IC/94/279
INTERNAL REPORT
(Limited Distribution)

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

TRACING A PLANAR ALGEBRAIC CURVE

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ABSTRACT

In this paper, an algorithm that determines a real algebraic curve is outlined. Its basic step is to divide the plane into subdomains that include only simple branches of the algebraic curve without singular points. Each of the branches is then stably and efficiently traced in the particular subdomain. Except for the tracing, the algorithm requires only a couple of simple operations on polynomials that can be carried out exactly if the coefficients are rational, and the determination of zeros of several polynomials of one variable.

MIRAMARE – TRIESTE
September 1994

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1. Introduction

In the papers [5] and [1], an efficient and reliable algorithm for finding the intersection of a cubic Bézier patch defined on a rectangle (on a triangle), and a plane was considered. With the help of Bezout resultants both results were extended in [2] to the general degree case. Here, the approach is carried one step further in order to provide a simple algorithm that computes a real algebraic curve without selecting its domain in advance. The basic idea is to use the topological properties of the given curve to divide the plane in subdomains in such a way that the structure of the restriction of the algebraic curve to each domain is rather simple. In particular, each subdomain can have only simple branches but no singular points. In each of subdomains then the curve sections are determined simply by tracing all the particular branches. This can be done very efficiently since there the tangent vector never vanishes. Numerical experiments show that the computation complexity is linear in the number of computed curve points, and for each point approximately six function evaluations are needed for quite a high precision.

2. The basic statements

Let

$$f : \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto f(x,y) := \sum_{i,j} f_{ij} x^i y^j = \sum_{i=0}^{n} a_i(y) x^i, \qquad (2.1)$$

$$a_i(y) := \sum_j f_{ij} y^j,$$

denote a given polynomial of degree $\leq n$ in the variable x with polynomial coefficients in y, i.e. for a fixed y

$$f \in \mathcal{P}_n := \{ \text{ a collection of real polynomials of degree } \leq n \}$$

and $a_i \in \mathcal{P}_m$ for some integer $m \geq 0$. For simplicity we shall throughout the paper assume

$$a_n(y) \not\equiv 0. \qquad (2.2)$$

Let S be the set of real zeros of f. The main step of the algorithm is to determine the sequence

$$Y := (y_i)_{i=1}^{\ell}, \quad -\infty < y_1 < \ldots < y_\ell < \infty \tag{2.3}$$

that pins down the changes of the topological structure of S in the y-direction. To be precise, the values $y_i \in \mathbb{R}$ are all such values which provide that all intersections between the straight line

$$y = \tilde{y}, \quad \tilde{y} \in (y_i, y_{i+1})$$

and each particular branch of the algebraic curve $f(x,y) = 0$ depend continuously on $\tilde{y}$. Additionally, each algebraic curve segment in (y_i, y_{i+1}) should contain no singular points, i.e. points at which the tangent vector vanishes. For example, consider $f(x,y) := (1-y)x^3 - y^2(1+y)$. At $(0,0)$ the algebraic curve is continuous, but not the tangent (figure 1). So the value 0 should be included in the corresponding Y. Similarly, with $f(x,y) := (1-y)x^2 - y^2(1+y)$, the values $-1, 1$ should be included in Y (Figure 2). The intersections of the curve with the line $y = \tilde{y}$ don't continue to both sides of these points. One might argue that the value $y = 0$ could be in Y considered extraneous since both branches of the curve could be traced along the tangent across the double (singular) point $(0,0)$. But the tangent vector at $(0,0)$ vanishes, and one does not know where to continue the curve from $(0,0)$ further. So 0 has also to be included in Y. The following theorem characterises (2.3) .

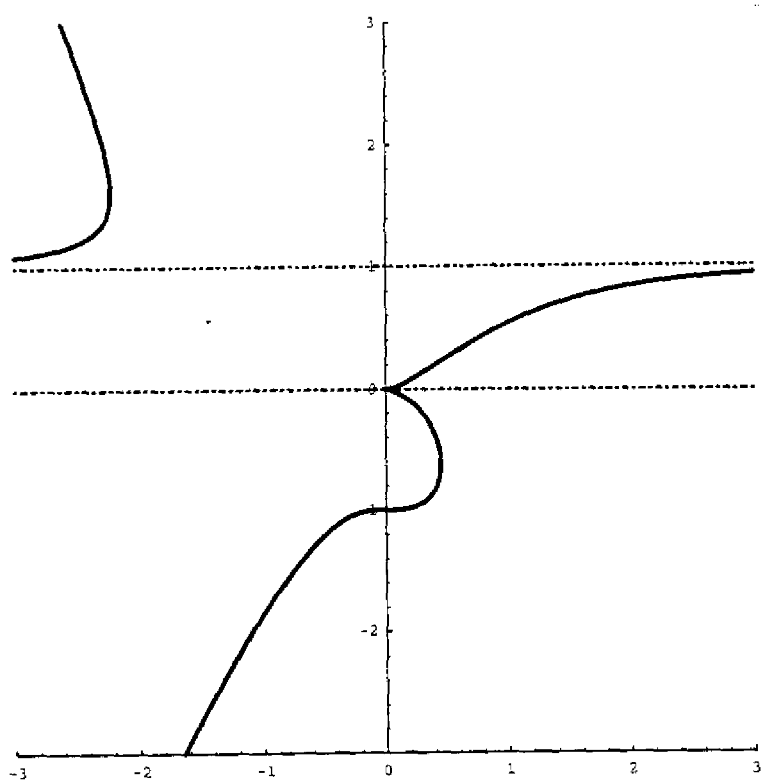

Figure 1: Change of the topological structure: the tangent changes its direction

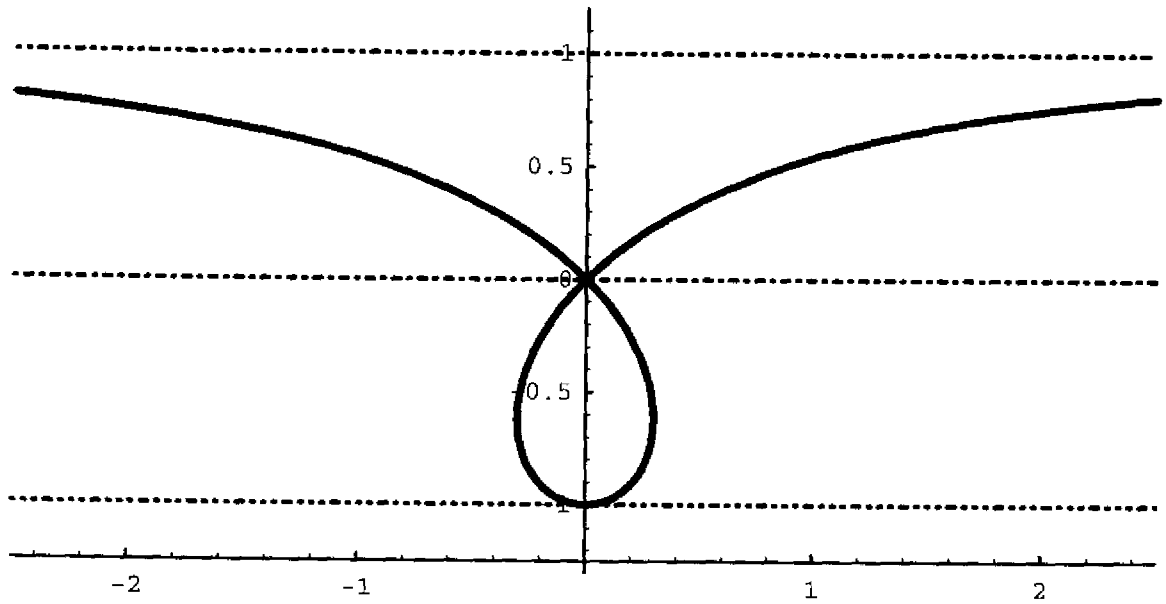

Figure 2: Change of the topological structure: the curve discontinuation

Theorem. Let $y = \tilde{y}$ be a line at which the topological structure of S changes. Suppose that it includes no singular points. Then at least one of the following conclusions must hold:

- The line $y = \tilde{y}$ is an asymptotic line of $f(x, y) = 0$.
- The polynomial of one variable $x \mapsto f(x, \tilde{y})$ has at least one zero of even multiplicity on this line.

Proof. If $y = \tilde{y}$ is not an asymptotic line then

$$\exists\ \tilde{x},\ -\infty < \tilde{x} < \infty$$

such that

$$\frac{\partial^i}{\partial x^i} f(\tilde{x}, \tilde{y}) = 0,\ i = 0, 1, \dots k - 1,$$

and

$$\frac{\partial^k}{\partial x^k} f(\tilde{x}, \tilde{y}) \neq 0$$

for some $1 \leq k \leq n$. If

$$\frac{\partial}{\partial y} f(\tilde{x}, \tilde{y}) \neq 0 \tag{2.4}$$

the implicit function theorem [4] provides us with the expansion of the algebraic curve near $(\tilde{x}, \tilde{y})$ as a function $y = y(x)$,

$$y(x) = \tilde{y} + \text{const } (x - \tilde{x})^k + \mathcal{O}((x - \tilde{x})^{k+1}), \text{ const } \neq 0.$$

The topology at the point concerned in this case changes iff k is even. If (2.4) is violated, $k = 1$ since $(\tilde{x}, \tilde{y})$ is not a singular point. But then one has expansion

$$x(y) = \tilde{x} + \text{const } (y - \tilde{y}) + \mathcal{O}((y - \tilde{y})^2), \text{ const } \neq 0$$

and the topology at $(\tilde{x}, \tilde{y})$ does not change. Thus at least one zero of f on the line concerned must be of even multiplicity. ■

It is now obvious how to construct the set Y. It should include all the y lines that contain the singular points (the set Y_1), the asymptotic lines (the set Y_2), and all the lines that contain points where the restriction of f has an even order zero (the set Y_3), and let $Y = Y_1 \cup Y_2 \cup Y_3$. Since f is a polynomial, this procedure is quite straightforward. Note that some algebraic curve branch might make no contribution to either of sets Y_1, Y_2, Y_3, i.e. it might be overlooked. A single branch curve defined by $f(x, y) := y^5 - x^3$ is an example of such a curve. However, due to the analytic nature of algebraic curves such a branch could be written as a function $x = x(y)$ that satisfies

$$\lim_{y \to \pm\infty} x(y) = \pm\infty$$

or

$$\lim_{y \to \pm\infty} x(y) = \mp\infty.$$

Thus if Y is not empty, such a branch would not be overlooked. If $Y = \emptyset$, any value can be added to Y to overcome this problem.

Let

$$\det R_n : \mathcal{P}_n \times \mathcal{P}_n \to \mathbb{R} : (p,q) \mapsto \det R_n(p,q) := \det(r_{ij}(p,q))_{i=0;j=0}^{n-1}$$

denote the Bezout resultant map ([3]). For a particular power basis representation

$$p(x) := \sum_{i=0}^{n} a_i x^i, \; q(x) := \sum_{i=0}^{n} b_i x^i$$

the elements of the symmetric matrix $R_n(p,q)$ read

$$r_{ij}(p,q) := \sum_{k=\max(n-i,n-j)}^{\min(n,2n-i-j-1)} \begin{vmatrix} a_k & a_{2n-i-j-k-1} \\ b_k & b_{2n-i-j-k-1} \end{vmatrix}. \tag{2.5}$$

If p,q are linearly independent, and at least one of exact degree n, they have a common zero iff the Bezout resultant $\det R_n(p,q)$ vanishes.

Let us denote

$$f_x(x,y) := \frac{\partial}{\partial x} f(x,y) = \sum_{i=0}^{n-1} (i+1) a_{i+1}(y) x^i,$$

$$f_y(x,y) := \frac{\partial}{\partial y} f(x,y) = \sum_{i=0}^{n} \frac{\partial}{\partial y} a_i(y) x^i,$$

and let

$$r_1(y) := \det R_n(f(\bullet,y), f_x(\bullet,y)),$$

$$r_2(y) := \det R_n(f(\bullet,y), f_y(\bullet,y))$$

be the polynomials obtained by taking the resultant map to the first variable.

Corollary. Let $f_x \not\equiv 0$. For any fixed y then the polynomial

$$x \mapsto f(x,y)$$

has a multiple zero iff

$$r_1(y) = 0.$$

Let also $f_y \not\equiv 0$. The tangent vector (f_x, f_y) to the curve $f(x,y) = 0$ vanishes for some x iff the polynomial

$$q_3 := \text{the greatest common divisor of } r_1, r_2$$

vanishes.

Proof. Since f, f_x are linearly independent if $f_x \not\equiv 0$ the first claim is obvious. Similarly for a tangent vector to vanish f, f_x as well as f, f_y must have the same common zero. But this implies that greatest common divisor has the same zero. ■

It is obvious that the resultant r_1 will vanish also at those lines that cover only singular points, or are asymptotic lines. However, it is worth to compute Y_1, Y_2 directly, since it can be done in a stable way (only the polynomials of low degree are involved). As a bonus, the degree of polynomial that determines Y_3 will be by this approach reduced too. In the following, the condition that determines when $y = \tilde{y}$ is an asymptotic line of $f(x,y) = 0$ will be recalled ([4]).

Lemma. Let $a_n \not\equiv 0$, and let

$$a_1, a_2, \ldots, a_n$$

T

have no common divisor. Then

$$a_n(\tilde{y}) = 0 \tag{2.6}$$

iff $y = \tilde{y}$ is an asymptotic line of $f(x,y) = 0$ or $(\pm\infty, \tilde{y})$ is an isolated singular point of the algebraic curve.

Proof. Clearly the function

$$\frac{f(x,y)}{x^n} = a_n(y) + \frac{a_{n-1}(y)}{x} + \ldots + \frac{a_0(y)}{x^n} =: a_n(y) + g(x,y)$$

for $x \neq 0$ defines the same curve as f. Since

$$\lim_{x \to \pm\infty} g(x, \tilde{y}) = 0$$

the condition (2.6) is necessary. On the other hand, g is of the same sign for y close enough to $\tilde{y}$, and $|x|$ large enough. If the equation $a_n(y) + g(x,y) = 0$ has a solution, it will have a solution for larger x. This implies $\tilde{y}$ is an asymptotic line. Otherwise one has an isolated singular point at infinity. ■

3. The algorithm

Let us now recall the complete algorithm. Here S will represent the computed (discrete) representation of the algebraic curve.

1. Find the greatest common divisor of $a_0, a_1, \ldots, a_n$, say q_0. If $q_0 \neq$ const, divide

   $$a_i := \frac{a_i}{q_0}, \; i = 0, 1, \ldots, n$$

   and add the real zeros of q_0 to the set S as lines parallel to the x-axes.

2. Compute the resultants r_1, r_2, and their greatest common divisor q_1. Calculate the lines with singular points

$$Y_1 = \{y \in \mathbb{R} \mid q_1(y) = 0\}.$$

While q_1 divides r_1, simplify $r_1 := \frac{r_1}{q_1}$.

3. Determine the asymptotic lines

$$Y_2 = \{y \in \mathbb{R} \mid a_n(y) = 0\}.$$

While a_n divides r_1, simplify $r_1 := \frac{r_1}{a_n}$.

4. Compute the real zeros of the reduced resultant

$$Y_3 = \{y \in \mathbb{R} \mid r_1(y) = 0\}.$$

Let $Y := Y_1 \cup Y_2 \cup Y_3 = \{ y_1 < y_2 < \ldots < y_\ell \}$. Add the set Y to S. Let $[y_0, y_{\ell+1}]$, $y_0 < y_1$, $y_{\ell+1} > y_\ell$ denote the region where the algebraic curve has to be computed. Put $Y := Y \cup \{y_0, y_{\ell+1}\}$.

5. Determine the algebraic curve for each of the subdomains

$$(y_i, y_{i+1}), \ i = 0, 1, \ldots, \ell.$$

Let us add a few comments. It is obvious that one could improve the selection of points in Y_3 by excluding the even order zeros (theorem). The price one would have to pay is to compute the real zeros of $f(x, y_i)$, and for each computed zero determine if it is odd or even.

Steps 2 and 3 simplify the resultant r_1. However, only synthetic division is used (that can be carried exactly in the field of rational numbers). For the sake of stability one avoids division by computed linear factors $y - y_i,\ y_i \in Y_j$ even thought r_1 could be this way further reduced.

The step 5 is carried out by the standard Newton iteration. At first, the starting points of each algebraic curve branch are determined by calculating the real roots

$$x_{ij}, j = 1, 2, \ldots, k_i$$

of $f(x, y_{i+\frac{1}{2}}) = 0$. Then, for each x_{ij} the simple curve branch is followed to the boundaries y_i, y_{i+1}. There are quite a lot known ways to follow a curve. The following is a simple, but efficient one. Let (x, y) be a given point on the curve. Then the Taylor expansion

$$f(x + \Delta x, y + \Delta y) = f(x, y) + f_x(x, y)\Delta x + f_y(x, y)\Delta y + \ldots$$

provides a very good starting guess for the next point of the curve $(x + \Delta x, y + \Delta y)$ which is h away from the previous (in the Euclidian norm $\|\ \|_2$), i.e.

$$\begin{pmatrix} \Delta x^{(0)} \\ \Delta y^{(0)} \end{pmatrix} := \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \frac{h}{\sqrt{f_x^2(x, y) + f_y^2(x, y)}} \begin{pmatrix} -f_y(x, y) \\ f_x(x, y) \end{pmatrix}.$$

Note that one makes the use of the fact that the tangent vector never vanishes. The initial guess usually needs only a couple of Newton corrections to obtain the required accuracy:

$$\begin{pmatrix} \Delta x^{(j+1)} \\ \Delta y^{(j+1)} \end{pmatrix} = \begin{pmatrix} \cos\omega^{(j)} & \sin\omega^{(j)} \\ -\sin\omega^{(j)} & \cos\omega^{(j)} \end{pmatrix} \begin{pmatrix} \Delta x^{(j)} \\ \Delta y^{(j)} \end{pmatrix}, \quad j = 0, 1, \ldots. \tag{3.7}$$

Here

$$\omega^{(j)} := \frac{f}{-\Delta y^{(j)} f_x + \Delta x^{(j)} f_y}$$

with the function and the derivatives evaluated at

$$(x + \Delta x^{(j)},\ y + \Delta y^{(j)}).$$

The iteration stops when the corrections

$$(\Delta x^{(j)},\ \Delta y^{(j)}),\ \ (\Delta x^{(j+1)},\ \Delta y^{(j+1)})$$

differ less then ε in the ℓ_2 norm.

The choice of dividing the plain along y direction was quite arbitrary. The role of x and y can be of course reversed. More generally, any division that covers all the plane would do. As an example take the slope parametrisation $y = sx$. The algorithm would then look for the set of slopes $\{s_i\}$ such that polynomial $x \mapsto f(x, s_i x)$ would have a double zero.

4. The examples

Let us first carry out the steps of the algorithm for the curves of figures 1 and 2, where the lines that belong to Y_i are denoted by i consecutive dots. As for the first one, $f(x,y) := (1-y)x^3 - y^2(1+y)$. Since $q_0 = 1$ we proceed to step 2. One obtains the resultants

$$r_1(y) = -27(1-y)^4 y^4 (1+y)^2,$$

$$r_2(y) = 8y^3(1+y-y^2)^3.$$

This gives $q_1(y) = y^3$, and $Y_1 = \{0\}$. $a_2(y)$ is equal to $1-y$. Thus $Y_2 = \{1\}$, and r_1 divided by a_2^3 now reads $r_1(y) = y(1+y)^2$. This reveals $Y_3 = \{-1, 0\}$. If one bothers to exclude the lines with only odd zeros, one computes that

$$f(x,-1) = 2x^3$$

has only one triple zero (at 0). This leaves Y as $\{0,1\}$. Otherwise $Y = \{-1,0,1\}$ would produce only a little additional work.

The second algebraic curve is defined as $f(x,y) := (1-y)x^2 - y^2(1+y) = 0$. The resultants are computed as

$$r_1(y) = 4(1-y)^3 y^2 (1+y),$$
$$r_2(y) = -4y^2(1+y-y^2)^2.$$

Their greatest common divisor is $q_1(y) = 4y^2$. That gives $Y_1 = \{0\}$. and simplifies r_1 to $(1-y)^3(1+y)$. Also $Y_2 = \{1\}$, and r_1 is further simplified to $1+y$, what provides the only point of Y_3, and $Y = \{-1, 0, 1\}$.

As to the step 5, let us take the first example, and follow the curve branch that makes its way up to the singular point $(0,0)$. Let us assume that one has chosen $y_0 = -20$, and the procedure follows the curve branch from $(-4.34127, -10)$ up to $(0,0)$ for different choices of the step

$$h = 0.001, 0.002, \ldots, 0.1$$

and three different precisions $\varepsilon = 10^{-4}, 10^{-5}, 10^{-6}$. The figure 3 clearly shows that for small values of h the average number of the function evaluations is three. Then it makes its way up to six, with the rate depending on ε. Each Newton step requires three evaluations (f, f_x, f_y), so the iteration (3.7) is repeated only (at most) twice on average in order to achieve the required precision. This numerical evidence shows that in practice the computational complexity is linear in the number of curve points computed.

At the end, we add a more elaborate example, i.e. an algebraic curve determined by a polynomial of total degree 9,

$$\begin{aligned} f(x,y) \;:=\; & 72x^4 - 72x^5 - 120x^6 + 120x^7 + 48x^8 - 48x^9 + 84x^3y - 140x^5y \\ & + 56x^7y - 138x^2y^2 - 102x^3y^2 + 284x^4y^2 + 116x^5y^2 - 152x^6y^2 \\ & - 8x^7y^2 - 21xy^3 + 98x^3y^3 - 84x^5y^3 + 30y^4 + 30xy^4 - 158x^2y^4 \\ & - 122x^3y^4 + 120x^4y^4 + 120x^5y^4 - 21xy^5 + 30y^6 + 30xy^6. \end{aligned}$$

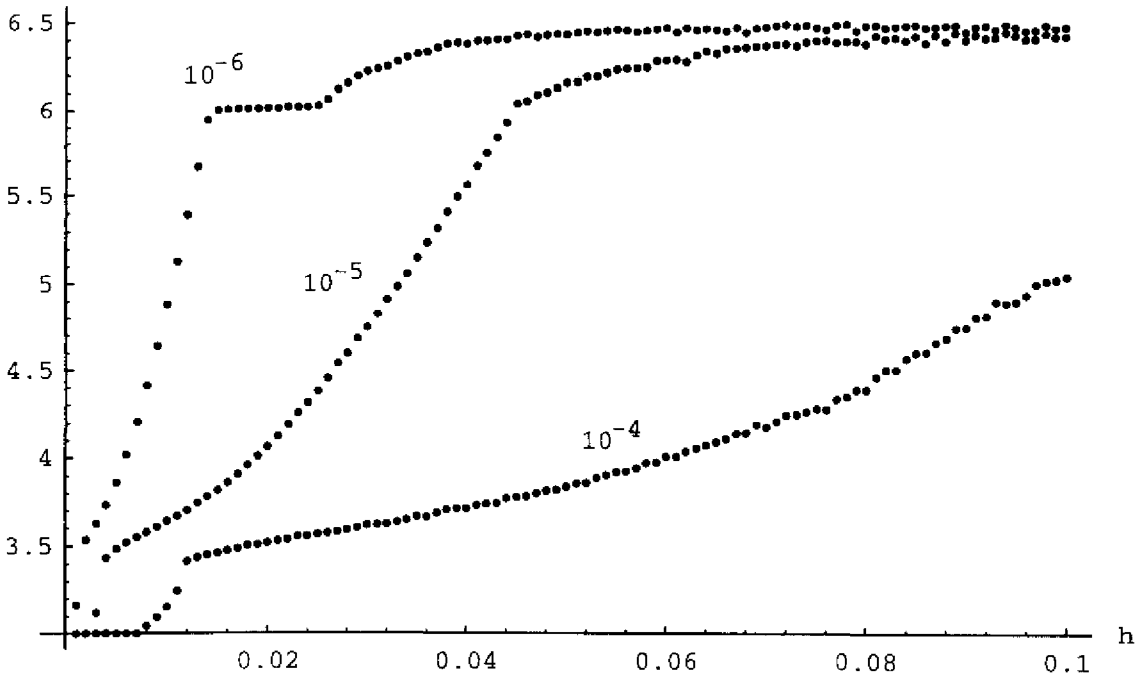

Figure 3: The average number of functions evaluations

The algorithm determines

$$
\begin{aligned}
Y_1 &= \{ -0.7026744505578 31,\ 0.0,\ 0.3607644111132052, \\
&\qquad 0.950765524387607 \}, \\
Y_2 &= \emptyset, \\
Y_3 &= \{ -1.0,\ -0.1368007417941486,\ 0.4159329879506296,\ 1.0 \}.
\end{aligned}
$$

The figure 4 shows the computed curve, as well as the lines $y \in Y$.

Acknowledgments

One of the authors (Y.Y.F.) would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste. (F.L.C.) supported by NSF and SF of National Educational Committe of China. (J.K.) supported by the Ministry of Science and Technology of Slovenia.

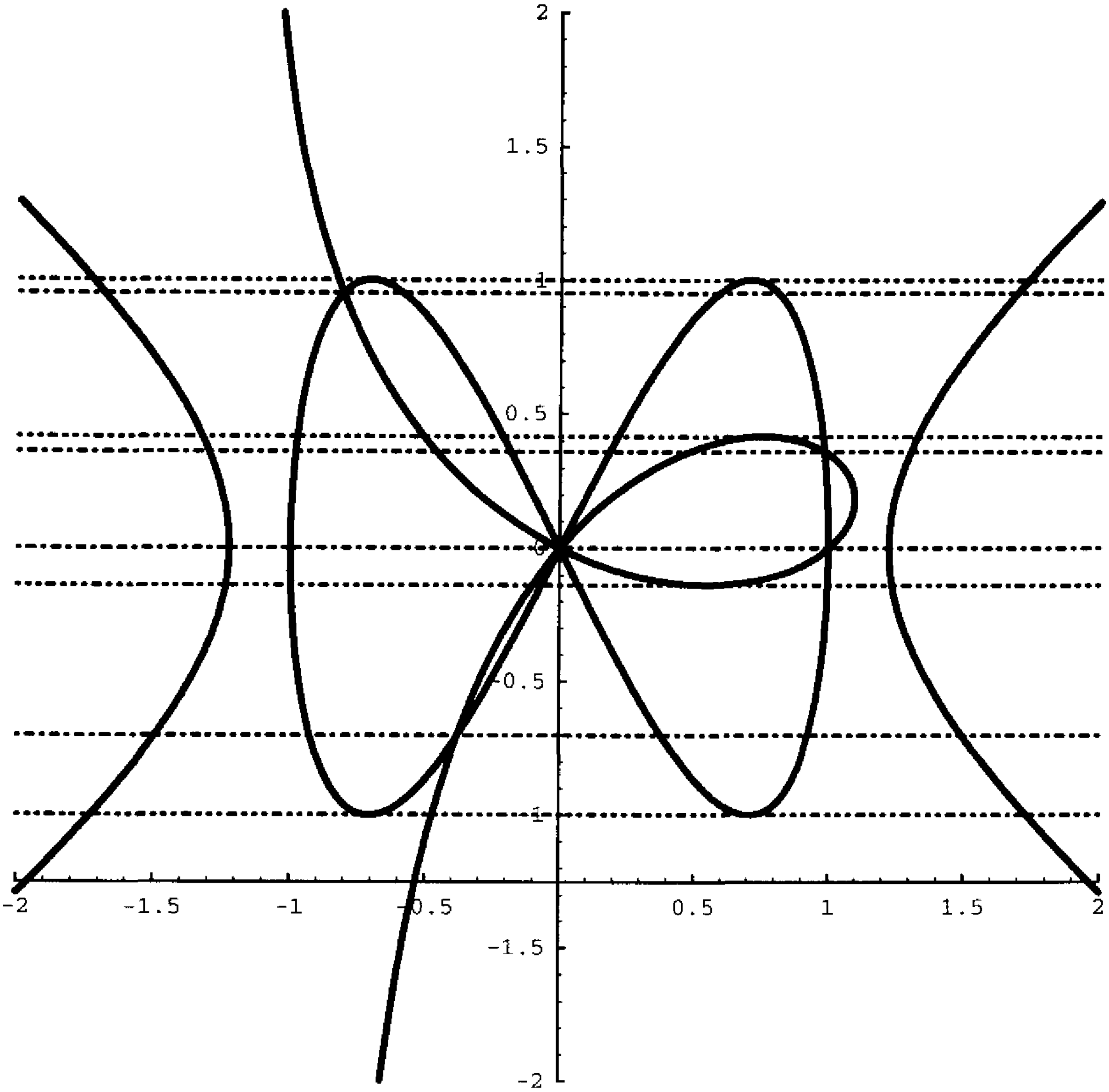

Figure 4: The algebraic curve of the third example

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