IC/94/314

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

EXACT SMOOTH CLASSIFICATION OF HAMILTONIAN VECTOR FIELDS ON SYMPLECTIC 2-MANIFOLDS

B.S. Krouglikov

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

MIRAMARE-TRIESTE

IC/94/314

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

EXACT SMOOTH CLASSIFICATION OF HAMILTONIAN VECTOR FIELDS ON SYMPLECTIC 2-MANIFOLDS

B.S. Krouglikov [1]
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

Complete exact classification of Hamiltonian systems with one degree of freedom and Morse Hamiltonian is carried out. As it is a main part of trajectory classification of integrable Hamiltonian systems with two degrees of freedom, the corresponding generalisation is considered. The dual problem of classification of symplectic form together with Morse foliation is carried out as well.

MIRAMARE – TRIESTE
October 1994

[1] Permanent address: Department of Differential Geometry and Application, Faculty of Mechanics and Mathematics, Moscow State University, Moscow 119899, Russian Federation. E-mail address: borkru@difgeo.math.msu.su

§0. Introduction.

In the work [BF] a complete trajectory classification on isoenergy surfaces of general integrable Hamiltonian systems with two degrees of freedom in category C^0 has been carried out. It turns out that a considerable part of the problem is an exact classification (in category C^0) of general Hamiltonian systems with one degree of freedom near the critical leaf of the foliation of the surface by the trajectories. There is also such a reduction in category of C^k-smooth maps, see [B]. To be more exact the problem is as follows.

Let us consider a connected compact orientable 2-manifold P^2 with the boundary $\partial P^2 = \cup S^1$ which possesses a Morse function (Hamiltonian) F with only one hyperbolic (elliptic case is much more easier and can be found in [I] in general (any degree of freedom) analytical case and in [E] in general smooth) critical value (we will suppose it to be zero) and which is constant on the boundary circles (we will suppose $F|_{\partial P^2} = \pm 1$). So P^2 contains a graph $K = F^{-1}(0)$ with vertices of multiplicity 4 as deformational retract and all the boundary annuli $P^2 \backslash K$ divide onto positive ($F > 0$) and negative ($F < 0$), to each edge of K annuli of different signs adjoining (such a P^2 one calls a letter). Let us consider a symplectic form (volume form) ω on P^2 and the corresponding Hamiltonian vector field $\mathrm{sgrad}_\omega F$. One has a problem of exact classification of germs of such fields on K. It was solved completely in category C^0 in the [BF] (i.e. there was carried out a classification of jets on $K \subset P^2$ of Hamiltonian vector fields of above indicated type up to a homeomorphism that moves trajectories to trajectories and preserves time). The present paper exhibits a new way of solving and by means of it the classification is obtained in category $C^k, k = 1, \ldots, \infty$, a part of theorems being true in the same wording also when $k = 0$.

The main theorem for the problem (let us call it (P_1)) will be preformulated in §3, its final form is theorem 5.7, in §§ 1, 2 all the necessary for that is defined. In §6 a generalization of the main theorem for (P_1) is given which permits one to extend the theory of trajectory classification to the case of systems with hyperbolic critical circles with nonorientable separatrix diagram (systems with stars).

Let us consider a more general problem, (P_2). Let us have a (connected) compact symplectic surface (V^2, ω). For any Morse function $F : V^2 \to R$, which is constant on boundary components $\partial V^2 = \cup S^1$, we associate the Hamiltonian flow $v = \mathrm{sgrad}_\omega F$. The problem $(P_2)_k$ is the classification of pairs (V^2, v) in the category $C^k, k = 1, \ldots, \infty$. The main theorem for $(P_2)_k$ is given in §7. In §8 we will consider the dual problem, $(P_3)_k$, i.e. there we will consider the classification of triples $(V^2, \mathcal{L}, \omega)$, where $\mathcal{L}$ is the foliation of V^2 nonsingular fiber of which are trajectories of some Hamiltonian system v from (P_2) tangent to the boundary if $\partial V^2 \neq \emptyset$ (to be more exact the foliation by connected components of level lines of some multivalued Morse function F).

We will write "smooth" instead of "C^∞ – smooth" and "have k-jet zero", $j_k() = 0$, instead of "have zero of order k".

§1. Reformulation of the problem (P_1) and necessary conditions.

Let us denote by $\Pi_{\omega,i}(F_0)$ the period of passing the line $\{F = F_0\}_i$ by the vector field $\mathrm{sgrad}_\omega F$. $\Pi_{\omega,i}(F_0)$ is finite when $F_0 \neq 0$, i.e. when $\{F = F_0\}_i$ is a closed line. It is clear that if there exists an isomorphism Φ between (P^2, F, ω) and $(\tilde{P}^2, \tilde{F}, \tilde{\omega})$, then $\Pi_{\Phi^*\tilde{\omega},\tilde{i}}(\Phi^*\tilde{F}) \equiv \Pi_{\omega,i}(F)$ (hereinafter by isomorphism one means a germ of it, i.e. an isomorphism between $F^{-1}(-\varepsilon_1, \varepsilon_2)$ and $\tilde{F}^{-1}(-\tilde{\varepsilon}_1, \tilde{\varepsilon}_2)$). The function $\Pi_{\omega,i}(F)$ is monotone by $|F|$ and $\Pi_{\omega,i}(0) = \infty$. Hence there always exists a homeomorphism Φ satisfying the condition above. In [B] A.V.Bolsinov found the existence condition for such a diffeomorphism of arbitrary smoothness C^k, $k = 1, \dots, \infty$. Because of the noted above we may think that $\tilde{P}^2 = P^2, \tilde{F} = F$. Besides, let Φ looked for preserve vertices, edges K and ends of P^2 (i.e. components of ∂P^2) (it is clear this is not an essential restriction). Then the first necessary condition must held:

$$\Pi_{\tilde{\omega},i}(F) = \Pi_{\omega,i}(F) \tag{1.1}$$

So we receive the following reduced formulation of the problem. Let us consider P^2 as above, $F : P^2 \to R$, 0 is a unique (hyperbolic) critical value for F. Let one have two symplectic forms ω and $\tilde{\omega}$ on P^2 that induce the same orientation on P^2 (the general case is easily reduced to that one). When does an automorphism Φ (of $C^k class$) P^2 preserving F, vertices and edges of K and mapping $\tilde{\omega}$ to ω (or equivalently $\Phi_*(\mathrm{sgrad}_\omega F) = \mathrm{sgrad}_{\tilde{\omega}} F$) exist ? Let us call the problem $(P_1{}')$. The first necessary condition (1.1) can be reformulated in the following way. Let $C_i(\varepsilon)$ be the i-th connectivity component of the set $F^{-1}(0, \varepsilon)$ (or $(\varepsilon, 0)$, if $\varepsilon < 0$). Then we will necessary have:

$$\mathrm{vol}_\omega C_i(F) = \mathrm{vol}_{\tilde{\omega}} C_i(F) \tag{1.1'}$$

Let us consider an arbitrary vertex A of the graph K. As it was showed in [CVV] there exist local coordinates (x, y) such that $F = xy$ and $\omega = \Lambda(F)\, dx \wedge dy$, $\Lambda(0) > 0$. It is easy to check the fulfillment of the following (see.[BF],B)

Lemma 1.1. *The time of passing near the vertex A of the flow* $\mathrm{sgrad}_\omega F$ *is equal to* $-\Lambda(F) \ln |F| + c(F)$ *where c is a smooth function.*

Remark 1.2. Two terms in the sum above are respectively called infinite (unsmooth) and finite parts of the period function.

Therefore if we expand $\Lambda(F)$ into a Taylor series $\sum_{i=0}^{\infty} \Lambda_i F^i$ we will obtain a system $\{\Lambda_i\}_{i=0}^{\infty}$ of invariants of autosymplectomorphism Φ. Let $\Lambda_{A,m}(t)$ denote the sum $\sum_{i=0}^{m} \Lambda_i t^i$. Then it is easy to check that the second necessary existence condition for C^k autosymplectomorphism Φ is

$$\Lambda_{A_j,k}(t) \equiv \tilde{\Lambda}_{A_j,k}(t) \tag{1.2}$$

for every vertex A_j.

§2. The beginning of proving the main theorem for $(P_1{}')$ and the third necessary condition.

In order to exist for C^k-isomorphism between (P^2, F, ω) and $(P^2, F, \tilde{\omega})$ besides necessary conditions (1.1) and (1.2) one must require the fulfillment of one more condition. We will formulate it later and now let us commence constructing the isomorphism. ω and $\tilde{\omega}$ can be connected by homotopy in the class of symplectic forms $\omega_t = t\tilde{\omega} + (1-t)\omega$ since ω and $\tilde{\omega}$ define the same orientation. $H^2(P^2) = H^2(K) = 0$, hence $\dot{\omega}_t = \tilde{\omega} - \omega = d\alpha$ and following [M] let us note that shift by the time t along the time-dependent vector field u_t, i.e. a family of automorphisms $\Phi_t, t \in [0, 1], \Phi_0 \equiv id, \frac{d}{dt}\Phi_t = u_t \circ \Phi_t$, where

$\omega_t(u_t, \star) = -\alpha(\star)$, maps ω_t into $\omega_0 = \omega$. Φ_1 being automorphism will be a consequence of the preservation of F, and for this evidently it is sufficient to require the preservation of F by the flow u_t with a successful choice of α. We will show below that it is possible to make such a choice for isomorphic triples (P^2, F, ω). The preservation of the function F by the shift u_t is equivalent to the condition $u_t = f_t\, \mathrm{sgrad}_\omega F$, i.e. $\alpha(\mathrm{sgrad}_\omega F) = 0$. Thus the problem $(P_1{}')$is reduced to a search of 1-form α of the class C^k such that

$$\begin{cases} d\alpha = \tilde{\omega} - \omega \\ \alpha(\mathrm{sgrad}_\omega F) = 0 \end{cases} \tag{2.1}$$

Lemma 2.1. *Let the conditions (1.1) and (1.2) hold. Then there exists a smooth 1-form α such that $d\alpha = \tilde{\omega} - \omega$, $\alpha(x) = 0\ \forall x \in K$ and α has (k+1)-jet zero in every vertex: $j_{k+1}\alpha(A_s) = 0$.*

Proof. As it was mentioned before there exists a 1-form α such that $d\alpha = \tilde{\omega} - \omega$. In order to satisfy the two other conditions we will change it by closed 1-forms. Let us orient the edges of the graph K by means of the flow $\mathrm{sgrad}_\omega F$ (or equivalently by means of $\mathrm{sgrad}_{\tilde{\omega}} F$). Consider $\alpha|_K$, we are going to make $[\alpha|_K] = 0 \in H^1(K; R) = H^1_{DR}(K)$ (since the graph has only double transversal selfintersection points it is easy to give a sense to the symbol). For any edge $e \subset K$ define $\alpha(e) = \int_e \alpha = a \neq 0$. Choose an interval $I \Subset e$ and find $f : I \to R$ — such a smooth function that it is identically zero at the beginning of I (it inherits an orientation from e) and is identically a at the end (Fig. 1). Extend it onto a neighborhood $U = I \times J$ of the interval I in P^2 (Fig. 2) by means of projection $U \to I$ and redefine $\alpha \stackrel{\mathrm{def}}{=} \alpha - df$, where 1-form df is the differential of f on U and 0 outside. Then for a new 1-form α we have $\alpha(e) = 0$. Doing such an operation for each edge e we will obtain $[\alpha|_K] = 0 \in H^1(K; R)$.

As it follows from $[\alpha|_K] = 0$ there exists a C^∞-function $f|_K : K \to R$ such that $d(f|_K) = \alpha|_K$. Extend $f|_K$ to a smooth f on P^2 so that $df(x) = \alpha(x), \forall x \in K$. It is possible to do that outside vertices since we might extend it onto $P^2 \setminus K$ along normals (for any metric) $n(x)$ to K with the condition $df(n(x)) = \frac{df}{dn}(x) = \alpha(n(x))$. The normals will intersect near the vertices and one needs an agreement condition for extensions. But this may be written as:

$$d(df)_A = d\alpha|_A = \tilde{\omega}_A - \omega_A = (\tilde{\Lambda}_0 - \Lambda_0)\ dx \wedge dy|_A = 0$$

for any vertex A. In detail: suppose K near A is given by $xy = 0$, (x, y) being local coordinates, and $\alpha = a\,dx + b\,dy$. Then $\frac{\partial}{\partial x}(f|_K) = a$, $\frac{\partial}{\partial y}(f|_K) = b$, i.e. $\frac{\partial f}{\partial x}(x, 0) = a(x, 0)$, $\frac{\partial f}{\partial y}(0, y) = b(0, y)$ and we may define f with $xy \neq 0$ by formula

$$\begin{aligned} f(x, y) = {} & f(x, 0) + f(0, y) - f(0, 0) + x(a(0, y) - a(0, 0)) \\ & + y(b(x, 0) - b(0, 0)) - zxy, \ z = \frac{\partial a}{\partial y}(0, 0) = \frac{\partial b}{\partial x}(0, 0) \end{aligned}$$

Now if $\alpha \stackrel{\mathrm{def}}{=} \alpha - df$ then $\alpha(x) = 0\ \forall x \in K$ and $d\alpha = \tilde{\omega} - \omega$.

The rest of the lemma is local. Namely we will show that near a vertex A of the graph there is a smooth 1–form $\tilde{\alpha}$ which coincides with α outside a small neighborhood of the point A, $j_{k+1}\tilde{\alpha}(A) = 0$, and which satisfies $d\tilde{\alpha} = d\alpha$. Let $\alpha_{(i)}$ be a 1–form, having i-jet 0 in zero and such that $d\alpha_{(i)} = d\alpha$, $i \leq k$. Consider $\alpha_{(i)} = \beta_i + \gamma_i$, where $\beta_i = \sum\limits_{k+l=i+1} a_{kl}x^k y^l\, dx + \sum\limits_{k+l=i+1} b_{kl}x^k y^l\, dy$, $j_{i+1}\gamma_i(0) = 0$. Since $d\alpha_{(i)}$ has k–jet 0 in zero, the same is also true for $d\beta_i$, and hence $d\beta_i = 0$ and $\beta_i = dh_{(i)}$, $h_{(i)}(0,0) = h_{(i)}(A) = 0$. Choose a Euclidean norm $|\ |$ in coordinates (x, y) in a neighborhood of the point A and let $\gamma_{\varepsilon_1,\varepsilon_2} : O(A) \to R$ be such a smooth function, $\varepsilon_1 < \varepsilon_2$, that $\gamma_{\varepsilon_1,\varepsilon_2}(z) \equiv 1, |z| < \varepsilon_1$ and $\gamma_{\varepsilon_1,\varepsilon_2}(z) \equiv 0$, $|z| > \varepsilon_2$. Denote $h_i^{(*)} = \gamma_{1/i,2/i} \cdot h_{(i)}$ and $\alpha_{(i+1)} \overset{\text{def}}{=} \alpha_{(i)} - dh_i^{(*)}$. If k is finite then the process will be finished in the finite number of steps and $\tilde{\alpha} = \alpha_{(k+1)}$. Otherwise, forms $\alpha_{(-1)} = \alpha, \alpha_{(0)}, \ldots$ converge to $\alpha_{(\infty)} = \alpha - \sum\limits_{i=0}^{\infty} dh_i^{(*)}$, which has the jet 0 in zero; the smoothness of $h = \sum\limits_{i=0}^{\infty} h_i^{(*)}$ in zero can be obtained in standard way, $\tilde{\alpha} = \alpha_{(\infty)}$. □

Remark 2.2. Any other 1–form $\hat{\alpha}$ satisfying the conditions of the lemma may be obtained this way: $\hat{\alpha} = \alpha - df$, where $f : P^2 \to R$ is such a smooth function that $df(x) = 0\ \forall x \in K$ and $j_{k+1}f(A_j) = 0$.

Let us now begin constructing the third invariant which will distinguish nonequivalent triples (P^2, F, ω). $P^2 \setminus K$ is a union of annuli C_i. Choose for every annulus $C_i = S^1 \times I \to I$ a transversal for some edge e_{j_i} section β_i ($\beta_i \cap e_{j_i} = pt_i$) and define a continuous function $\varkappa : \bar{C}_i \to R$ this way:

$$\varkappa|_K = 0; \quad \varkappa|_{\beta_i} = g_i; \quad g_i \in C^\infty(\beta_i); \quad g_i(pt_i) = 0; \quad \varkappa = \int_l \alpha + g_i, \tag{2.2}$$

where l is an interval of the integral trajectory of the field $\operatorname{sgrad}_\omega F$, connecting the point with a point on the section β_i. Let us check out the correctness of this definition. Let $C_i(\varepsilon) = \{F = \varepsilon\}_i$ and $\partial C_i(\varepsilon) = \gamma_i(\varepsilon) - \gamma_i(0)$ be a union of two boundary circles, $\gamma_i(0) \subset K$, $\gamma_i(\varepsilon) \subset C_i \setminus K$, oriented by means of $\operatorname{sgrad}_\omega F$. Then by the Stocks' formula:

$$\int_{\gamma_i(\varepsilon)} \alpha = \int_{\gamma_i(0)} \alpha + \iint_{C_i(\varepsilon)} (\tilde{\omega} - \omega) = \operatorname{vol}_{\tilde{\omega}} C_i(\varepsilon) - \operatorname{vol}_\omega C_i(\varepsilon) = 0.$$

C^{k+1}-smoothness of the function $\varkappa$ on $\bar{C}_i$ outside vertices of the graph K is evident and in the vertices follows from the (k+1)–jet of α being equal to zero:

Lemma 2.3. *Let α be a smooth 1-form on R^2 with s-jet 0 in zero and be vanishing when $xy = 0$. Consider a function $\varkappa : \{x \geq 0\} \to R$ defined by conditions $\varkappa = 0$ when $x(1-x)y = 0$ and $\varkappa(x,y) = \int_l \alpha$, l is an interval of trajectory of the vector field $x\partial_x - y\partial_y$ connecting the points (x,y) and (1,xy). Then $\varkappa$ is C^s-smooth in the half-plane.*

Thus we have received a continuous function $\varkappa : P^2 \to R$.

For each edge e_m let us choose sections s_m transversal to the flow $\operatorname{sgrad}_\omega F$, we will call them cuts. Each cut has a positive part s_m^+, where $F \geq 0$, and a negative one s_m^-; $s_m^+ \cap s_m^- = s_m^0 \in e_m$, $s_m^+ \cup s_m^- = s_m$(see Fig. 3). Expand $\varkappa$ into a Taylor series by F

on $s_m^\pm$ in a neighborhood of the point s_m^0. We can do it up to the (k+1)-th term since $\varkappa|_{\bar{C}_i}$ is C^{k+1}-smooth:

$$\varkappa^\pm = \sum_{i=1}^{k+1} \delta_i^\pm F^i + \bar{o}(|F|^{k+1}) \text{ or } \doteq \sum_{i=1}^{\infty} \delta_i^\pm F^i, \ k = \infty.$$

Definition 2.4. $\delta_k^m(t) = \sum\limits_{i=1}^{k+1} (\delta_i^+ - \delta_i^-) t^{i-1}$, where m counts edges.

Associate to a system $(P^2, F, \omega, \tilde{\omega})$ graph Γ, dual to K, the vertices of which are boundary circles $S_j^1 \subset \partial P^2$ and the edges are oriented (by means of F) cuts. One may think $\Gamma \subset \tilde{P}^2$, where $\tilde{P}^2$ is P^2 with the boundary components glued by discs D^2: $\tilde{P}^2 = P^2 \bigcup\limits_{\partial P^2} (\cup D^2) = P^2/\{S_j^1 \ni x \sim y \in S_j^1\}$. Calling vertices and edges of the graph Γ respectively 0- and 1-cells and neighborhood of vertices of the graph K — 2-cells we obtain a cell decomposition of $\tilde{P}^2$ that is dual by Poincaré to the decomposition (vertices of K), (edges of K), $\tilde{P}^2 \setminus K = \cup D^2$. Associate now to every oriented cut s_m the polynomial (or the series) $\delta_k^m(t)$. It is easy to see that obtained cochain $\delta_k(\omega, \tilde{\omega})$ (with the values in polynomials $R_k[t] \simeq R^k$ or in series $R[t] \simeq R^\infty$) is a cocycle. Moreover, holds the following

Proposition 2.5. *A system $(P^2, F, \omega, \tilde{\omega})$ uniquely defines an element $Z_k(\omega, \tilde{\omega}) = [\delta_k(\omega, \tilde{\omega})] \in H^1(\tilde{P}^2; R_k[t])$.*

Indeed, the cocycle δ does not change when one replaces $\alpha \mapsto \alpha - df$, $f \in C^\infty(P^2)$ (see remark 2.2), replaces g_i and the cuts s_m. And if the sections β_j on the annulus C_j are replaced, a polynomial or series is added to every cut incident to the annulus C_j (a j–th vertex of the graph Γ), the same to all, i.e. the cocycle changes by coboundary.

Proposition 2.6. *The following formulas hold:*

1. $Z_k(\omega, \omega) = 0$
2. $Z_k(\omega, \omega') + Z_k(\omega', \omega'') = Z_k(\omega, \omega'')$
3. *If an automorphism Φ preserves vertices of K then $Z_k(\omega, \Phi^*\omega) = 0$; in other words: $Z_k(\omega, \omega') = Z_k(\Phi^*\omega, \omega')$.*

Proof. It is needed only for the last formula. We can represent ω and $\omega' = \Phi^*\omega$ in the forms $\omega = d\alpha$, $\omega' = d\alpha'$, $\alpha' = \Phi^*\alpha$. Let $\hat{\omega}$ be a 2–form with the same invariants Π and Λ as in the case of ω and so localized near the vertices that its support does not intersect neither cuts nor sections. Then:

$$Z_k(\omega, \hat{\omega}) = Z_k(d\alpha, \hat{\omega}) = Z_k(d\alpha', \hat{\omega}) = Z_k(\omega', \hat{\omega}). \qquad \square$$

Remark 2.7. The proposition suggests an idea that $Z_k(\omega, \omega') = Z_k(\omega) - Z_k(\omega')$. As we shall see in the next section this is true.

Thus we have found the last necessary condition:

$$Z_k(\omega, \tilde{\omega}) = 0, \tag{2.3}$$

and now we are able to formulate the main theorem.

§3. THE MAIN THEOREM FOR (P_1').

Theorem 3.1. *Triples (P^2, F, ω) and $(P^2, F, \tilde{\omega})$ C^k- equivalent ($k = 1, \dots, \infty$) iff (1.1),(1.2) and (2.3) hold.*

Proof. The necessity was investigated in §§1,2 so let us pass to the sufficiency. As explained above it is sufficient to find a 1-form of the smoothness C^k such that (2.1) holds. Consider α from the lemma 2.1. Since $0 = Z_k(\omega, \tilde{\omega}) = Z_k(0, d\alpha)$ there exists a function $\varkappa$ on P^2, C^∞-smooth on $P^2 \setminus K$ and C^{k+1}-smooth on the edges.

Lemma 3.2. *If a function is C^k-smooth in half-planes $\{x \geq 0\}$, $\{y \geq 0\}$, $\{x \leq 0\}$, $\{y \leq 0\}$, then it is C^k-smooth in R^2.*

As follows from the (obvious) lemmas 2.3 and 3.2 the function $\varkappa$ is C^{k+1}-smooth on P^2 so $\alpha \stackrel{\text{def}}{=} \alpha - d\varkappa$ is one looked for. $\square$

Definition 3.3. Let $I_k(P^2, F, \omega, \tilde{\omega})$ denote a set $\{\{\Pi_{\omega,i}(F) - \Pi_{\tilde{\omega},i}(F)\}, \{\Lambda_{A_j,k}(t) - \tilde{\Lambda}_{A_j,k}(t)\}, Z_k(\omega, \tilde{\omega})\}$.

Theorem 3.1 states that $(P^2, F, \omega) \stackrel{C^k}{\sim} (P^2, F, \tilde{\omega})$ is equivalent to $I_k(P^2, F, \omega, \tilde{\omega}) = 0$. It can be reformulated in a different, more common, manner. Let us denote

$$\Upsilon_k = \{(\Phi, \omega_1, \omega_2) |\ \Phi \in Aut_k(= Aut_{C^k})(P^2), \Phi^* F = F, \Phi^* \omega_2 = \omega_1\}$$

$$\Omega_k = \{(\omega_1, \omega_2) |\ \omega_i \text{ is a symplectic form on } P^2,\ I_k(P^2, F, \omega_1, \omega_2) = 0\}$$

Let $p : \Upsilon_k \to \Omega_k$ be the natural projection.

Theorem 3.4. *$(\Upsilon_k, \Omega_k, p)$ is a Serre fibration.*

Lemma 3.5.

(1) *Let M^n be closed. Then a map $d : \Omega^s(M^n) \to Im\, d \subset \Omega^{s+1}(M^n)$, mapping the form ω to $d\omega$, is a projection of Serre fibration.*

(2) *Let us consider (P^2, K) and $\tilde{\Omega}^1 = \{\alpha \in \Omega^1 |\ \alpha(x) = 0\ \forall x \in K\}$. Then $d : \tilde{\Omega}^1(P^2) \to Im\, d \subset \Omega^2(P^2)$ is a projection of Serre fibration.*

(3) *If $\tilde{\tilde{\Omega}}^1 = \{\alpha \in \tilde{\Omega}^1 |\ \alpha(\text{sgrad}_\omega F) = 0\}$ (where one can substitute ω either by ω_1 or by ω_2) then $d : \tilde{\tilde{\Omega}}^1 \to Im\, d \subset \Omega^2$ is a projection of Serre fibration.*

The proof is conducted by means of localization and the fact that $d^{-1}(*)$ is contractible.

Proof of the theorem. Let us at first show that any path in Ω_k is lifted to the path in Υ_k with a preassigned beginning. Let $(\omega_{1t}, \omega_{2t})$ be a path and $\Phi^* \omega_{20} = \omega_{10}$. Clearly without lost of generality one might believe that ω_{10} and ω_{20} induce the same orientation on P^2 and moreover $\omega_{10} = \omega_{20}$, $\Phi_0 = id$. Consider forms $\omega_{\tau t} = (2 - \tau)\omega_{1t} + (\tau - 1)\omega_{2t}$.. As above there are diffeomorphisms $\Phi_{\tau t}$ such that $\Phi_{1t} = id\ \forall t$ and $\Phi^*_{\tau t} \omega_{\tau t} = \omega_{1t}$, where $\Phi_{\tau t}$ is a shift by the time τ along the trajectories of the vector field $u_{\tau t}$ and $u_{\tau t}$ is defined from the equation

$$\omega_{\tau t}(u_{\tau t}, *) = -\alpha_t(*),\ d\alpha_t = \omega_{2t} - \omega_{1t},\ \alpha_t(x) = 0\ \forall x \in K,\ \alpha_t(\text{sgrad}_\omega F) = 0.$$

These conditions define α_t uniquely, up to a term $c(F)dF$, where by $c(F)$ is denoted a function which is constant on the leaves of fibration P^2 by trajectories of the vector field sgrad$_\omega F$; similarly, $u_{\tau t}$ is defined uniquely up to a term $c(F)\,\text{sgrad}_{\omega_{\tau t}} F$. Using lemma 2.5.(3) we chose C^k-form α_t smooth by t, $\alpha_0 = 0$. Then $u_{\tau t}$ and $\Phi_{\tau t}$ are (C^k-)smooth by τ, t and we let $\Phi_t = \Phi_{2t}$, $\Phi_0 = id$.

General path lifting theorem is deduced now easily because every component of $p^{-1}((\omega, \omega))$ is contractible. Indeed, let $\Phi^* F = F$, $\Phi^* \omega = \omega$ and $\Phi \sim id$ in the class of such maps. Clear, $\Phi_* \text{sgrad}_\omega F = \text{sgrad}_\omega F$, so Φ is uniquely determined by images of transversals sticking out of some vertices, one for an annulus. So far as the last set is contractible, tout est fait. $\square$

Let us now fulfil the promise of remark 2.7. Let us have (P^2, F, K) and a symplectic form ω. Following lemma 1.1 we can write

$$\Pi_{\omega,i}(F) = -\sum_{j=1}^{m_i} \Lambda_{A_j,k}(F) \ln |F| + c_i^k(F),$$

where m_i is an amount of vertices of the graph K on the boundary C_i, $\Lambda_{A_j,k}(F) = \sum_{r=0}^k \Lambda_r^{(j)} F^r$, $c_i^k \in C^k[0, \varepsilon)$. In case $k = \infty$ for $\Lambda_{A_j,k}(F)$ we take any function with the Taylor series $\sum_{r=0}^\infty \Lambda_r^{(j)} F^r$. Further, similarly to lemma 2.1 we can write $\omega = d\alpha$, where $\alpha|_K \equiv 0$. Let us consider like in §2 an arbitrary set of cuts $\{s_m\}$ and sections $\{\beta_i\}$. Fix some annulus C_i. Suppose it is positive — C_i^+. There are on it a section β_i and semicuts $s_{1i}^+, \dots, s_{m_i i}^+$. Denote a space along the field sgrad$_\omega F$ between s_{ji}^+ and $s_{(j+1)i}^+$ by $U_{ji} = [s_{ji}^+, s_{(j+1)i}^+) \subset \overline{C_i^+}$, $\overline{C_i^+} = \bigsqcup_j U_{ji}$. Suppose e.c. $\beta_i \in U_{1i}$. Define a function $\varkappa : \overline{C_i^+} \to R$:

$$\varkappa = 0 \text{ on } K \cap \overline{C_i^+};\ \varkappa|_{\beta_i} = g_i;\ g_i \in C^\infty(\beta_i);\ g_i(pt_i) = 0;\ \varkappa = \int_l \alpha + g_i \text{ on } U_{1i},$$

where l is an interval of integral trajectory of the field sgrad$_\omega F$ in U_{1i} connecting the point with a point on the section β_i. Further, for $x_0 \in s_{2i}^+$ define $\varkappa(x_0) = \lim\limits_{\substack{x \to x_0 \\ x \in U_{1i}}} \varkappa(x) + \Lambda_{A_1,i}(F(x_0)) \ln F(x_0) - \dfrac{1}{m_i} c_i^k(F(x_0))$ and for $x_1 \in U_{2i}$ let $\varkappa(x_1) = \varkappa(x_0) + \int_l \alpha$, where l is an interval of integral trajectory of the vector fields sgrad$_\omega F$ in U_{2i} connecting the points x_1 and $x_0 \in s_{2i}^+$. For $x_0 \in s_{3i}^+$ define $\varkappa(x_0) = \lim\limits_{\substack{x \to x_0 \\ x \in U_{2i}}} \varkappa(x) + \Lambda_{A_2,i}(F(x_0)) \ln F(x_0) - \dfrac{1}{m_i} c_i^k(F(x_0))$ and so on. When we make a complete turnover and again define $\varkappa|_{U_{1i}}$, we will obtain the same value.

Let us repeat the procedure for all annuli and similarly to the definition 2.4 associate to every (oriented) cut s_m a polynomial (series) $\delta_k^m(t)$. But this time the obtained 1-chain is not a cycle, where one considers (0,1 and 2) chains of the standard cell decomposition of $\tilde{P}^2$: (vertices of K)$_0$, (edges of K)$_1$, (glued discs)$_2$: $\delta_k(\omega) = \sum\limits_m \delta_k^m(t) e_m \in C_1(\tilde{P}^2; R_k[t])$. If we vary possible arbitrariness in the definition of $\delta_k(\omega)$ (see prop.2.5), we will receive uniquely determined element $[\delta_k(\omega)] \in C_1/B_1\,(\tilde{P}^2; R_k[t])$. To get (as earlier) cohomologic invariant let us act like [BF]: $C_1/B_1 = C_1/Z_1 + Z_1/B_1 = B_0 + H^1$. Let us define a scalar product in $C_1(\tilde{P}^2; R_k[t])$ with respect to which 1-chains $\{t^l e_m\}_{l=0,\dots,k}$, m counts edges, are an orthonormal basis. One can uniquely decompose $\delta_k(\omega) = w_k(\omega) + v_k(\omega)$, where $v_k(\omega) \in Z_1(\tilde{P}^2; R_k[t]) = \{c \in C_1 | \partial c = 0 \in C_0\}$, $w_k \perp v_k$.

Definition 3.6. $\Delta_k(\omega) \stackrel{\text{def}}{=} \partial\omega_k = \partial\delta_k \in B_0(\tilde{P}^2; R_k[t])$ and $Z_k(\omega) \stackrel{\text{def}}{=} [v_k(\omega)] \in Z_1/B_1 = H_1 = H^1(\tilde{P}^2; R_k[t])$.

Remark 3.7. δ_k is uniquely recovered by v_k and Δ_k, so $[\delta_k]$ is uniquely recovered by Z_k and Δ_k.

0-chain with coefficients in $R_k[t]$ this is a collection of polynomial (series) on the vertices of the graph K. Near each vertex A_j two positive annuli I_j and III_j, and two negative annuli II_j and IV_j pass. Let m_{I_j} be a number of vertices of K on ∂I_j, and $c^k_{I_j}(F)$ be a Taylor series up to k-th term of the finite part of the period function (see above) and similarly for II, III, IV. It is straightforward to check

Lemma 3.8. *$\Delta_k(\omega) = \sum_j f_j A_j$, where* $f_j = \frac{c^k_{II_j}(F)}{m_{II_j}} + \frac{c^k_{IV_j}(F)}{m_{IV_j}} - \frac{c^k_{I_j}(F)}{m_{I_j}} - \frac{c^k_{III_j}(F)}{m_{III_j}}$.

Corollary 3.9. *If (1.1) and (1.2) hold then $\Delta_k(\omega) = \Delta_k(\tilde{\omega})$.*

Proposition 3.10. *If (1.1) and (1.2) hold then $Z_k(\omega,\tilde{\omega}) = Z_k(\omega) - Z_k(\tilde{\omega})$.*

Proof. The difference of $\varkappa$-s for ω and $\tilde{\omega}$ while the sections are the same is clearly the function $\varkappa$ for the pair $(\omega,\tilde{\omega})$ whenever the form α is a difference of the forms. □

Definition 3.3′. $I_k(P^2, F, \omega) \stackrel{\text{def}}{=} \{\{\Pi_{\omega,i}(F)\}, \{\Lambda_{A_j,k}(t)\}, Z_k(\omega)\}$.

Then $I_k(P^2, F, \omega, \tilde{\omega}) = I_k(P^2, F, \omega) - I_k(P^2, F, \tilde{P}^2)$ and we obtain:

Theorem 3.1′. $(P^2, F, \omega) \stackrel{C^k}{\sim} (P^2, F, \tilde{\omega})$ *iff* $I_k(P^2, F, \omega) = I_k(P^2, F, \tilde{\omega})$.

§4. Elimination of redundant information.

So to classify forms ω on (P^2, F, K) we associate with them the following data : $\Pi_{\omega,i}(F)$, $\Lambda_{A_j,k}(F)$ and $Z_k(\omega)$. But this information is redundant. In order to see this let us consider the letter-atom B (Fig. 3). Take any interior circle $\{F = \text{const}\}_i$. We have $\Pi_{\omega,i}(F) = -\Lambda_k(F)\ln|F| + c(F)$ where c is a C^k-smooth function on $[0,\varepsilon)$ (or on $(\varepsilon, 0]$). Therefore we can recover Λ-invariant. But this is not the case with the letter C_2 (Fig. 4) because any period $\Pi_{\omega,i}(F)$ contains unsmooth part in the form $-(\Lambda^{(1)}_k(F) + \Lambda^{(2)}_k(F))\ln|F|$, i.e. the possibility to separate $\Lambda^{(1)}$ and $\Lambda^{(2)}$ is given only when we determine one of them. Let us find out how many of Λ-s and where we must leave on the vertices so that by means of them and Π we can recover the rest. Since we can deal with the coefficients in $\Lambda(t) = \sum \Lambda_i t^i$ independently the question is as follows : suppose there are numbers λ_j on vertices of the graph K. Associate to any boundary circle $\gamma_i \in \partial P^2$, $i = 1, \dots, n$ a number $\mu_q \stackrel{\text{def}}{=} \sum k_j\lambda_j$ where j counts vertices lying on the boundary of the i-th annulus and $k_j = 0, 1, 2$ is a multiplicity. The question is how many of λ_j and where one must leave to be able to recover by means of them and $\{\mu_i\}$ the whole set $\{\lambda_j\}$.

Let us formulate at first the formal answer. Let us enumerate the vertices of K : $A_1, \dots, A_m$, $n - m = \chi(\tilde{P}^2)$. Form a matrix $\mathbf{A}=(a_{ji})_{m\times n}$:

$$a_{ji} = \begin{cases} 0, \text{ if } \gamma_i \text{ does not pass near } A_j \\ 1, \text{ if pass once} \\ 2, \text{ if twice} \end{cases}$$

Clear that $\vec{\lambda}\cdot\mathbf{A} = \vec{\mu}$, thus holds

Lemma 4.1. *If $N \stackrel{\text{def}}{=} m - \text{rk}\mathbf{A}$, then μ recovers λ. Otherwise marks λ must be leaved on vertices $A_{i_1}, \dots, A_{i_N}$ where $(i_1, \dots, i_N)$ is such a minimal collection that the rows of* $\mathbf{A}$ *with the numbers $i \notin \{i_1, \dots, i_N\}$ are linearly independent.*

Remark 4.2. $N > 0$ whenever P^2 is not planar. It it obvious for a surface $\tilde{P}^2$ of genus $g > 1$: rk$\mathbf{A} \le n = m + 2 - 2g < m$. But it holds also when $g = 1$, rk$\mathbf{A} < n$, since the sum of the columns corresponding to positive annuli equals to the sum of the columns corresponding to negative ones and equals to the column of twos.

However application of the lemma is laborious so inconvenient. Let us consider the flat case, $g = 0$. Any graph K dealed with is a number of circles, immersed to P^2, which following [BF] we will call atomic. Denote them by $\beta_1, \dots, \beta_q$.

Proposition 4.3. *Let $g = 0$. To recover $\{\lambda\}$ one must leave $(q-1)$ marks λ_j on the vertices $A_1, \dots, A_{q-1}$ where $A_j \in \beta_{s_{1j}} \cap \beta_{s_{2j}}$ and the set of transposition (s_{1j}, s_{2j}) forms a basis in S_q.*

Proof. Let us at first show that if $q = 1$ then one can recover λ by μ. Take an arbitrary vertex A_j and let us move along the circle in any direction. When we return into the vertex, we will have an imbedded circle with an "angle"(Fig. 5). It bounds a number of domains and we can associate integers to them so that their sum by domains having an angle in a vertex of our circle with "angle" is zero for every such a vertex other than the "angle" while the only domain incident to the "angle" has +1. To see this uncoil the loop allowing only degenerations of common type A) and B) (see Fig. 6), put ± 1 and coil it back changing the numbers adequately (see Figs. 6,7). Now λ_j is a sum of μ_i, corresponding to domains and multiplied by corresponding numbers.

Let us now have two atomic circles β_1 and β_2. Since the intersection number $\beta_1 \cdot \beta_2 = 0$, the number of intersections β_1 and β_2 is even. Move along β_2 and change λ in the points in turn by +x,-x. Then, obviously, none of μ_i changes (Fig. 8) so the minimal number of leaving λ is $N \ge 1$. Let one determine λ_j for some $A_j \in \beta_1 \cap \beta_2$. There are at least three different annuli among I-IV incident to A_j. Let I≠III be opposite (of the same sign) annuli. Made an operation indicated on the Fig. 9. As a result instead of two atomic circles we get one, lost the vertex A_j and instead of annuli I and III get an annulus V with the mark $\mu_V = \mu_I + \mu_{III} - 2\lambda_j$. Therefore $N \le 1$, i.e. $N = 1$. Similarly for every q : $q - 1 \le N \le q - 1$. □

In general case let us denote by $\gamma H_1(\tilde{P}^2; R) \subset H_1(\tilde{P}^2; R)$ a subgroup generated by all the cycles $[\beta_1], \dots, [\beta_q]$.

Proposition 4.4. *The minimal number of leaving marks λ is* $(q-1) + 2g - \dim\gamma H_1(\tilde{P}^2; R)$.

For the proof and the rule of placing marks λ one may see [BF],part II,§10.

§5. Simplification of the Π-invariant and the main theorem for (P_1).

Actually in Π-invariant there contains also redundant information. To see this change some function $\Pi_{\omega,i}(F)$ by function with k-jet 0 in zero (when $F = 0$). Then there exists C^k-automorphism P^2 preserving the foliation $\{F = \text{const}\}$, superposing Π old and new and C^k not differing from Id on K and therefore preserving Λ- and Z-invariants. So only k-jet of the finite part c of the period function Π has sense (and besides then one would not have to make somewhat disagreeable operation of eliminating redundant

Λ–information). Such jets at the end of §3 were united in ΔZ–invariant. To apply the method of §2 we have supposed in §1 that Φ preserves F. Now we may remove that restriction. But at first we must explain what happens with Λ, Δ and Z-invariants when F C^k-changes, the changes being independent on every annulus so far that the global transformation is C^k-smooth. As a matter of fact basing on smoothness arguments we can without loss of generality consider only global changes $\tilde{F} = \tilde{F}(F)$ on the whole P^2.

Let $\tilde{F} = \tilde{F}(F) = \sum_{i=1}^{k} a_i F^i + \bar{o}(|F|), a_i \neq 0$. If we substitute it into $\tilde{\Lambda}(\tilde{F}) \ln |\tilde{F}| = \sum_{i=0}^{k} \tilde{\Lambda}_i \tilde{F}^i \ln |\tilde{F}|$, we will obtain $\sum_{i=0}^{k} \Lambda_i F^i \ln |F| + c_k(F)$ where c_k is C^k-smooth in zero. Thus we will receive the rule of changing the Λ-invariant: $\sum_{i=0}^{k} \tilde{\Lambda}_i \tilde{F}^i \mapsto \sum_{i=0}^{k} \Lambda_i F^i$. For example, when k=1, substituting $\tilde{F} = aF + \bar{o}(|F|)$ into $(\tilde{\Lambda}_0 + \tilde{\Lambda}_1 \tilde{F}) \ln |F|$, we obtain

$$(\tilde{\Lambda}_0 + \tilde{\Lambda}_1(aF + \bar{o}(|F|)))(\ln |a| + \ln |F| + \ln(1 + \bar{o}(1))) = (\Lambda_0 + \Lambda_1 F) \ln |F| + c_k(F),$$

from where $\Lambda_0 = \tilde{\Lambda}_0, \Lambda_1 = a\tilde{\Lambda}_1$, i.e. Λ-invariant in category C^1 is a collection $\{\Lambda_{A_j,0}\}$ and a collection $(\Lambda_{A_1,1} : \ldots : \Lambda_{A_m,1})$ of numbers Λ_l thinking of up to multiplication by a nonzero number.

Let us note that in this way Λ-invariant in category C^0 can be obtained. Actually, changing $\tilde{F} = F^\alpha$, $\alpha > 0$, we receive $\tilde{\Lambda}_0 \ln |\tilde{F}| = \alpha \tilde{\Lambda}_0 \ln |F| = \Lambda_0 \ln F$, i.e. the invariant is a collection $(\Lambda_{A_1,0} : \ldots : \Lambda_{A_m,0})$ (see [BF]).

Δ-invariant changes even easier: one takes a polynomial (series) in the vertex j, substitutes into it the series for $\tilde{F}$ by F and cuts down the resulting series if $k < \infty$. The similar operation is done with a representative of a class Z, the result enclosing into square brackets.

Definition 5.1. $Inv_k(P^2, F, \omega) \stackrel{\text{def}}{=} \{\{\Lambda_{A_j,k}\}_j, \Delta_k, Z_k\}$.

Now we understand what means C^k-automorphism Φ, preserving the foliation $\{F = \text{const}\}$, maps a collection $Inv_k(P^2, F, \omega)$ to $Inv_k(P^2, F', \omega')$ and hence understand what means Φ maps $Inv_k(P^2, F, \omega)$ to $Inv_k(P^{2\prime}, F', \omega')$. Therefore we can present the main theorem.

Theorem 5.2. $(P^2, F, \omega) \stackrel{C^k}{\sim} (P^{2\prime}, F', \omega')$ *iff* $Inv_k(P^2, F, \omega) = Inv_k(P^{2\prime}, F', \omega')$.

Proof. To the one side this theorem is a consequence of theorem 3.1′ and the lemma 3.8. Let now one have C^k-diffeomorphism Φ which superposes the foliations and maps Inv_k to Inv'_k. We must prove C^k-equivalence of the systems (P^2, F, ω) and $(P^2, \Phi^* F', \Phi^* \omega)$. We can take α as in lemma 2.1 with $\tilde{\omega} = \Phi^* \omega'$. Further, since Δ and Z-invariants coincide, using remark 3.7, we can choose to be coincident 1-chains $\delta_k(\omega)$ and $\delta_k(\Phi^* \omega')$ generating them, therefore the difference of the functions $\varkappa$ and $\varkappa'$ generating them is C^{k+1}-smooth (see §2 and the proof of the theorem 3.1). In this case we can let $\alpha \stackrel{\text{def}}{=} \alpha - d\varkappa$ and note that such an α satisfies (2.1). □

The equality $Inv_k = Inv'_k$ means the existence of a C^k-isomorphism mapping $Inv_k(F)$ to $Inv'_k(F')$. And even though when it exists it is unique up to diffeomorphism which is C^k-zero on K and can be easily found, one may desire to have more effective method of comparing $Inv_k(F)$ and $Inv'_k(F')$. And it is possible upon good reparametrization of the foliation $\{F = \text{const}\}$.

Proposition 5.3. *There exists a smooth transformation* $F \mapsto \mathring{F}$ *on the half-interval* $[0, *)$ *or* $(*, 0]$, $\operatorname{sgn} \mathring{F} = \operatorname{sgn} F$, *such that* $\Pi(F) = -\Lambda(F) \ln |F| + c(F)$ *becomes* $-\mathring{\Lambda}(\mathring{F}) \ln |\mathring{F}|$.

Proof. Let $\Lambda(F) = \Lambda_0 + \ldots, c(F) = c_0 + \ldots$. Let us change $F \mapsto a_1 F$. Then $\Pi(F) \mapsto -\Lambda(a_1 F)(\ln |F| + \ln a_1) + c(a_1 F) = -(\Lambda_0 + \ldots) \ln |F| - \Lambda_0 \ln a_1 + c_0 + \ldots$. Therefore in order to kill the constant term in $c(F)$ we change $F \mapsto a_1 F$ where $a_1 = e^{c_0/\Lambda_0}$. Let us kill the first order term in $c(F) = c_1 F + \ldots$ changing $F \mapsto F + a_2 F : \Pi(F) \mapsto (-\Lambda_0 + \ldots)(\ln |F| + \ln(1 + a_2 F)) + c(F + a_2 F) = (-\Lambda_0 + \ldots) \ln |F| - (\Lambda_0 a_2 F + \ldots) + c_1 F + c_1 a_2 F^2 + \ldots$. We take $a_2 = c_1/\Lambda_0$. In order to kill the second order term in $c(F) = c_2 F^2 + \ldots$, let us change $F \mapsto F + a_3 F^3$ etc. As a result we obtain the formal change $F \mapsto \sum_{i=1}^{\infty} a_i F^i$ which kills all the Taylor series terms for $c(F)$. Let us change $F \mapsto \mathring{F} = \mathring{F}(F)$, where $\mathring{F}(F)$ is arbitrary smooth with the Taylor series $\sum_{i=1}^{\infty} a_i F^i$. Then $\mathring{\Pi}(\mathring{F}) = \Pi(F(\mathring{F})) = -\hat{\Lambda}(\mathring{F}) \ln |\mathring{F}| + \hat{c}(\mathring{F})$, where $j_\infty \hat{c}(0) = 0$. We set

$$\mathring{\Lambda}(\mathring{F}) = \hat{\Lambda}(\mathring{F}) - \frac{\hat{c}(\mathring{F})}{\ln |\mathring{F}|}. \quad \square$$

Proposition 5.4. *If another function* $\mathring{F}'(F)$ *satisfies proposition 5.3 then* $\mathring{F}$ *and* $\mathring{F}'$ *express one by another and* $\mathring{F}' = \mathring{F} + q(\mathring{F})$, $j_\infty q(0) = 0$.

Proof is going on as in 5.3 and we obtain that the Taylor series terms for $\mathring{F}'(\mathring{F})$ are $a_i = \delta_{i1}$. □

Corollary 5.5. *There exists in a small neighborhood of* $K \subset P^2$ *a unique, up to addition* $q(F)$, $j_\infty q(0) = 0$, *function* $\mathring{F}(F)$ *such that* $\frac{d\mathring{F}}{dF} > 0$ *and for any preassigned annulus* $C_j \subset P^2$ *the period function for the vector field* $v = \operatorname{sgrad}_\omega F$ *satisfies* $\mathring{\Pi}_{j,\omega}(\mathring{F}) = \Pi_{j,\omega}(F(\mathring{F})) = -\mathring{\Lambda}_{j,\omega}(\mathring{F}) \ln |\mathring{F}|$.

Thus choosing an end of the letter-atom (P^2, K) we uniquely determine the jet of (smooth) parametrization $\Gamma = P^2/\mathcal{L}$. So we may uniquely substitute F to $t = \mathring{F}$, $F = F(t)$ in $Inv_k(P^2, F, \omega)(= Inv_k(P^2, v = \operatorname{sgrad}_\omega F))$ and express all polynomials (series) by t. After that the coincidence of objects must become literal, so we obtain the final form of the main theorem for (P_j).

Definition 5.6. Let us denote by $Inv_k(P^2, v)_j = \mathring{Inv}_k(P^2, F, \omega)_j$ the set $Inv_k(P^2, F, \omega)$ parametrized by $\mathring{F} = \mathring{F}_j(F)$ for arbitrary (F, ω) satisfying $v = \operatorname{sgrad}_\omega F$.

Theorem 5.7. $(P^2, \operatorname{sgrad}_\omega F) \stackrel{C^k}{\sim} (P^{2\prime}, \operatorname{sgrad}_{\omega'} F')$ *iff* $Inv_k(P^2, \operatorname{sgrad}_\omega F)_* = Inv_k(P^{2\prime}, \operatorname{sgrad}_{\omega'} F')_*$, *where by equality we mean one induced by isomorphism* $\Gamma \sim \Gamma'$ *that maps marked ends to marked ones.*

§6. Generalization: stars.

In the theory of topological classification besides letter-atoms described in the introduction there are letter-atoms with stars (see [F],[BMF]). A star is a vertex on a graph of degree two such that for (P^2, K) (with vertices of multiplicity 2 and 4) there exists a letter-atom $(\hat{P}^2, \hat{K})$ and an involution σ on it such that $\sigma^*\hat{F} = \hat{F}$, $\sigma^*\hat{\omega} = \hat{\omega}$, $(\hat{P}^2, \hat{K})/\sigma = (P^2, K)$, fixed points of σ and only they mapping upon factorisation $\hat{\sigma} : \hat{P}^2 \to P^2$ into stars ; near them the map σ is a reflection: there are coordinates (x, y) satisfying $F = xy$, $\omega = \omega(xy)\,dx \wedge dy$, $\sigma(x, y) = (-x, -y)$ (see Fig.5). A neighborhood of the singular fiber in the three-dimensional submanifold is $P^2 \times I/\{(x, 0) \sim (\sigma(x), 1)\}$. One can construct $(\hat{P}^2, \hat{K})$ by (P^2, K) the following way: K is disposed in R^3_+ being tangent to $R^2 = \partial R^3_+$ along $R^1 \subset R^2$ in stars. Further one takes a reflection K from R^2, replaces stars by crosses and at last draws $\hat{P}^2$ inverting it on angle π while passing through R^2 (see Fig.11). It is easy to see that although $(\hat{P}^2, \hat{K})$ depends on the embedding (P^2, K) into R^3_+, the neighborhood of the singular fiber in threedimensional submanifold $P^2 \times I/\sim$ does not.

One can include systems with stars into the theory of trajectory classification. For doing this one must only know how to classify exactly germs of Hamiltonian systems with involution on a singular leaf in $\hat{P}^2$ up to C^k-diffeomorphism commuting with involution. As in §1 we can think $\Phi^*\hat{F} = \hat{F}$. Define $\Pi_{\omega,i}(F)$: if $\tilde{\gamma}_i(F) = \hat{\sigma}^{-1}\gamma_i(F)$ is a pair of circles let $\Pi_{\omega,i}(F)$ be the time of passing any of them by the flow $\hat{\sigma}_*^{-1}\,\mathrm{sgrad}_\omega F = \mathrm{sgrad}_{\hat{\omega}}\hat{F}$ and if it is a single circle then a half of it. Similarly for every vertex A_j (including stars) one defines $\Lambda_{j,k}(t)$ to be the Λ-invariant of any $\hat{\sigma}$-inverse image. To define $Z_k(\omega_1, \omega_2)$ let us consider a 1-form $\hat{\alpha}$, given by the lemma 2.1 with respect to forms $\hat{\omega}_1$ and $\hat{\omega}_2$. Pass to σ-invariant form $\hat{\alpha} \stackrel{\text{def}}{=} \dfrac{\hat{\alpha} + \sigma^*\hat{\alpha}}{2}$ and define as above a function $\hat{\varkappa} : \hat{P}^2 \to R$. Replace it also by σ-invariant one $\hat{\varkappa} \stackrel{\text{def}}{=} \dfrac{\hat{\varkappa} + \sigma^*\hat{\varkappa}}{2}$. Since $\sigma^*\hat{\varkappa} = \hat{\varkappa}$ we can pull it down to the base. One gets a function $\varkappa : P^2 \to R$, continuous on P^2 and smooth outside the graph K. Choosing an arbitrary set of sections $\{s_m\}$ we define a 1-cochain $\delta_k(t)$ like in the definition 2.4. Easy to see this cochain does not depend on to what side of the star one takes the section. Therefore applying arguments of the proof of the proposition 2.5 we obtain an element $Z_k(\omega_1, \omega_2) \in H^1(\hat{P}^2; R_k[t])$. The proposition 2.6 holds true ; to prove 3) one lifts Φ to $\hat{\Phi} : \hat{P}^2 \to \hat{P}^2$ acting on vertices like σ, $[\hat{\Phi}, \sigma] = 0$.

Theorem 6.1. *Let (P^2, F, ω_1) be a letter-atom (with stars) and (P^2, F, ω_2) be another. They are C^k-equivalent iff $I_k(P^2, F, \omega_1, \omega_2) = 0$.*

Proof. Since the necessity is done above let us pass to the sufficiency. Because of $Z_k(\omega_1, \omega_2) = 0$ we can find sections and initial values of $\hat{\varkappa}$ on them such that $\hat{\varkappa} \in C^{k+1}(\hat{P}^2)$ (see the proof of the theorem 3.1). Then averaging, $\hat{\varkappa} \stackrel{\text{def}}{=} \dfrac{\hat{\varkappa} + \sigma^*\hat{\varkappa}}{2}$, and replacing $\hat{\alpha}$ by $\hat{\alpha} - d\hat{\varkappa}$ we get a σ-invariant $\hat{\alpha}$ such that $d\hat{\alpha} = \hat{\omega}_1 - \hat{\omega}_2$ and $\hat{\alpha}(\mathrm{sgrad}_{\hat{\omega}}\hat{F}) = 0$. Therefore an isotopy for the unit time along the vector fields $\hat{u}_t$, $(t\hat{\omega}_2 + (1-t)\hat{\omega}_1)(\hat{u}_t, *) = \hat{\alpha}(*)$, moves $\hat{\omega}_2$ to $\hat{\omega}_1$ and commutes with σ. □

Passing to the definition of the ΔZ-invariant let us note that cuts s_m must be put on every edge including ones containing vertices of multiplicity 2 (stars) as a boundary. The defining rule for function $\varkappa$ on the cuts is now slightly changing. While passing an edge containing stars we subtract an accumulated infinite part $(-\Lambda_{A,i}(F(x_0)) \ln |F(x_o)|$ — see §3) and all the finite parts $(\dfrac{1}{m_i} c_i^k(F(x_0))$) corresponding to the stars of the edge being on the first met edge ; being on the others of that edge we subtract only infinite

parts. Owing to this definition there is no difference among cuts on the edge taken for determining $\delta_k^m(t)$. Thus we can correctly define $Z_k(\omega), \Delta_k(\omega)$ and prove:

Theorem 6.2. *Theorems 5.2 and 5.7 hold true in the star case.*

In conclusion of the section let us consider interrelations between $\mathrm{rk}\, H_1(\hat{P}^2)$ and $\mathrm{rk}\, H_1(P^2)$ i.e. between the genera of the surfaces. Suppose a letter-atom (P^2, K) contains n 4-multiple vertices, p stars and m boundary circles. One may easily see that for Euler characteristic of the "glued" surface the following formula holds : $\chi(\tilde{P}^2) = m - n$. For the "unfolded" atom $(\hat{P}^2, \hat{K})$: the number of the vertices of $\hat{K}$ is (2n+p), the number of the boundary circles in $\partial\hat{P}^2$ is $\hat{m} \in [2m - p, 2m - \dfrac{1-(-1)^p}{2}]$, the Figs. 12 and 13 showing that $\hat{m}_{min}$ and $\hat{m}_{max}$ are achieved. One may show any $\hat{m} \in [\hat{m}_{min}, \hat{m}_{max}]$ is achieved. These estimates imply:

$$\chi(\hat{\tilde{P}}^2) \in [2\chi(\tilde{P}^2) - 2p, 2\chi(\tilde{P}^2) - p - \frac{1-(-1)^p}{2}]$$

Since the genus $g = 1 - \frac{\chi}{2}$, we have proved

Proposition 6.3. *If g is the genus of $\tilde{P}^2$ and g_* is the genus of $\hat{\tilde{P}}^2$ then $g_* \in [2g + [\frac{p-1}{2}], 2g + p - 1]$, all possible values being achieved.*

So it is possible to have a situation as in Fig. 14 when $g = 0, g_* > 0$.

§7. Exact smooth classification of Hamiltonian vector fields on symplectic 2-manifolds.

In this paragraph we solve (P_2) in the category C^k, $k = 1, \ldots$. For $k = 0$ it was solved in [BF]. We reduce (P_2) to (P_1) solved in §§1–5. Let us have connected compact symplectic surface V^2 and a Hamiltonian system $v = \mathrm{sgrad}_\omega F$ on it with the Morse Hamiltonian which is constant on the boundary components. Let $\mathcal{L}$ be the foliation of P^2 by connected components of level lines of F. $V^2/\mathcal{L}$ is a connected graph Γ, vertices of degree 1 of which correspond to elliptic critical points of F (the notation is A) or to the boundary components (the notation is ∂). Other letters are used for multiple vertices. Let us note that a small neighborhood of singular leaf (vertex of the graph Γ) is precisely the letter-atom, see §0. The weight of the atom is by definition the number of critical points (vertices, in our case of multiplicity 4) on the singular leaf, on the graph $K \subset P^2$. One may see the list of atoms of small weight in [BMF]. We will use the notation from there. Letter-atoms can be nonflat, see Fig.15. Edges of Γ correspond to annuli $S^1 \times I$, foliated by nonsingular leaves of $\mathcal{L}$, by circles. An example of Γ is shown on Fig.16. We will associate invariants to edges and vertices of Γ.

For every edge e_m we have a function f_m, a period of the trajectory of the vector field v. Let us associate to e_m the C^k-conjugation class $[f_m]_k$ of the function f_m on the interval I. If for example we bound ourselves to the class of Morse functions $f_k : I \to R_+$, then $[f_m]$ is classified by the consequence of maximums and minimums together with the values of f_m in them. Let us consider the letter-atom A. In its neighborhood there exist coordinates action-angle (x, y): $\omega = dx \wedge dy$, $F = F(x^2 + y^2)$, $\left.\dfrac{dF}{dt}\right|_{t=0} \neq 0$, see.[E],[I],[K]. The period is $\Pi_\omega(t) = \dfrac{d}{dt}\mathrm{vol}_\omega\{x^2 + y^2 \le F^{-1}(t)\} = \pi\dfrac{d}{dt}F^{-1}(t)$. So the germ of the vector field v on the letter A is classified by the germ of the function Π_ω in the point $a = F(0)$, $\Pi_\omega(a) > 0$. If we consider only systems with $\Pi_\omega(t)$ such that $j_\infty \Pi_\omega(a) \neq j_\infty$const, then the germ is classified by the collection

$$\{\Pi_\omega(a), k_0 = \min\{k \geq 1 | \left.\frac{d^k}{dt^k}\right|_{t=a} \Pi_\omega(t) \neq 0\}, \operatorname{sgn} \left.\frac{d^{k_0}}{dt^{k_0}}\right|_{t=0} \Pi_\omega(t)\}.$$

In the general case let us denote by $Inv_k(A, v)$ the class of C^k-conjugation of the germ of v at A. The classification of the germ of v on ∂ can be easily carried out. Let us denote the result by $Inv_k(\partial, v)$. If L is any other letter, the classification of the germ is the association of the invariants from the definition 5.6, see theorem 5.7. So for every letter $L \in \{\partial, A, B, \dots\}$ we associate a collection of invariants $Inv_k(L, v)_*$ which classify the germ of v on $L \subset V^2$.

Definition 7.1. $Inv_k(V^2, v) \stackrel{\text{def}}{=} \{\{[f_m]_k\}_m, \{Inv_k(L, v)_*\}_L\}$

The discussion above together with the classificational theorem for (P_1) imply, evidently

Theorem 7.2. $(V^2, v) \stackrel{C^k}{\sim} (V^{2'}, v')$ *iff* $Inv_k(V^2, v) = Inv_k(V^{2'}, v')$.

In theorem 7.2 we understand the equality as earlier. In detail, let us fix some isomorphism $\Gamma \sim \Gamma'$. It induces an isomorphism between edges and between the letter-atoms. The isomorphism between the edges must send the conjugation classes of period functions to the conjugation classes and the isomorphism between the letters must send Inv_k to Inv'_k as it was understood in theorem 5.7 and earlier. We write $Inv_k(V^2, v) = Inv_k(V^{2'}, v')$ if there exists an isomorphism $\Gamma \to \Gamma'$ inducing the equality. Since the set $\pi_0(Isomorph(\Gamma, \Gamma')$ is finite, the process is quite effective.

§8. The dual problem.

Let us now consider the problem (P_3). Note that $(P_3)_\infty$ was solved in [DMT] for the simplest case when $\mathcal{L}$ is generated by a simple Morse function on a closed surface, i.e. all the vertices of Γ are A and B. In this case all the letters are flat, therefore Δ- and Z invariants are zero and the classification is not hard.

Set as earlier $\Gamma = V^2/\mathcal{L}$, where $\mathcal{L}$ is Morse foliation. If we denote by C the set of singular leaves of $\mathcal{L}$, then $V^2 \setminus C$ is, evidently, the collection of annuli $S^1 \times I$ corresponding to the edges of the graph Γ. Their ω-volumes are invariants with regard to transformations from the classificational problem (P_3). Thus to every edge e_m we associate a number b_m. Let us associate now invariants to letters, vertices of Γ, other than ∂. For A in its neighborhood there exists a special Morse function $\mathring{F} = x^2 + y^2$, $\omega = dx^2 \wedge dy^2$ and we associate to it $Inv_k(A, \omega) =$(the germ of the function $\Pi(\mathring{F})$ in zero). Let L be a hyperbolic letter and F be a Morse function generating $\mathcal{L}$. Similar to the function $\Pi_v(F)$ we consider the function $S_\omega(t) = \text{vol}_\omega\{F \in (F_c, t)\}$ in a neighborhood of L (instead of conjugation class for Π on an edge we associate to the monotone function S only the variation). If $F = F_c$ is a critical value, then S is defined by the conditions $\Pi_\omega(F) = \text{sgn}(F - F_c)\frac{d}{dF}S_\omega(F), S_\omega(F_c) = 0$. Therefore $S_\omega(F)$ has the form $S_\omega(F)_j = -\Lambda_j(F)\ln|F - F_c| + c_j(F)$, $\Lambda_j(F_c) = c_j(F_c) = 0$ on the j-th annulus from $P^2 \setminus K$, where (P^2, K) is a letter-atom corresponding to L. As earlier let us take a special function $\mathring{F} = \mathring{F}_j(F)$ for the j-th annulus of L such that $\mathring{S}_\omega(\mathring{F}) = S_\omega(F(\mathring{F})) = -\mathring{\Lambda}(\mathring{F})\ln|\mathring{F}|$ and let us correspond to L an invariant $Inv_k(L, \omega)_j = I\mathring{n}v_k(P^2, \mathring{F}, \omega)_j$ like in definition 5.6.

Definition 8.1. $Inv_k(V^2, \omega) \stackrel{\text{def}}{=} \{\{b_m\}_m, \{Inv_k(L, \omega)_*\}_L\}$.

Quite similar to the theorem 7.2 one may prove

Theorem 8.2. $(V^2, \omega) \stackrel{C^k}{\sim} (V^{2'}, \omega')$ *iff* $Inv_k(V^2, \omega) = Inv_k(V^{2'}, \omega')$, *where the equality is understood as in 7.2.*

Acknowledgement

The work was raised by the question of A.V. Bolsinov. I thank him for a series of fruitful discussions and remarks as well as A.T. Fomenko for the attention to the work. I would also like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

REFERENCES

[B] A.V.Bolsinov, *Smooth trajectory equivalence of integrable Hamiltonian systems with two degrees of freedom* (to appear).

[BF] A.V.Bolsinov, A.T.Fomenko, *Trajectory equivalence of integrable Hamiltonian systems with two degrees of freedom. Classificational theorem. Parts I,II*, Math.Sbornik (to appear).

[BMF] A.V.Bolsinov, S.V.Matveev, A.T.Fomenko, *Topological classification of integrable Hamiltonian systems with two degrees of freedom. List of systems of small complexity*, Uspechi Mat.Nauk **45 no.2(272)** (1990), 59–77; English transl. in Russian Math.Survey **45 no.2** (1990), 59–94.

[CVV] Y.Colin De Verdiere et J.Vey, *Le lemme de Morse isochore*, Topology **18** (1979), 283–293.

[DMT] Jean-Paul Dufour, Pierre Molino et Anne Toulet, *Classification des systemes integrables en dimension 2 et invariants des modeles de Fomenko*, Comptes Rendus de l'Academie des Sciences de Paris **318** (1994), no. 10, 949–952.

[E] Ellіason L.H., *Normal forms for Hamiltonian systems with Poisson commuting integrals. Elliptic case*, Comment.Math.Helv. **65** (1990), no. 1, 4–35.

[F] A.T.Fomenko, *Symplectic geometry. Methods and applications*, Izdat.Moscov.Univer., Moscow, 1988; English transl. of a first draft, in two halves, *Symplectic geometry*, Gordon and Breach,New York, 1988, and*Integrability and nonintegrability in geometry and mechanics*, Kluwer,Dordrecht, 1988.

[I] Ito H., *Action-angle coordinates at singularities for analytic integrable systems*, Math.Z. **206** (1991), 363–407.

[K] B.S.Krouglikov, *Diploma of MSU, part I.*

[M] Moser J., *On the volume elements on a manifold*, Trans.Amer.Math.Soc. **120** (1965), no. 2, 286–294.

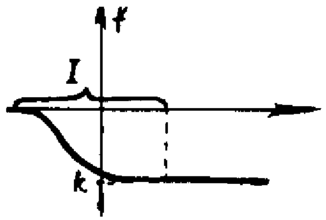

Fig.1

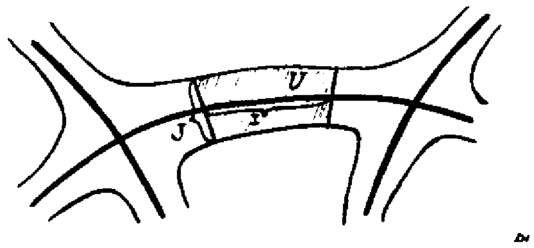

Fig.2

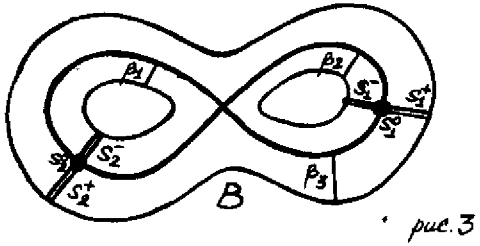

Fig.3

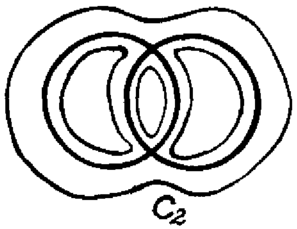

Fig.4

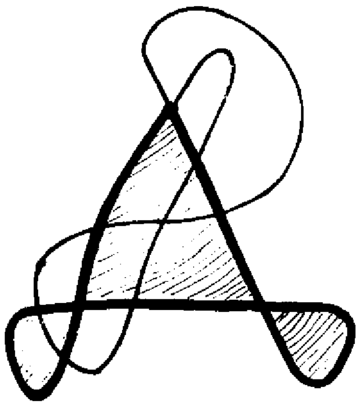

Fig.5

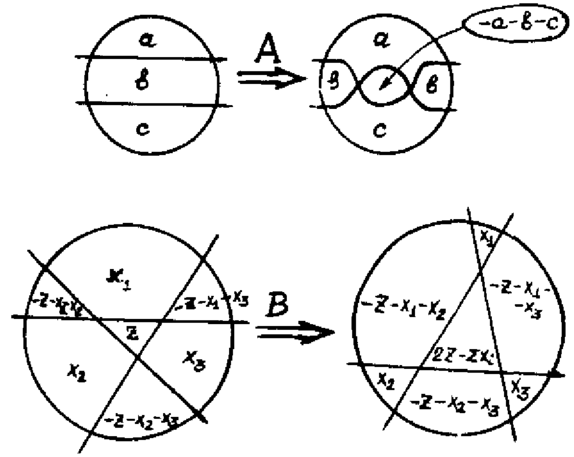

Fig.6

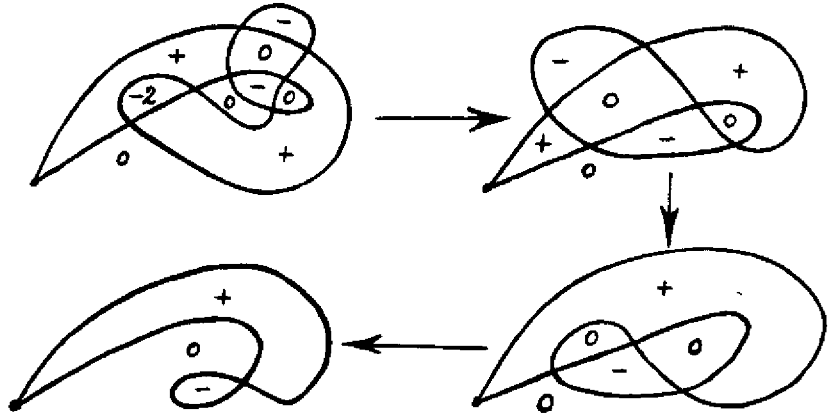

Fig.7

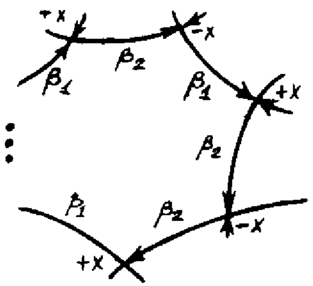

Fig.8

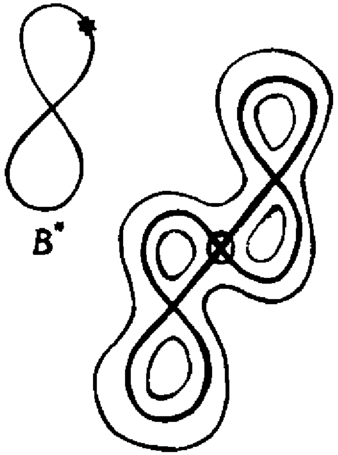

Fig.10

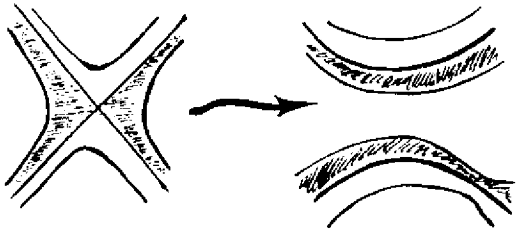

Fig.9

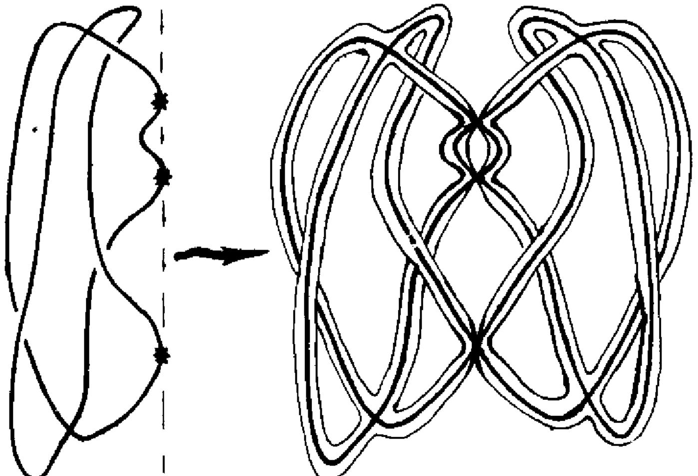

Fig.11

Fig.12

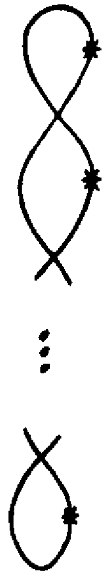

Fig.13

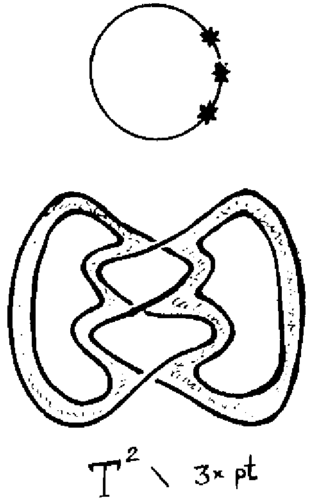

Fig.14

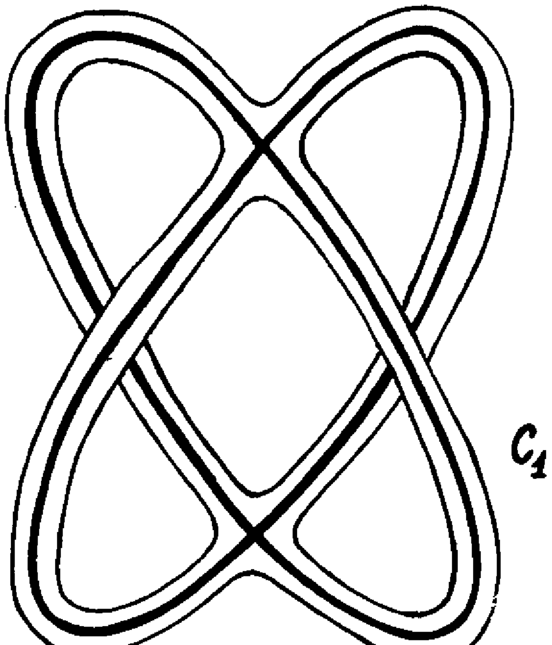

Fig.15

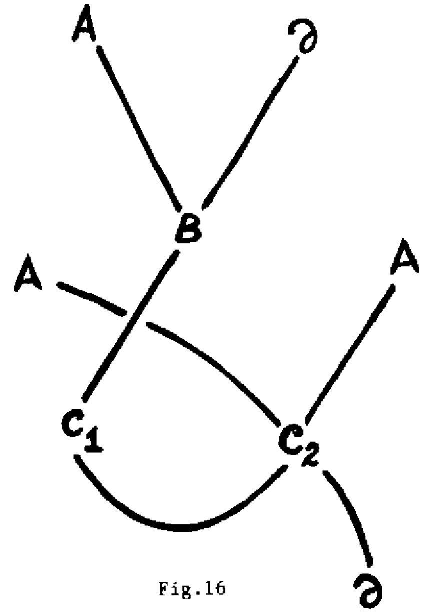

Fig.16