac{1}{2}(q+1)n_2 + \frac{1}{2}(q-1)n_1} \\ &\quad \nu^{\frac{1}{2}(q+1)n_1 - \frac{1}{2}(q-1)n_2} \exp\left[\frac{(1-q)\pi(r-1)(n_1 - n_2)i}{q}\right]. \end{aligned} \quad (73)$$

Then, after substituting (69)-(72) in the expression for the selfenergy (65) and performing elementary integrals over \vec{x} and \vec{y} the following formula can be obtained

$$\begin{aligned} \epsilon^c &= \frac{e^2}{r_0} \frac{V_N}{(2\pi r_0^2)} \sum_{\vec{Q}} \Delta^c(\vec{Q}) \Delta^*(\vec{Q}) \exp\left(-\frac{r_0^2 \vec{Q}^2}{4}\right) \\ &\quad \left[\frac{(1 - \delta_{\vec{Q}, 0})}{r_0 |\vec{Q}|} \exp\left(-\frac{r_0^2 \vec{Q}^2}{4}\right) - (\pi/2)^{1/2} I_0\left(\frac{r_0^2 \vec{Q}^2}{4}\right) \right], \end{aligned} \quad (74)$$

where $\Delta^c(\vec{Q})$ is defined in (73) and is proportional to the eigenvalue of the function ϕ^c .

The final result gives the selfenergy corrections in the Hartree-Fock approximation in a form closely resembling the one determining the energy per particle. Relation (74) becomes useful for doing calculations of the band spectra because the order parameter Δ is explicitly known for the considered class of solutions. An interesting property which follows for the special kind of states with p of the q bands (above defined) filled is that the band energies are fully degenerated over the indices σ and ν . This occurs because (49) for $\Delta(\vec{Q})$ is different from zero only for even values of n_1 and n_2 , a condition which makes Δ independent of σ and ν . Therefore, the band spectrum for any value of q shows precisely q branches for a fixed value of k . A similar structure could be expected for the states of type b) in Section 4.

6 Energy per particle at various filling schemes at $\nu = \frac{1}{3}$

In this section we present results of the numerical calculation of the energy per particle for the few kind of states defined in Section 4. The numerical evaluations associated to general values of the filling factors, as well as the higher corrections to the Hartree-Fock result will be considered elsewhere.

The calculations were performed by means of the standard formula [6]

$$\begin{aligned} \epsilon &= \frac{e^2}{r_0} \frac{1}{2\nu} \sum_{\vec{Q}} |\Delta(\vec{Q})|^2 \exp\left(-\frac{r_0^2 \vec{Q}^2}{4}\right) \\ &\quad \left[\frac{(1 - \delta_{\vec{Q}, 0})}{r_0 |\vec{Q}|} \exp\left(-\frac{r_0^2 \vec{Q}^2}{4}\right) - (\pi/2)^{1/2} I_0\left(\frac{r_0^2 \vec{Q}^2}{4}\right) \right], \end{aligned} \quad (75)$$

in which expression (51), (59) and (64) were substituted for $\Delta(\vec{Q})$ in concordance with the types of states defined in Section 4. The limits for the summation over \vec{Q} were expanded until the satisfaction of the sum rule (45) up to approximately one percent for the cases a) and b). As for the Tao-Thouless like state c) the continuous approximation for the density of filled states $f(\vec{p})$ disregards the higher harmonics, not allowing the sum rule to be satisfied. Since the higher harmonics should not contribute appreciably to the energy per particle the calculated value should not be affected by the continuous approximation.

The energy per particle resulting values for each of the states were

$$\epsilon_a = -0.258 \frac{e^2}{r_0}, \quad (76)$$

$$\epsilon_b = -0.318 \frac{e^2}{r_0}, \quad (77)$$

$$\epsilon_c = -0.327 \frac{e^2}{r_0}. \quad (78)$$

The result in case a) is not lowering appreciably even the Hartree-Fock exchange energy of the Tao-Thouless state which is $-\sqrt{\frac{8}{3}} \frac{e^2}{r_0}$. Such a result would be related with the way of filling of the one-electron orbits in this case. It corresponds to fill out four disconnected

regions in the space of the wave vectors \vec{p} . In the originally proposed parent states one third of states are filled with uniform density.

On another hand the case b) corresponds to fill all the states having a \vec{p} contained in a circular neighborhood of the origin which produces a lower result.

Lastly, the outcome in case c) illustrates that by modulating the the density of filled orbitals in the momenta space the Hartree-Fock contribution to the energy can be improved in the class of Tao- Thoules states with modulated filling.

It can be noticed that the energy per particle of all these states are higher than the $-0.388\frac{e^2}{\epsilon_0}$ result of the lowest energy CDW state at $1/3$ value of the filling factor. However the overlapping among the localized states (Wannier) which could be defined using the Bloch functions (26) is expected to be greater than the one in the CDW states. Therefore the role of higher order corrections could be enhanced. Such perturbations, as it was mentioned before are negligible at least for the best CDW wavefunction [9]. The corresponding evaluation in the states discussed in [14], [15] has not been carried out yet up to our knowledge.

7 Summary

The exact solutions of the Hartree-Fock variational problem introduced in [1] was further considered here. Simple analytical and summation formulae are presented for the order parameter and the selfenergy bands of arbitrary states. The analytic results given may help in simplifying the treatment of higher order corrections.

Energy per particle calculation for $\nu = 1/3$ for a few states obtained by various rules for filling the one electron states are given. The results are not competing with the corresponding energy per particle value in the lowest energy CDW state. In spite of this fact, the high electron overlapping due to the periodicity of the wavefunction in a lattice having one flux quantum per cell, should allow the next order perturbative corrections to lower appreciably the results. Such higher order contributions become negligible in the case of the lowest energy CDW states.

In conclusion, the argumentation given in this work reinforces the view, expressed in [1], about the possibilities for the connections between this class of Hartree-Fock states and previous analysis suggesting the presence of crystal like properties in the FQHE ground states [2,4,23]. This question needs for further studies. One of them is the calculation of the energy per particle against filling factor dependence. Another one is the calculation of the next perturbative corrections to the energy per particle, a parameter which is a crucial characteristic in deciding the relevance of a ground state candidate. These matters will be considered in the further continuation of the work.

8 Acknowledgments

The author wishes to acknowledge the remarks and discussions given by Drs. A. Gonzalez, M. De Dios, A. McDonald, N. Read, E. Tossati, S. Fantoni, and the members of the Theoretical Physics Groups of the Faculty of Physics at Havana University and the Institute of Cybernetics, Mathematics and Physics. He would also like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality

at the International Centre for Theoretical Physics, Trieste. The benefits of the TWAS Research Grant Scheme are also deeply acknowledged.

References

- [1] A. Cabo, *Phys. Lett.* A171 (1992) 90.
- [2] B. I. Halperin, Z. Tesanovic and F. Axel, *Phys. Rev. Lett.* 57 (1986) 922.
- [3] Z. Tesanovic, F. Axel and B.I. Halperin, *Phys. Rev.* B39 (1989) 8525.
- [4] S. Kivelson, C. Kallin, D. P. Arovas and J. R. Schrieffer, *Phys. Rev. Lett.* 56 (1986) 873.
- [5] D. H. Lee, G. Baskaran and S. Kivelson, *Phys. Rev. Lett.* 59 (1987) 2467.
- [6] H. Fukuyama, D. M. Platzman and P.W. Anderson, *Phys. Rev.* B19 (1979) 5211.
- [7] D. Yoshioka and H. Fukuyama, *J. Phys. Soc. Jpn.* 44 (1978) 1035.
- [8] Y. Kuramoto and R. R. Gerhardt, *J. Phys. Soc. Jpn.* 51 (1982) 3810.
- [9] D. Yoshioka and P. A. Lee, *Phys. Rev.* B27 (1983) 4986.
- [10] A. H. MacDonald, *Phys. Rev.* B28 (1983) 6713.
- [11] A. H. MacDonald, *Phys. Rev.* B29 (1984) 3057.
- [12] D. Levesque, J. J. Weiss and A. H. MacDonald, *Phys. Rev. B* 30 (1984) 1056.
- [13] A. H. MacDonald, *Phys. Rev.* B30 (1984) 4392.
- [14] A. H. MacDonald and D. B. Murray, *Phys. Rev.* B32 (1985) 2291.
- [15] F. Claro, *Solid State Comm.* 53 (1985) 27.
- [16] F. Claro, *Phys. Rev.* B35 (1987) 7980.
- [17] F. Tao and Y. S. Wu, *Phys. Rev.* B30 (1984) 1097.
- [18] R. Tao and D. J. Thouless, *Phys. Rev.* B28 (1983) 1142.
- [19] R. Tao, *Phys. Rev. B* (1984) 636.
- [20] R. Tao, *J. Phys. C: Solid State* 17 (1984) L419.
- [21] D. J. Thouless, *Phys. Rev.* B31 (1985) 8305.
- [22] G. F. Giuliani and J. J. Quinn, *Phys. Rev.* B31 (1985) 3451.
- [23] S. T. Chui, *Phys. Rev.* B32 (1985) 1436.
- [24] S. T. Chui, *Phys. Rev.* B32 (1985) 8438.
- [25] S. T. Chui, T. M. Hakim and K. B. Ma, *Phys. Rev.* B33 (1986) 7110.
- [26] R. Ferrari, *Phys. Rev.* B42 (1990) 4598.
- [27] R. Ferrari, *University of Trento Preprint, U.T.F.* 238 (1991).
- [28] A. P. Prudnikov, Y. A. Brichkov and O. I. Marichev, *Integrals and Series. Elementary Functions* (Nauka, Moscow, 1981).