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Magnetized Pair Bose Gas: Relativistic Superconductor

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Abstract

We investigate the magnetized Bose gas at temperatures above pair threshold. New magnetization laws are obtained for a wide range of field strengths, and the gas is shown to exhibit the Meissner effect. Some related results for the Fermi gas, a relativistic paramagnet, are also discussed.

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As one of the fundamental quantum systems, the Bose gas has provided insight into a variety of exotic physical phenomena, from liquid He^4 superfluidity, to superconductivity [1]. Schafroth [2] showed how the nonrelativistic Bose gas exhibits the Meissner effect, that is, total expulsion of an external magnetic field, and thereafter the Bose gas has played a role in the understanding of superconductivity in metals. More recently, in seminal work Haber and Weldon [3,4] have developed the statistical mechanics of the relativistic Bose gas with no external fields, and applied this to a study of spontaneous symmetry breaking [5].

It is now apparent that astrophysics and cosmology provide venues where high temperatures and large magnetic fields play a significant role. For example, white dwarfs, neutron stars and supernovae [6] are examples of exotic stellar objects where fields and temperatures can be of the order of the mass scale of their constituent particles. Furthermore, it has been suggested that fields of the order of $\sim 10^{23}$ G [7], and possibly $\sim 10^{33}$ G [8] existed at the electroweak phase transition, where temperatures were $\sim 10^{15}$ K. Given this, and that the mass of the pion, for example, is in field/temperature units $\sim 10^{18}$ G/ $\sim 10^{12}$ K, and that of the electron is $\sim 10^{13}$ G/ $\sim 10^{10}$ K, a study of the statistical mechanics, and in particular the magnetic properties, of the Bose and Fermi gases above pair threshold and over a wide range of field strengths is necessary to give an insight into these physical scenarios. The interplay between temperature, field strength, and mass scales provides several parameter regions in which to do so. Previously, Miller and Ray [9] made an attempt to study the magnetized pair Bose gas, but their results proved inconclusive.

The thermodynamic potential $\Omega = -T \log Z$ for a pair fermion $(+)$ or pair boson $(-)$ gas is

$$-\beta\Omega = \pm \sum_{\mathbf{p}} \log\left[1 \pm e^{-\beta(E(p)-\mu)}\right] + \mu \to -\mu \quad , \tag{1}$$

where μ is the chemical potential, $\beta = 1/T$, and $E(p)$ is the single-particle energy spectrum in the field (of strength B). For spinless bosons, the spectrum is given by

$$E_n^2(p) = p^2 + (2n+1)eB + m^2 \quad . \tag{2}$$

where e is the boson charge and n is the Landau-level quantum number. Passing sums to integrals using the density of states in a magnetic field [10] for a $(d+1)$-dimensional flat spacetime of volume V gives

$$\frac{\Omega_d}{V} = \frac{eB}{2^{d-2}\pi^{d/2}\Gamma\left(\frac{d}{2}-1\right)\beta}\sum_{n=0}^{\infty}\int_0^{\infty} dp\, p^{d-3}\log\left[1-e^{-\beta(E_n(p)-\mu)}\right] \quad + \quad \mu \to -\mu \quad . \tag{3}$$

This integral may be computed to yield the sum form

$$\frac{\Omega_d}{V} = -\frac{\overline{B}m^{d+1}}{2^{(d-1)/2}\pi^{(d+1)/2}}\sum_{n=0}^{\infty}\sum_{j=1}^{\infty}\left[(2n+1)\overline{B}+1\right]^{(d-1)/4}\frac{e^{j\overline{\beta}\overline{\mu}}}{\left(j\overline{\beta}\right)^{(d-1)/2}}$$
$$\times K_{(d-1)/2}\left(j\overline{\beta}\left[(2n+1)\overline{B}+1\right]^{1/2}\right) + \ \overline{\mu}\to -\overline{\mu} \ , \tag{4}$$

where $K_\nu(z)$ is a modified Bessel function, and we have introduced the dimensionless quantities $\overline{\beta} = m\beta$, $\overline{\mu} = \mu/m$ and $\overline{B} = eB/m^2$.

Although Ω_d cannot be computed exactly, it is possible to develop high-T asymptotic expansions using the Mellin transform technique. We reserve a full discussion for Ref. [11], where we have given a detailed study of the pair quantum gases. Here, we give the Mellin integral representation for Ω_d

$$\frac{\Omega_d}{V} = -\frac{m^{d+1}}{\pi^{(d+1)/2}}\sum_{k=0}^{\infty}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\left(\frac{2}{\overline{B}}\right)^{(s-d-1)/2+k}\frac{\zeta(s)\,\Gamma\left(\frac{s}{2}+k\right)\Gamma\left(\frac{s-d+1}{2}+k\right)}{\overline{\beta}^{s}\,\Gamma(2k+1)}\overline{\mu}^{2k}$$
$$\times\zeta\left(\frac{s-d+1}{2}+k,\frac{1}{2}+\frac{1}{2\overline{B}}\right) \quad , \tag{5}$$

where $\mathrm{Re}\, c > d+1$, $\zeta(z,a)$ is the Hurwitz zeta function [12,13], defined for $\mathrm{Re}\, z > 1$ by the series representation

$$\zeta(z,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^z} \quad , \tag{6}$$

and analytically continued elsewhere by an appropriate integral representation, $\zeta(z) = \zeta(z,1)$ is the Riemann zeta function, and $\Gamma(z)$ is the gamma function.

We note that the analytic structure of the Landau-level sum in (4) is now contained within its functional representation, the Hurwitz zeta function appearing in (5). We may now develop asymptotic expansions for Ω_d in the region $\overline{B}\,\overline{\beta}^2 \ll 1$ (that is $eB/T^2 \ll 1$) by closing the contour of the Mellin integral to the left and using Cauchy's theorem. This region of parameter space corresponds to both the weak-field ($\overline{B} \ll 1 \ll \overline{\beta}^{-2}$) and intermediate-field ($1 \ll \overline{B} \ll \overline{\beta}^{-2}$) regions, although there will be some sensitivity to the size of $\overline{B}$ due to differing asymptotics for the Hurwitz zeta function in each of these regions. That expansions for both are obtainable more or less simultaneously is a reflection of the physics, as under the condition $eB/T^2 \ll 1$ the thermal energy is much greater than the energy gap between Landau levels so that the (anti)bosons are distributed throughout them. In the strong-field $\overline{B}\,\overline{\beta}^2 \gg 1$ region, the field is large enough to suppress pair production and occupation of levels other than $n = 0$, and so the physics has changed dramatically. It is therefore not surprising that the Mellin integral representation is much less useful for obtaining the asymptotics for this region; more of this later.

The $\overline{B}\,\overline{\beta}^2 \ll 1$ expansion for Ω_d in odd spatial dimensions (i.e. $d = 2l + 1$, where $l = 1, 2, 3, \ldots$) is

$$\begin{aligned}\frac{\Omega_{2l+1}}{V} = -\frac{m^{2l+2}}{2^{2l+1}\pi^{l+1}} \Bigg\{ &\sum_{k=0}^{l+1} \frac{2^{2l+2}}{\overline{\beta}^{2l-2k+2}}\zeta(2l-2k+2)\,\frac{\Gamma(l+1)}{\Gamma(2k+1)}\overline{\mu}^{2k} \\ &+\sum_{q=0}^{l-1}(-1)^q\left(\frac{2}{\overline{\beta}}\right)^{2l-2q}\frac{\zeta(2l-2q)\,\Gamma(l-q)}{\Gamma(q+1)}\left(2\overline{B}\right)^{q+1}\zeta\left(-q,\frac{1}{2}+\frac{1}{2\overline{B}}\right) \\ &+\frac{\left(-2\overline{B}\right)^{l+1}}{\Gamma(l+1)}\left[\frac{1}{2}\zeta\left(-l,\frac{1}{2}+\frac{1}{2\overline{B}}\right)\left\{\log\left(\frac{8\pi^2}{\overline{B}\,\overline{\beta}^2}\right)-\gamma+\psi(l+1)\right\}\right. \\ &\left.+\frac{d}{ds}\zeta\left(\frac{s}{2}-l,\frac{1}{2}+\frac{1}{2\overline{B}}\right)\Bigg|_{s=0}\right]+\frac{\pi^{1/2}}{\overline{\beta}}\left(2\overline{B}\right)^{l+1/2}\Gamma\left(\frac{1}{2}-l\right)\zeta\left(\frac{1}{2}-l,\frac{1+\overline{B}-\overline{\mu}^2}{2\overline{B}}\right) \\ &+\mathcal{O}\left(\overline{B}\,\overline{\beta}^2\right)+\ldots\Bigg\}\ . \end{aligned} \tag{7}$$

where γ is Euler's constant, and $\psi(z)$ is the digamma function. We can obtain the other thermodynamic quantities from Ω_d, such as the charge density ρ_d and the magnetization (density) M_d via

$$\rho_d = -\frac{1}{V}\frac{\partial\Omega_d}{\partial\mu} \qquad \text{and} \qquad M_d = -\frac{1}{V}\frac{\partial\Omega_d}{\partial B}\ . \tag{8}$$

The first issue to tackle is Bose-Einstein condensation. May [14] showed that the nonrelativistic magnetized Bose gas condenses only for $d \geq 5$, and Haber and Weldon [5] demonstrated that the critical behavior of the gas in no fields carries over from the nonrelativistic to the relativistic regime. We can now complete the picture. It is easy to see from (1) that the condensation condition in the field is $\overline{\mu}^2 = 1 + \overline{B}$. Taking the μ-derivative of (7) (which requires some of the properties of $\zeta(z, a)$ presented in Ref. [11]), we find that the term which entirely controls the analyticity of ρ_d is the one containing

$$\zeta\left(\frac{3}{2} - l, \frac{1 + \overline{B} - \overline{\mu}^2}{2\overline{B}}\right) \quad ,$$

which diverges as $\overline{\mu}^2 \rightarrow 1 + \overline{B}$ for $l < \frac{3}{2}$, so that the gas does not condense for $d = 3$, but does for all odd $d \geq 5$. Furthermore, we have evaluated the corresponding expansions for $d = 2l$ in [11] , where the analyticity of ρ_{2l} is determined by a derivative of a Hurwitz zeta function, and as expected the gas does not condense for $d = 4$ but does for higher even spatial dimensions. Hence, we have seen that the critical behavior of the nonrelativistic magnetized Bose gas carries over into the magnetized, relativistic regime.

We now move to the question of whether a relativistic analog to tne Schafroth superconducting state exists. The magnetization in weak fields below the zero-field condensation temperature T_c for the nonrelativistic gas was shown by him [2] to be

$$M_3 = -\mu_0 \rho_3 \left[1 - (T/T_c)^{3/2}\right] + \mathcal{O}(B) + \ldots \quad , \tag{9}$$

where μ_0 is the Bohr magneton. The effective field in the medium, $B_{eff} = B + 4\pi M$, therefore vanishes below a critical value and so the gas exhibits the Meissner effect, even though it does not condense in the field. Relativistically, we now give the magnetization for weak fields $\overline{B} \ll 1$ around the zero-field condensation temperature T_c (so that $\overline{\mu}(T_c) = 1$):

$$M_3 = -\frac{9}{\sqrt{2}} \zeta\left(-\frac{1}{2}, \frac{1}{2}\right) \mu_0 \rho_3 \overline{\beta}_c \overline{B}^{1/2} \{1 + \mathcal{O}(t) + \ldots\} + \mathcal{O}\left(\overline{B} \log \overline{\beta}_c\right) + \ldots \quad , \tag{10}$$

where $\zeta\left(-\frac{1}{2}, \frac{1}{2}\right) \simeq 0.061$, $t = (T - T_c)/T_c$, and we have used $\rho_3 \simeq m^3/\left(3\overline{\beta}_c^2\right)$, with $\overline{\beta}_c = m/T_c$.

This is a totally new magnetization law. In the relativistic regime we do not find a magnetization of the Schafroth form (9). Therefore, field expulsion due to a macroscopic fieldindependent ground-state magnetization has been lost when $T \gtrsim m$, so that the Schafroth result (9) applies only to low temperatures, where a remnant of the macroscopically occupied $T = 0$ ground state mimics a condensate phase up to the (nonrelativistic) zero-field condensation temperature. This is a reflection of Bose statistics, as at low temperatures even without true condensation the occupation density of the lowest energy level is macroscopic. Without condensation in the field, we see here that at high temperatures this remnant ground state does not survive. However, the relativistic magnetization law (10) has its own remarkable features, for upon evaluating the corresponding B_{eff} we find that the external field will be expelled if it does not exceed the critical value

$$\overline{B}_c \simeq 4.8 \times 10^{-3} \left(\overline{\beta}_c\right)^{-2} \quad . \tag{11}$$

Clearly, increasing the temperature T_c will increase the size of this critical field, and in fact if $\overline{\beta}_c \lesssim 0.07$, then $\overline{B}_c$ will be $\mathcal{O}(1)$ so that *all* external fields conforming to the original constraint $\overline{B} \ll 1$ will be expelled. The mechanism for this manifestation of the Meissner effect is now not a remnant ground state, but pair production. It can be seen from studying the various forms for Ω_d, and argued from simple physics, that the contribution of bosons and antibosons to the net magnetization is additive. While quantum field theory tells us that ρ must be held fixed, the freedom to produce pairs allows M to be so large and diamagnetic as to totally expel the field. This is superconductivity through sheer weight of numbers.

What then happens in the limit $T/m \to \infty$, when pair production is profuse? This corresponds to $\overline{\mu} \to 0$, and the magnetization is then

$$M_3 = -\frac{\mu_0 \rho_3}{4\pi\overline{\beta}} \overline{B}\,\overline{\beta}_c^2 + \mathcal{O}\left(\overline{B}\,\overline{\beta}_c^2 \log \overline{\beta}\right) + \dots \quad . \tag{12}$$

so that the field is expelled if $\overline{\beta} \lesssim 0.01$. As the light cone is approached, the macroscopic magnetization is ever increasing due to the overwhelming production of pairs, now leading again to field expulsion.

Even if the magnitude of the field is increased, so that now $m^2 \ll eB \ll T^2$ (which for pions means that $B \gtrsim 10^{19}$ G), the leading-order magnetization law at $\overline{\mu} = 1$, Eqn. (10), still holds, so that fields above the mass scale can be expelled. The magnetization does change in the $T/m \to \infty$ limit in this transition from weak to intermediate fields, and becomes

$$M_3 = \frac{9\sqrt{2}}{4\pi}\zeta\left(-\frac{1}{2}\right)\mu_0\rho_3\left(\frac{\overline{\beta}_c}{\overline{\beta}}\right)\overline{\beta}_c\,\overline{B}^{1/2} + \mathcal{O}\left(\overline{B}\,\overline{\beta}_c^2\log\overline{B}\,\overline{\beta}^2\right) + \ldots \quad . \tag{13}$$

so that with $\zeta\left(-\frac{1}{2}\right) \simeq -0.21$, the field will be expelled if $\overline{\beta}\,\overline{B}^{1/2} \lesssim 0.003$, which is a far more stringent condition on the temperature than we saw in the weak-field case. It is sensible, though, that more pairs need to be produced to expel a stronger field.

As discussed previously, when the field is strong enough to dominate both the mass and the temperature ($m^2 \ll T^2 \ll eB$), then antiboson production is suppressed, and nearly all of the bosons are in the lowest Landau level. The Mellin integral representation of (5) is then not useful for determining the magnetization; in fact we must perform a large parameter expansion on the modified Bessel functions appearing in (4). The result we obtain is the temperature-independent law

$$M_3 \simeq -\frac{\mu_0\rho_3}{\overline{B}^{1/2}} \quad . \tag{14}$$

This is actually valid for all temperatures $T^2 \ll eB$, including $T = 0$, and this can be understood as the field strength imposing zero-temperature behavior on the gas for all temperatures below the field scale. Field expulsion can still occur, but only under the most extreme conditions. For example, it can be shown that a particle density of $\rho \simeq 10^{62}$ cm^{-3} is required to expel the mooted 10^{33} G field at the electroweak phase transition. As a *caveat*, it is of course important to consider the vacuum and interactions when fields are of this extremely large magnitude.

We have shown, in marked contrast to the nonrelativistic Bose gas, that the relativistic magnetized Bose gas exhibits unique magnetization laws, leading in every region of parameter space to a Meissner effect, and thus is truly a relativistic superconductor.

In Ref. [11], we have also made a parallel study of the magnetized pair spin-1/2 Fermi gas, with energy spectrum

$$E_{n,\sigma}^2 = p^2 + (2n - \sigma + 1)\,eB + m^2 \quad , \tag{15}$$

where σ has the values -1 and 1 for spin-down and spin-up states respectively. Following the procedure that we used to study the magnetized pair Bose gas, we can successfully develop the statistical mechanics of its fermionic analog, and we briefly mention some of our results here. The magnetization in weak fields is

$$M_3 = \frac{\mu_0 \rho_3}{\pi^2} \overline{B}\, \overline{\beta}_0^2 \left[\log\left(\frac{\pi}{\overline{\beta}}\right) - \gamma + \mathcal{O}\left(\overline{\beta}^2\right) + \ldots \right] , \tag{16}$$

and in the intermediate-field region

$$M_3 = \frac{\mu_0 \rho_3}{8\pi^2} \overline{B}\, \overline{\beta}_0^2 \left[\left| \log \overline{B}\, \overline{\beta}^2 \right| + \mathcal{O}(1) + \ldots \right] , \tag{17}$$

where $\overline{\beta}_0 = m/T_0$. T_0 is a characteristic temperature defined by $\overline{\mu}(T_0) = 1$, and $\rho_3 \simeq m^3 / \left(3\overline{\beta}_0^2\right)$. The gas is paramagnetic, increasingly so as a function of temperature. This can again be linked to the pair mechanism, as the paramagnetic contribution of spin alignment to the field of newly-created pairs is able to overcome their diamagnetic Landau moment.

In the strong-field region, the result around T_0 is

$$M_3 = \frac{\mu_0 \rho_3}{\overline{B}} \left\{ \frac{\pi^2}{3\overline{\beta}_0^2} (1 + 2t) + \log\left(\frac{\overline{\beta}_0}{\pi}\right) - \gamma - \frac{1}{2} - t \right.$$
$$\left. + \tau(3) \left(\frac{\overline{\beta}_0}{4\pi}\right)^2 (5 - 2t) + \mathcal{O}\left(\overline{\beta}_0^4\right) + \ldots \right\} + \mathcal{O}\left(t^2\right) + \ldots . , \tag{18}$$

where $t = (T - T_0)/T_0$, $\tau(z) = (1 - 2^{1-z}) \zeta(z)$, and now $\rho_3 \simeq m^3 \overline{B} / (2\pi^2)$. In the case of fermions, the magnetization is dominated by the particles in the lowest Landau level with spin aligned with the field. The temperature sensitivity of (18) compared to the Bose analog (14) is due to the exclusion principle, as these $n = 0$, spin-aligned fermions are forced to occupy linear momentum states above the ground state.

In conclusion, we can state that the pair gases, through the interplay between pair creation, temperature, field strength, statistics and (in the case of fermions) spin, have remarkable magnetic properties. The next challenge is an investigation of the effect of interactions on this behavior.

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REFERENCES

[1] V. L. Ginzburg, J. Stat. Phys. **1**, 3 (1969).

[2] M. R. Schafroth, Phys. Rev. **100**, 463 (1955).

[3] H. E. Haber and H. A. Weldon, Phys. Rev. Lett. **46**, 1497 (1981).

[4] H. E. Haber and H. A. Weldon, J. Math. Phys. **23**, 1852 (1982).

[5] H. E. Haber and H. A. Weldon, Phys. Rev. D **25**, 502 (1982).

[6] Ya. B. Zeldovich and I. D. Novikov, *Relativistic Astrophysics Vol. 1: Stars and Relativity* (University of Chicago Press, Chicago, 1971).

[7] T. Vachaspati, Phys. Lett. **B265**, 258 (1991).

[8] J. Ambjørn and P. Olesen, *Neils Bohr Institute preprint NBI-HE-93-17* (1993).

[9] D. E. Miller and P. S. Ray, Phys. Rev. A **33**, 1990 (1986).

[10] K. Huang, *Statistical Mechanics* (Wiley, New York, 1987).

[11] J. Daicic, N. E. Frankel, R. M. Gailis and V. Kowalenko, *University of Melbourne preprint UM-P-93/44*, submitted for publication.

[12] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1980)

[13] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, 1927).

[14] R. M. May, J. Math. Phys. **6**, 1462 (1965).