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ON COMPACT MULTIPLIERS OF TOPOLOGICAL ALGEBRAS

Noor Mohammad

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Noor Mohammad[1]
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

It is shown that if the maximal ideal space $\Delta(A)$ of a semisimple commutative complete metrizable locally convex algebra contains no isolated points, then every compact multiplier is trivial. Particularly, compact multipliers on semisimple commutative Frechét algebras whose maximal ideal space has no isolated points are identically zero.

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[1]Permanent Address: Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan.

1 Introduction and Preliminaries

The aim of this note is to investigate a necessary condition under which a compact multiplier on a complete metrizable locally convex (in particular, Frechét) algebra becomes trivial. For the related concepts and the properties not explicitly given here, we refer to [3]. However, for the sake of completeness we recall some basic notions needed for the subsequent analysis.

In what follows, A stands for a complex algebra and all the topological vector spaces under consideration are assumed to be Hausdorff, without mentioning them explicitly. A complete metrizable locally convex algebra is called a B_0-algebra, whereas a complete metrizable locally multiplicatively convex algebra is said to be a *Frechét algebra* [3]. Given any semisimple commutative B_0-algebra A, then following [2] a *multiplier* is a mapping $T : A \to A$ satisfying $Tx \cdot y = x.\ Ty$ for each $x, y \in A$. We denote the set of all multipliers on A by $M(A)$. Here we note that each $T \in M(A)$ is linear and continuous ([2], Corollary 2.3). Recall that $T \in M(A)$ is compact if it maps bounded sets into compact sets.

We denote the set of all non-zero continuous multiplicative linear functionals on A by $\Delta(A)$. It is assumed that $\Delta(A)$ is non-empty and separates points of A. We endow $\Delta(A)$ with the Gelfand topology, i.e., the induced ω^*-topology from the topological dual A^* of A.

2 Main Result

Theorem 2.1. Let A be a semisimple commutative B_0-algebra and T a compact multiplier on A. If the maximal ideal space $\Delta(A)$ of A contains no isolated points, then T is identically zero.

Proof. It is well-known ([2], Theorem 2.5) that for each $T \in M(A)$ there exists a complex-valued continuous function φ_T on $\Delta(A)$ such that

$$(Tx)^\wedge(f) = \varphi_T(f)\hat{x}(f)$$

for all $f \in \Delta(A)$ and all $x \in A$. Here the function φ_T is defined by $\varphi_T(f) = f \circ T(x)$, where $x \in A$ is chosen such that $f(x) = 1$. We claim that for each $f \in \Delta(A)$, $\varphi_T(f)$ is an eigenvalue of the adjoint T^* of T. In fact, for $f \in \Delta(A)$, let μ_f denote the linear functional in A^* given by $\mu_f(x) = \hat{x}(f)$ (Note that the Gelfand transform $\hat{x}$ of x is a continuous function). Then for each $x \in A$, we have

$$(T^*\mu_f)(x) = \mu_f(Tx) = (Tx)^\wedge(f) = \varphi_T(f)\hat{x}(f)$$

$$= \varphi_T(f)\mu_f(x)\ .$$

Thus $T^*\mu_f = \varphi_T(f)\mu_f$, which proves that $\varphi_T(f)$ is an eigenvalue of T^*.

By assumption, T is compact and hence T^* is compact ([5], Chapter VIII, Sec. 2, p. 152). Therefore, the spectrum $\sigma(T^*)$ of T^* is either a finite set or a sequence which converges to 0 ([5], Chapter VIII, Proposition 3). Furthermore, $\sigma(T^*)$ has also the property that every $0 \neq \lambda \in \sigma(T^*)$ is an eigenvalue of finite multiplicity, i.e. $\dim(\ker(\lambda I - T^*)) < \infty$. Assume that f_0 is an arbitrary element of $\Delta(A)$ which is not an isolated point. We claim that $\varphi_T(f_0) = 0$. For this, let us assume that $\varphi_T(f_0) \neq 0$. There is $x_0 \in A$ with $f_0(x_0) = 1$.

Since φ_T is a continuous function on $\Delta(A)$ and f_0 is a limit point of $\Delta(A)$, for each positive integer n, there exists an element $f_n \in \Delta(A)$, $f_0 \neq f_n$, with

$$|\varphi_T(f_n) - \varphi_T(f_0| = |f_n(Tx_0) - f_0(Tx_0)| < \frac{1}{n} \quad \text{(see the proof of Theorem 2.5 [2])} .$$

Since each non-zero eigenvalue of T^* has finite multiplicity, it follows that $\varphi_T(f_n) = \varphi_T(f_0)$ for only finitely many n. Therefore, $\varphi_T(f_0)$ is a limit point of $\{\varphi_T(f_n)\} \subset \sigma(T^*)$. But as we have seen before, 0 is the only possible limit point of $\sigma(T^*)$, hence the contradiction. This proves that $\varphi_T(f_0) = 0$. Since, by hypothesis, no element in $\Delta(A)$ is an isolated point, we conclude that $\varphi_T(f) = 0$ for all $f \in \Delta(A)$. Hence

$$(Tx)^\wedge(f) = \varphi_T(f)\hat{x}(f) = 0$$

for all $f \in \Delta(A)$ and all $x \in A$. Thus $Tx \in \cap_{f\in\Delta(A)}\ker\ (f) = \mathrm{rad}(A)$, and by the semisimplicity of A, $Tx = 0$ for all $x \in A$. This completes the proof.

Corollary 2.1. Let A be a semisimple commutative Frechét algebra such that $\Delta(A)$ has no isolated points. If T is a compact multiplier, then T is trivial.

As an immediate consequence of Theorem 2.1, we derive the following result due to Kamowitz [4], in the particular case of Banach algebras.

Corollary 2.2. Let A be a semisimple commutative Banach algebra and T a compact multiplier on A. If the maximal ideal space $\Delta(A)$ of A contains no isolated points, then $T = 0$.

Remark. If the maximal ideal space $\Delta(A)$ has isolated points, then there exists a non-zero compact multiplier on A. To see this, let f_0 be an isolated point of $\Delta(A)$. By the Shilov idempotent theorem [1], there is an idempotent element e in A satisfying $\hat{e}(f) = 1$ if and only if $f = f_0$. Then the operator T defined by $Tx = ex = \hat{x}(f_0)e$ is clearly a non-zero multiplier which is compact since its range is one-dimensional.

We now give two examples of algebras where the maximal ideal space $\Delta(A)$ contains isolated points. In each of these examples we describe a non-zero compact multiplier.

Example 1. Consider the set s of all complex sequences with coordinatewise operations. Then s is a unital semisimple commutative Frechét algebra possessing an orthogonal basis $\{e_n : n \geq 1\}$ (see [3], Chapter II, Example 3.4). Each $x \in s$ can be expressed uniquely as $x = \sum \lambda_n(x)e_n$, where each coordinate functional λ_n is a continuous multiplicative linear functional. Furthermore, $\Delta(A) = \{\lambda_n\}$ is homeomorphic with the discrete space $\mathbb{N}$ of natural numbers ([3], Chapter III, Theorem 3.1). Note that, by definition, each e_n is idempotent and $xe_n = \lambda_n(x)e_n$. Also, $\hat{e}_n(\lambda_n) = \lambda_n(e_n) = 1$ for all $n \in \mathbb{N}$. For some fixed e_m in s, define the operator T by

$$Tx = xe_m = \lambda m(x)e_m = \hat{x}(\lambda_m)e_m .$$

Then clearly T is a non-zero compact multiplier.

Example 2. Let $H(D)$ denote the algebra of all holomorphic functions defined on the open disc $D = \{z \in \mathbb{C} : |z| < 1\}$. With the Cauchy-Hadamard product and the compact-open topology, $H(D)$ is a unital semisimple commutative B_0-algebra possessing an orthogonal basis $\{e_n : n \geq 0\}$, where $e_n(z) = z^n$ for any $z \in D$ ([3], Chapter III, p. 97). Repeating the similar arguments as given in example 1, we can construct a non-zero compact multiplier on $H(D)$.

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