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ABSTRACT

We study the commutativity of certain class of rings, namely rings with unity 1 and right s-unital rings under each of the following properties $[yx^m - x^n f(y)x^p, x] = 0$, $[yx^m + x^n f(y)x^p, x] = 0$, where $f(t)$ is a polynomial in $t^2\mathbb{Z}[t]$ varying with pair of ring elements x, y and m, n, p are fixed non-negative integers. Moreover, the results have been extended to the case when m and n depend on the choice of x and y and the ring satisfies the Chacron's Theorem.

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1 Introduction

There is a multitude of conditions each of which implies the commutativity of certain rings. The equivalence of few such conditions to that of commutativity of rings was established by Komatsu and Tominaga [11] and Ash'raf et al. [6]. The list of these equivalent conditions was further enlarged by author jointly with Abujabal and Samman in [4]. The major purpose of this paper is to extend the work of Abujabal [5] and Ashraf et al. [7]. In fact, several commutativity theorems can be obtained as corollaries to our result, for instance [1, Theorem], [2, Theorem], [3, Theorem], [4, Theorem], [10, Theorem], [11, Theorem], [12, Theorem] and [13, Theorem].

Throughout this paper, R will represent an associative ring (not necessarily with 1). $\mathbb{Z}[t]$ is the totality of polynomials in t with coefficient in $\mathbb{Z}$ the ring of integers. As usual, for any x, y in R the symbol $[x, y]$ will denote the commutator $xy - yx$. In section 2 we investigate the commutativity of rings with unity 1 under the ring properties:

(B_1) For every x, y in R there exists $f(x) \in X^2\mathbb{Z}[X]$, such that $[yx^m - x^n f(y)x^p, x] = 0$, where m, n and p are fixed non-negative integers.

(S_1) For every x, y in R there exists $f(X) \in X^2\mathbb{Z}[X]$, such that $[yx^m + x^n f(y)x^p, x] = 0$, where m, n and p are fixed non negative integers.

(B_2) For every x, y in R there exist non negative integers m, n, p and $f(X) \in X^2\mathbb{Z}[X]$ such that $[yx^m - x^n f(y)x^p, x] = 0$.

(S_2) For every x, y in R there exist non negative integers m, n, p and $f(X) \in X^2\mathbb{Z}[X]$ such that $[yx^m + x^n f(y)x^p, x] = 0$

(CA) For every x, y in R there exist $f(X)$, $g(X)$ in $X^2\mathbb{Z}[X]$ such that $[x - f(x), y - g(y)] = 0$.

The above results obtained are further extended to the right s-unital rings in the subsequent section 3.

2 Commutativity of Rings with Unity 1

The following theorems are main results of this paper.

Theorem 1. Let R be a ring with unity 1. If R is satisfying any one of the conditions (B_1) and (S_1), then R is commutative (and conversely).

Theorem 2. Let R be a ring with unity 1 satisfying (CA). If R satisfies any one of the conditions (B_2) and (S_2), then R is commutative (and conversely).

In order to develop the proof of the above theorem first we consider the following type of rings.

(1) $\begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

$(1)_r$ $\begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime

(1)$_\ell$ $\begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}$, p a prime

(2) $M_\sigma(F) = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} \Big/ a, b \text{ in } F \right\}$, where F is a finite field with a nontrivial automorphism σ.

(3) A non-commutative division ring.

(4) $S = \langle 1 \rangle + T$, T as non-commutative radical subring of S.

(5) $S = \langle 1 \rangle + T$, T is a non-commutative subring of S such that

$$T[T,T] = [T,T]T = 0 \ .$$

In [14], Streb gave a nice classification for non-commutative rings, which yields a powerful tool in obtaining a number of commutativity theorems (cf. [1], [2], [3], [11] and [12]). it follows from the proof of [14, Corollary (1)], that if R is non commutative ring with unity 1, then there exists a factor subring of R which is of type (1), (2), (3), (4) and (5). This observation gives the following result which plays the key role in our subsequent study.

Lemma 1. let P be a ring property which is inherited by factor subrings. If no rings of type (1), (2), (3), (4) or (5) satisfy (P), then every ring with unity 1 satisfying P is commutative.

For the sake of convenience, we state the following well-known result due to Herstein [8].

Lemma 2. Let R be a ring in which for every x, y in R there exists $f(x) \in X^2\mathbb{Z}[X]$ such that $[x - f(x), y] = 0$. Then R is commutative.

Proof of Theorem 1

Let R be a ring satisfying (B_1). First we shall show that no rings of type (1), (2), (3), (4) or (5) satisfies (B_1). Let us begin with the ring of type (1). Then in M_2 (GF(p)), p a prime, we observe that $[e_{12}e_{12+22}^m - e_{12+22}^n f(e_{12})e_{12+22}^p, e_{12+22}] \neq 0$ for all integers $m \geq 0$, $n \geq 0$, $p \geq 0$ and $f(X) \in X^2\mathbb{Z}[X]$. Hence the ring of type (1) does not satisfy (B_1).

Suppose that B is the ring of type (2). Taking $x = \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix}$ $(\sigma(a) \neq a)$, $y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in $(B = M_\sigma(F))$.

One observes that

$$[yx^m - x^m f(y)x^p, x] = -[x,y]x^m = (\sigma(a) - a)(\sigma(a))^m y \neq 0$$

for all integers $m \geq 0$, $n \geq 0$, $p \geq 0$ and $f(X) \in X^2\mathbb{Z}[X]$. Hence no rings of type (2) satisfy the condition (B_1).

Now, consider the ring R of type (3). If x is a unit of R, then for every y in R choose $f(x) \in X^2\mathbb{Z}[X]$ such that $[yx^{-m} - x^{-n}f(y)x^{-p}, x^{-1}] = 0$. This implies that

$$[yx^{-m} - x^{-n}f(y)x^{-p}, x] = 0 \ .$$

Thus it follows that

$$x^n[x,y]x^p = [x, f(y)]x^m \qquad (i)$$

Further, we consider $g(x) \in x^2\mathbb{Z}[x]$ such that $[f(y)x^m - x^n g(f(y))x^p, x] = 0$ This becomes

$$x^n[x, g(f(y))]x^p = [x, f(y)]x^m \qquad (ii)$$

Combining (i) and (ii), we get $x^n[x, g(f(y)]x^p = x^n[x, y]x^p$.

But since x is a unit, $[x, y - h(y)] = 0$ for $g(f(y)) = h(y) \in Y^2\mathbb{Z}[Y]$. Hence by Lemma 2, R is commutative which gives a contradiction. Thus a ring R of type (3) does not satisfy (B_1).

Suppose that R has a factor subring of type (4). Let $a, b \in T$. Since $1 - a$ is unit, using above paragraph there exists $f(X) \in X^2\mathbb{Z}[X]$ such that $[a, b - f(b)] = -[1 - a, b - f(b)] = 0$.

Thus T is commutative by Lemma 2. This is a contradiction, so R does not satisfy a factor subring of type (4) which satisfy (B_1).

Finally, we consider $S = \langle 1 \rangle + T$, where T is a non-commutative subring of S such that $T[T, T] = [T, T]T = 0$. Let R has a factor subring S. Choose $a, b \in T$ such that

$$[a, b] = (a + 1)^n[a, b](a + 1)^p = [a, f(b)](a + 1)^m = 0 .$$

This is a contradiction.

Hence, we have seen that no rings of type (1), (2), (3), (4) or (5) satisfy (B_1). Thus by Lemma 1, R is commutative.

Remark 1. If R satisfies the property (S_1) then the proof will follow on the same lines as above.

Proof of Theorem 2. Suppose that R satisfies the property (B_2). First, we assume that the rings of type (1). Then $[e_{12}e_{22}^m + e_{22}^n f(e_{12})e_{22}^p, e_{22}] \neq 0$ for any integers $m > 0$, $n \geq 0$, $p \geq 0$. and $f(X) \in X^2\mathbb{Z}[X]$. This shows that no rings of type (1) satisfying (B_2).

Using similar arguments as above one can prove that no rings of type (2) satisfy the condition (B_2). From this fact with the Corollary 1 of [11], we obtain the required result.

Remark 2. The proof for the case R satisfies (S_2), will follow on the same lines as above.

Since there are non-commutative rings with R^2 being central, neither of these conditions guarentees the commutativity of arbitrary rings. However, we extend the study to the class of rings, which are called right s-unital. The results obtained here generalize [1], [2], [3], [6], [9], [10], [12].

3 Extensions to right s-unital rings

An associative ring R is called right (resp. left) s-unital if $x \in xR$ (resp. $x \in Rx$) for each $x \in R$. Further R is called s-unital if it is both right as well as left s-unital, i.e. $x \in Rx \cap xR$ for each $x \in R$.

If R is an s-unital (resp. right or left s-unital) ring, then for any finite subset F of R, there exists an element $e \in R$ such that $xe = ex = x$ (resp. $xe = x$ or $ex = x$) for all $x \in F$. Such an element is called the pseudo (left or pseudo right) identity of F in R. Before we go ahead with our task, we state the following lemma due to Komatsu, Nishinaka and Tominaga [12] in order to make our paper self-contained as far as possible.

Lemma 3. Let R be a right s-unital ring and not left s-unital. Then R has a factor subring of type $(1)_r$.

If R is a right s-unital ring satisfying (B_1), then a careful scrutiny of the first paragraph of the proof of Theorem 1 shows that no rings of type $(1)_r$ satisfy (B_1). hence by Lemma 3, R is s-unital and in view of Proposition 1 of [9], R has unity 1. Thus R is commutative by Theorem 1, which gives the following

Theorem 3. Let R be a right s-unital ring satisfying any one of the conditions (B_1) and (S_1). Then R is commutative (and conversely).

Corollary 1. Let $m \geq 0, n \geq 0$ be fixed integers. Suppose that if R is a right s-unital ring in which for every x, y in R there exists integer $r = r(x,y) > 1$ such that either $[yx^m - x^n y^r x^p, x] = 0$ or $[yx^m + x^n y^r x^p, x] = 0$. Then R is commutative (and conversely).

From [11, Corollary 1], similarly we can prove the following result as follows:

Theorem 4. Let R be a right s-unital ring satisfying (CA). Suppose further that R satisfies any one of the conditions (B_2) and (S_2). Then R is commutative (and conversely).

Remark 3. One might conjecture a possible generalization of Theorems 1 and 2 when R is left s-unital ring. The following example shows that there is a non-commutative left s-unital ring satisfying either (B_1) or (S_1).

Example 1. Let $R = \{\alpha = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \gamma = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \delta = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ be a subring of 2×2 matrices over GF(2). Then for any fixed positive integers m, n, p and $f(x)$ in $X^2\mathbb{Z}[X]$, R satisfies (B_1) or (S_1). However R is not commutative, although R is left s-unital.

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