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ABSTRACT

In this paper, the relations between monodromy group and dressing group for Liouville equation in uniformization theorem are discussed. The representation of monodromy transformation, acting on the chiral components of the solution of Liouville equation, is obtained. The non-trivial exchange algebra for monodromy transformation is calculated.

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1. Introduction

In the paper[1], that there exists a closed relation between monodromy group and dressing group for the Liouville equation in uniformization theorem was pointed out. In this paper we discuss such a relation in more depth level. We found that a monodromy transformation acting on the chiral components of the uniformization solution of Liouville equation(USLE), up to the SL(2,R) transformations, is one to one correspondent to a dressing transformation for USLE. In terms of the representation of monodromy transformation, we calculated the exchange algebra for monodromy operators. The paper is organized as follows: in section two, we briefly describe the properties of monodromy group related to the uniformization theorem of Riemann surfaces. We explain that all USLE, which are correspondent to the conformal inequivalent class of Riemann surfaces, are in the orbit of the monodromy group of Fuchsian equation. In section three, we present the representation of monodromy transformation. The relations between monodromy group and dressing group for USLE are discussed. In section four, the exchange algebra for monodromy group is given. Finally we make the conclusion and some remarks.

2. Monodromy group and uniformization solutions of the Liouvill equation

Let us consider the case of Riemann surfaces X with high genus. By the uniformization theorem, X can be realized as a quotient space H/Γ, where H is the upper half plane and $\Gamma \subset PSL(2,R)$ is a Fuchsian group. We denote the uniformization map J: $H \rightarrow X$. The inverse map J^{-1} is proved to be a linearly polymorphic function (L.P. function) on X [2], which transforms linear fractionally under the action of the related fundamental group. By mapping the *Poincaré* metric of H on X in terms of J, we get an unique metric e^{ϕ} on X, which satisfies the curvature condition $R=-1$ [3] as:

$$e^{\phi(z,\bar{z})} = \frac{|(J^{-1})'(z)|^2}{(ImJ^{-1}(z))^2} \tag{1}$$

In other words the metric(1) is a unique solution up to SL(2,R) transformation of J^{-1} of the Liouville equation on X:

$$\partial_z \partial_{\bar{z}} \phi(z,\bar{z}) = \frac{1}{2} e^{\phi(z,\bar{z})} \tag{2}$$

It is well known that the L.P. function J^{-1} can be realized by taking the ratio of two linearly independent solutions of the Fuchsian equation, which plays the important role in the uniformization theorem. This equation has an invariant form on the Riemann surface X:

$$\frac{d^2\eta}{dp^2} + \frac{1}{2}S(p)\eta(p) = 0 \tag{3}$$

for all point $p \in X$. Here S(p) is a Schwarz connection and $\eta(p)$ the multi- value $-\frac{1}{2}$ differentials on X.

The monodromy problem arises as we move an arbitrary but fixed pair of solutions $\eta_i, i = 1,2$ around a non-trival circle on X. We let ξ denote a line vector of solutions, then ξM, $M \in GL(2,C)$ is a line vector for a pair of new solutions. Without loss of generality, we may assume $Det(M) = 1$. This is because one may easily check that the Wronskian $W(p) = \eta_1(p)\eta_2'(p) - \eta_2(p)\eta_1'(p)$ is a constant. Therefore we conclude that $M \in SL(2,C)$.

Suppose $\gamma \in \pi_1(X)$, and π_1 is the fundemental group of X. It is known that $\gamma \to M(\gamma)$ defines a homomorphism [2] $M : \pi_1(X) \to GL(2,C)$. Then the set of $M(\gamma)$ is the monodromy group of the Fuchsian equation (3). From the reason mentioned above, we are able to assume this monodromy group is a subgroup of SL(2,C). Exactly speaking, there is a homomorphism between $\pi_1(X)$ and PSL(2.C) [3]. In this case, each M is a matrix with three independent complex parameters.

If X is of genus g, there are 2g generators for $\pi_1(X)$. By homomorphism, we are allowed to choose 2g matrices $A_i, B_i, i = 1,2,...,g$ as the generators of the monodromy group M such that $\prod_{i=1}^{g} A_iB_iA_i^{-1}B_i^{-1} = I$. On the other hand, it is noticed that there is certain freedom in the choice of a pair of solutions of the Fuchsian equation, from which we get the other solutions under the monodromy tranformations. The different choice leads to a similarity transformation such that $A(or B) = S^{-1}A(or B)S$. Since the matrix S contains three essential complex parameters, it is obvious that 2g generators of the monodromy group M depend on $6g-6$ complex parameters.

It is reasonable to think that the space of above $6g-6$ complex parameters is related to the space of $3g-3$ moduli parameters of the compact Riemann surface[4]. To see this, let us mention the so called Schwarz equation in Fuchsian uniformization

[3]

$$S(J^{-1}) = S_0 + \sum_{i=1}^{3g-3} \lambda_i Q^i \tag{4}$$

where $S(J^{-1})$ is the connection appearing in eq.(3), S_0 is a specific connection, λ_i the accessory parameters, and Q_i the base of the regular quadratic differentials on X. The space spanned by Q_i is the cotangent space of the Teichmuller space. From eq.(4), it is clear that the difference of $S(J^{-1}) - S_0$ can be expanded by the base of the regular quadratic differentials. The Fuchsian equation (3) will be completly determined if we fix the $3g-3$ expansion coefficients, i.e. the accessory parameters.

If η_1 and η_2 is a pair of linearly independent solutions of eq.(3), then the ratio of η_1/η_2 satisfies eq.(4). Since the ratio of the pairs of all other topological inequivalent solutions, obtained by monodromy transformation, also satisfy eq.(4), the representation space of the monodromy group should depend on both the moduli parameters, in the sense of that there exists a monodromy mapping from the vector bundle $TQ \cong T_g \times C^{3g-3}$ of regular quadratic differentials over Teichmuller space T_g to the parameter space M of the monodromy group [2]. In fact, this mapping is a local diffiomorphism. We explain it briefly as follows.

As we mentioned above, the solution of Liouville equation (2) is unique on X to within the transformation of PSL(2,R), which is acting on the L.P. function J^{-1}. It is obvious that the Fuchsian uniformization group PSL(2,R) is isomorphic with the monodromy group Γ for J^{-1} in eq.(2). In this case, there are 6g-6 real parameter in the group Γ. This isomorphism χ, i.e. the Mobius transformation with the real parameters, depends holomorphically on the base of the regular quadratic differentials [5].

It is known that the deformations of the complex structure on Riemann surface X is determined by Beltrami differentials μ. Suppose X^μ is determined by Beltrami equation $(\bar{\partial} + \mu\partial)f = 0$, then the uniformization group for X^μ is $\Gamma^\mu = f \circ \Gamma \circ f^{-1}$ with the condition $\mu(\bar{z}) = (\bar{\mu(z)})$, and $X^\mu = H/\Gamma^\mu$. By the isomorphism χ^μ, we obtain the monodromy group Γ^μ. (Here we use Γ^μ to express both the Fuchsian group and the monodromy group.) In general case (without the condition $\mu = \bar{\mu}$), Γ^μ is a quasi-Fuchsian group, which is isomorphic with the monodromy group M of the Fuchsian equation(3) if we let the Mobius transformation in M keep a directed Jordan curve C fixed [5]. In terms of the regular quadratic differentials and the Beltrami differentials, we may construct a space Q isomorphism to the space TQ.

By the monodromy mapping $p : TQ \rightarrow M$, We thus set up the relation between M and Q.

We normalize the solutions of eq.(3) such that the Wronskian

$$W(z) \equiv \eta_1'(z)\eta_2(z) - \eta_2'(z)\eta_1(z) = 1 \tag{5}$$

with the initial conditions

$$\eta_1(z_0) = \eta_2'(z_0) = 1$$

In this case, the solution of Liouville equation (2) can be generally expressed as

$$e^{-\phi} = A\eta_1\bar{\eta}_1 + C\eta_1\bar{\eta}_2 + \bar{C}\eta_2\bar{\eta}_1 + B\eta_2\bar{\eta}_2 \tag{6}$$

here $4(AB - C\bar{C}) = -1$, and A, B are real parameters. If $A = B = 0$ and $C = 1/2i$, we recover the solution (1) in the form

$$e^{-\phi} = Im(\eta_1\bar{\eta}_2) \tag{7}$$

with the condition of $W(z) = 1$. By the uniformization theorem, we know that the solution of eq.(2) is unique to within the transformation of PSL(2,R) for a fixed Riemann surface X. Therefore one may think that **the solutions (6) with two sets of different fixed parameters are just correspondent to two distinct Riemann surfaces X.** We take some space here to explain it.

We denote a disc by U on a high genus Riemann surface $X \cong H/\Gamma$ with a minus constant curvature $R = -1$. The Liouville eq. (2) is equivalent to the condition $R = -1$, if we consider a local conformal flat metric on X. According to [11], there is an unique uniformization solution of eq. (2) on X, which has the form (1). A L.P. function f_1 defined by the ratio of a pair of linearly independent solutions of the eq. (3) is related to this uniformization solution. We may keep the curvature of the surface $(X - U)$ unchanged by removing U from X. It is reasonable to consider the L.P. function f_1 as a coordinate parameter on U, since it is endowed with a Poincare metric. By lifting U on the complex plane C, we have a disk D in C with the coordinate z. We may get a new disk $\tilde{D}$ in C with the coordinate w by a fractor transformation. If this is a complex fractor transformation, conformal equivalently we have a new disk $\tilde{U}$, on which the coordinate parameter f_2 is related to f_1 by

a complex fractional transformation. Since both of $\tilde{U}$ and $(X - U)$ are orientable surfaces, we may attach them together by a standard process of surgery [13]. As sewing $\tilde{U}$ with $(X - U)$ again, because the action of the meromorphic vector fields,the local coordinate reparametrization on overlap between $\tilde{U}$ and $(X - U)$ will change the complex structure on X [12], such that f_2 can be smoothly continued to cross the sewing boundary, and it becomes a L.P. function on X, in terms of which we may construct the uniformization solution of Liouville equation on deformed Riemann surface X with $R = -1$. Since every L.P. functions related to the uniformization solution can be expressed by the ratio of a pair of linearly independent solutions of the Fuchsian equation, the L.P. function f_2 must be related to the original L.P. function f_1 by a monodromy transformation with complex parameters. (Here we denote J^{-1} by f.). We thus explain that **a quasiconformal deformation of the Riemann surface X is related to a complex monodromy transformation acting on the solution vector of the Fuchsian equation.**

By observation, there exists a matrix m with three complex parameters a, b, c, d, $ad - cb = 1$, which is the element of the monodromy group M of Fuchsian equation (3), by which the solution (7) is related with the solution (6). In terms of the matrix M, the solution (6) can be re-expressed by

$$e^{-\phi} = Im(\eta_1^{*}\bar{\eta}_2{}^{*})$$

or

$$e^{\phi} = \frac{|\,(J^{-1*})'\,|^2}{(Im J^{-1*})^2}$$

where $J^{-1*} = (aJ^{-1} + b)/(cJ^{-1} + d)$. The set of the parameters in (6) depends on $6g - 6$ complex parameters of monodromy group M in the sense of the relations

$$\begin{aligned} A &= \tfrac{1}{2i}(a\bar{c} - \bar{a}c), \\ B &= \tfrac{1}{2i}(b\bar{d} - \bar{b}d), \\ C &= \tfrac{1}{2i}(a\bar{d} - \bar{b}c). \end{aligned} \tag{8}$$

On the other hand, We notice that the general form of the uniformization solutions completly depend on three real parameters through the formula (6). This means there are three real parameters that can be freely chosen in the right hand of (8) even the parameters A, B, and C are fixed. This means that half of $6g - 6$ complex parameters in monodromy group are just the parameters of Fuchsian group, and other half are related to the moduli of the compact Riemann surfaces with high genus.

3. The dressing transformations and the represatation of the monodromy group

The uniformization solution of eq.(2) is unique for a fixed Riemann surface X. In the Teichmuller space of the Riemann surfaces, every point is correspondent to a uniformization solution of eq.(2). All of these solutions satisfy the condition of $R = -1$. Therefore the solution space of the Liouville equation in uniformization theory is the space of the uniformization solutions. We denote this space by T. From the discussion in Section 2, we know that the uniformization solutions of the Liouville equation have the general form (6). It apppears as the result of the monodromy transformation acting on a fixed but arbitrary pair of linearly independent of the Fuchsian equation discussed above. This means that the monodromy group M maps one uniformization solution of the Liouville equation to another uniformization solution. On the other hand, the Liouville equation as an integrable system has the so-called dressing symmetry, which is also the symmetry in the solution space of the Liouville equation. Hence it is natural to explore the connections between the dressing symmetry and the monodromy group for Liouville equation in the uniformization theory of Riemann surfaces.

Eq. (1) admits a Lax representation. It is shown that the Liouville equation can arise from the so-called zero curvature condition

$$F_{xt} = \partial_x A_t - \partial_t A_x + [A_x, A_t] = 0 \tag{9}$$

where

$$A_x = \frac{1}{2}\pi H + \frac{1}{2}e^{\phi}(E_+ + E_-)$$
$$A_t = \frac{1}{2}\partial_x \phi H + \frac{1}{2}e^{\phi}(E_+ - E_-)$$

and $H, E_\pm$ are the generators of the Lie algebra sl(2,R). Eq.(9) is the compatibility condition of the auxiliary linear equation

$$(\partial_\mu - A_\mu)\Psi(x,t) = 0 \tag{10}$$

The dressing transformations are associated to a factorization problem in the group $G \subset SL(2,C)$ for the Liouville theory. Any elements $g \in G$ admits a decomposition $g = g_-^{-1}g^+$ with $g_\pm \in B_\pm = HN_\pm$. Here $G = N_-HN_+$ is the Gaussian

decomposition and $B_\pm$ is the Borel subgroup. The dressing transformations, as the gauge tranformations preserving the form of the lax connection, are defined as

$$\Psi(x) \to \Psi^g(x) = \Theta_\pm \Psi(x) g_\pm^{-1} \tag{11}$$

with $\Theta_-^{-1}\Theta_+ = \Psi(x) g \Psi^{-1}(x)$. It induces a gauge transformation on the lax connection A_μ such that

$$\begin{aligned} A_x^g &= \Theta_\pm A_x \Theta_\pm^{-1} + (\partial_x \Theta_\pm)\Theta_\pm^{-1} \\ &= \tfrac{1}{2}\pi^g H + \tfrac{1}{2}e^{\phi^g}(E_+ + E_-) \\ A_t^g &= \Theta_\pm A_t \Theta_\pm^{-1} + (\partial_t \Theta_\pm)\Theta_\pm^{-1} \\ &= \tfrac{1}{2}\partial_x \phi^g H + \tfrac{1}{2}e^{\phi^g}(E_+ - E_-) \end{aligned} \tag{12}$$

In this way, we get a new solution[7,8]

$$\phi^g = \phi - \Delta_+^g = \phi + \Delta_-^g$$

where $\Theta_\pm = K_\pm^g M_\pm^g$, and $K_\pm = exp(\Delta_\pm^g)$, with $M_\pm^g \in N_\pm$, and $K_\pm^g \in H$.

Suppose the finite dimensional Lie algebra in our case corresponding to the group G is $\mathcal{G}$. Given a highest weight vector $| \lambda_{max} >$, we define[9]:

$$\begin{aligned} \xi(x) &=< \lambda_{max} \mid e^{-\Phi(x)} T(x) \\ \bar{\xi}(x) &= T^{-1}(x) e^{-\Phi(x)} \mid \lambda_{max} > \end{aligned} \tag{13}$$

where $\Phi = \phi H$, H is the base of the Cartan subalgebra of $\mathcal{G}$, and $T(x)$ is the transport matrix.

$$T(x) = Pexp(-\int_0^x A_x dx)$$

By noticing $T(\gamma x) = T(x + 2\pi) = T(x)T(2\pi), \gamma \in \pi_1(X)$, we find that the monodromy group Γ has a representation in terms of the transport matrix T

$$\begin{aligned} \xi^m(x) &= \xi(x)\mathrm{m} \\ &= \xi(x + 2\pi) \\ &=< \lambda \mid e^{-\Phi(x+2\pi)} T(x + 2\pi) \\ &=< \lambda \mid e^{-\Phi(x)} T(x) T(2\pi) \\ &= \xi(x) T(2\pi) \end{aligned} \tag{14}$$

and

$$\begin{aligned}\bar{\xi}^m(x) &= \mathrm{m}^{-1}\bar{\xi}(x) \\ &= \bar{\xi}(x+2\pi) \\ &= T^{-1}(2\pi)\bar{\xi}(x)\end{aligned} \tag{15}$$

where $m \in \Gamma$. From eqs. (14) and (15), we know that

$$e^{-2\Phi^m} = (\xi^m \bar{\xi}^m) = (\xi\bar{\xi}) = e^{-2\Phi}$$

which shows that the uniformization solution is exactly unique to within the transformations of the monodromy group wth real parameters. The Poisson bracket for the monodromy matrix T is well known

$$T \otimes T = -[r_{\pm}, T \otimes T]$$

Here $r_{\pm}$ is the solution of the classical $Y-B$ equation.

As we consider the monodromy properties of the fields ξ and $\bar{\xi}$ under the action of the monodromy group M, and $M \subset PSL(2,C)$. The field ϕ, under the action of the group M, will be mapped from one point to another point in the space of the uniformization solutions. Suppose $m \in M$, then $m : \phi \longrightarrow \phi^m$ with

$$e^{-2\Phi^m} = \xi m \bar{m}^{-1} \xi$$

The general form of ϕ^m has been shown in formula (6). The matrix $m\bar{m}^{-1}$ has a form

$$\begin{pmatrix} C & -A \\ B & -\bar{C} \end{pmatrix}$$

with $AB - C\bar{C} = 1$. There is an unique decomposition $m\bar{m}^{-1} = g_-^{-1} g_+$, such that g_- and g_+ have inverse components on the Cartan torus. Recall that for every element $g \in G$, there is a couple $(g_-, g_+) \in G^*$, and the group G^* is the dual of the dressing group G. The dressing transformation map the field ϕ into the field ϕ^g, meanwhile it induces an action on the fields ξ and $\bar{\xi}$ such that

$$\xi^g = \xi g_-^{-1} \quad and \quad \bar{\xi}^g = g_+ \bar{\xi}$$

In terms of the fields ξ and $\bar{\xi}$, ϕ^g is expressed as

$$\begin{aligned}e^{-2\Phi^g} &= \xi^g \bar{\xi}^g \\ &= \xi g_-^{-1} g_+ \bar{\xi} \\ &= \xi g \bar{\xi}\end{aligned}$$

Now we consider a subset Ω in G. To let the elements $g \in \Omega$ have the form of the matrix $m\bar{m}^{-1}$, and

$$m = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

a, b, c, d are complex parameters satisfying $ad - cb = 1$. We then find that $\Phi^m = \Phi^g$, if $g \in \Omega$, by which we get the representation of the monodromy group M

$$\begin{aligned}\xi^m(x) &= \xi(x)\mathrm{m} \\ &=< \lambda_{max} \mid e^{-\Phi^m(x)} T^m(x) \\ &=< \lambda_{max} \mid e^{-\Phi^g(x)} T^g(x+2\pi) \\ &= \xi^g(x) T^g(2\pi) \\ &= \xi(x) g_-^{-1} \Theta_{\pm} T(2\pi) g_{\pm}^{-1}\end{aligned} \tag{16}$$

and

$$\begin{aligned}\bar{\xi}^m(x) &= \bar{\mathrm{m}}^{-1}\bar{\xi}(x) \\ &= (T^m)^{-1}(x) e^{-\Phi^m(x)} \mid \lambda_{max} > \\ &= (T^g)^{-1}(2\pi)\bar{\xi}^g(x) \\ &= g_{\pm} T^{-1}(2\pi) \Theta_{\pm}^{-1} g_+ \bar{\xi}(x)\end{aligned} \tag{17}$$

where $T(x)$ is a solution of the eq.(10), and $T^g = \Theta_{\pm}^g T g_{\pm}^{-1}$. It is easy to check that

$$\xi^m(x)\bar{\xi}^m(x) = \xi^g(x)\bar{\xi}^g(x) \tag{18}$$

Now let us consider the successive monodromy transformations on ξ and $\bar{\xi}$. By the formulas (16) and (17), we find

$$\begin{aligned}
\xi^{mn}(x) &= \xi(x)\mathrm{m}\bullet\mathrm{n} \\
&= < \lambda_{max} \mid e^{-\Phi^{mn}(x)} T^{mn}(x) \\
&= < \lambda_{max} \mid e^{-(\Phi^{g})^{n}(x)} (T^{g}(x)T^{g}(2\pi))^{n} \\
&= < \lambda_{max} \mid e^{-\Phi^{h}(x)} (T^{m}(x-2\pi))^{n} T^{g}(2\pi) \\
&= \xi^{h}(x)T^{h}(2\pi) \\
&= \xi(x)h_{-}^{-1}T^{h}(2\pi)
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
\bar{\xi}^{mn}(x) &= (\bar{\mathrm{m}}\bullet\bar{\mathrm{n}})^{-1}\bar{\xi} \\
&= (T^{mn})^{-1}(x)e^{-\Phi^{mn}(x)} \mid \lambda_{max} > \\
&= (T^{h}(2\pi)(T^{h})^{-1}(x)e^{-\Phi^{h}(x)} \mid \lambda_{max} > \\
&= (T^{h}(2\pi))^{-1}h_{+}\bar{\xi}
\end{aligned} \tag{20}$$

Like eq.(18), we also have

$$\begin{aligned}
\xi^{mn}\bar{\xi}^{mn} &= \xi(\mathrm{m}\bullet\mathrm{n})(\bar{\mathrm{m}}\bullet\bar{\mathrm{n}})^{-1}\bar{\xi} \\
&= \xi^{h}\bar{\xi}^{h} \\
&= \xi h_{-}^{-1}h_{+})\bar{\xi}
\end{aligned} \tag{21}$$

Here we find that the successive monodromy transformations are still related to a dressing transformation. It should be noticed that the algebraic structures of the monodromy and dressing group are different. To remember $g = g_{-}^{-1}g_{+}$, $h = h_{-}^{-1}h_{+} \in G$ with (g_{-}, g_{+}), $(h_{-}, h_{+}) \in G^{*}$, the product in the dual group G^{*} is defined by[8]

$$:(g_{-}h_{-})^{-1}g_{+}h_{+} :=: (g_{-}^{-1}g_{+})(h_{-}^{-1}h_{+}):$$

If

$$g_{-}^{-1}g_{+} = \mathrm{m}\bar{\mathrm{m}}^{-1}$$

$$h_{-}^{-1}h_{+} = \mathrm{n}\bar{\mathrm{n}}^{-1}$$

such kinds of successive dressing transformations are not correspondent to a monodromy transformation, in other words

$$:(\mathrm{m}\bullet\mathrm{n})(\bar{\mathrm{m}}\bullet\bar{\mathrm{n}})^{-1} :\neq: (\mathrm{m}\bar{\mathrm{m}}^{-1})\bullet(\mathrm{n}\bar{\mathrm{n}}^{-1}):$$

4. The exchange algebra for the monodromy group

The Lie - Poisson algebraic structure of the dressing group G naturally induces the Poisson brackets for the monodromy transformations. It is known that the exchange algebra for the dual group G^{*} is [8]

$$\begin{aligned}
\{g\otimes g\}_{G^{*}} &= +(g\otimes I)\, r_{+}\, (I\otimes g) + (I\otimes g)\, r_{-}\, (g\otimes I) \\
&\quad -(g\otimes g)\, r_{\pm} - r_{\mp}\, (g\otimes g) \\
\{g_{+}\otimes g_{+}\}_{G^{*}} &= -[r_{\pm}, g_{+}\otimes g_{+}] \\
\{g_{-}\otimes g_{-}\}_{G^{*}} &= -[r_{\mp}, g_{-}\otimes g_{-}] \\
\{g_{-}\otimes g_{+}\}_{G^{*}} &= -[r_{-}, g_{-}\otimes g_{+}]
\end{aligned} \tag{22}$$

If $g \in \Omega$, the elements g also can be factorized into $m\bar{m}^{-1}$, which satisfy the first equation in (22). On the other hand, we have obtained the representation of the elements m in the monodromy group M, i.e. $\mathrm{m} = g_{-}^{-1}\Theta_{\pm}^{g}T(2\pi)g_{\pm}^{-1}$ and $\bar{\mathrm{m}}^{-1} = g_{\pm}T^{-1}(2\pi)(\Theta_{\pm}^{g})^{-1}g_{+}$. In terms of the eq.(22) and the Poisson bracket of T, we then deduce the exchange algebra of the monodromy group M for the high genus Riemann surfaces.

$$\begin{aligned}
\{m\otimes m\}_{G^{*}} &= -[r_{\mp}, m\otimes m] - m\otimes m\, (E_{+}\otimes E_{-} - E_{-}\otimes E_{+}) \\
\{\bar{m}\otimes\bar{m}\}_{G^{*}} &= -[r_{\pm}, \bar{m}\otimes\bar{m}] - \bar{m}\otimes\bar{m}\, (E_{+}\otimes E_{-} - E_{-}\otimes E_{+}) \\
\{m\otimes\bar{m}\}_{G^{*}} &= -r_{-}\, (m\otimes\bar{m}) \\
\{\bar{m}\otimes m\}_{G^{*}} &= -r_{+}\, (\bar{m}\otimes m)
\end{aligned} \tag{23}$$

Under the monodromy transformation, the chiral fields ξ and $\bar{\xi}$ are changed into the fields ξ^{m} and $\bar{\xi}^{m}$, which are related to the dressing chiral fields ξ^{g} and $\bar{\xi}^{g}$

separately from the eqs. (16) and (17), i.e.

$$\xi^m = \xi^g T^g, \qquad \bar{\xi}^m = (T^g)^{-1}\bar{\xi}^g$$

To recall that the exchange algebra for the chiral fields ξ and $\bar{\xi}$[10]

$$\begin{aligned}
\{\xi(x) \otimes \xi(x')\} &= \xi(x) \otimes \xi(x')\ [\ \theta(x-x')\ r_+ + \theta(x'-x)\ r_-\ ] \\
\{\bar{\xi}(x) \otimes \bar{\xi}(x')\} &= [\ \theta(x-x')\ r_- + \theta(x'-x)\ r_+\ ]\ \bar{\xi}(x) \otimes \bar{\xi}(x') \\
\{\xi(x) \otimes \bar{\xi}(x')\} &= -\xi(x) \otimes I\ r_-\ I \otimes \bar{\xi}(x')
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
\{T \otimes T\} &= -[\ r_\pm, T \otimes T\ ] \\
\{\xi(x) \otimes T\} &= \xi(x) \otimes T\ r_- \\
\{\bar{\xi}(x) \otimes T\} &= -I \otimes T\ r_+\ \bar{\xi}(x) \otimes I
\end{aligned} \tag{25}$$

The exchange algebra for $\xi^g(x), \bar{\xi}^g(x)$ and T^g keeps the same form of (24) and (25), thus the Poisson brackets for the fields ξ^m and $\bar{\xi}^m$ are calculated as follows (It should be reminded that here the parameter x appearing in the exchange algebra on Riemann surface is the local parameter on time-level line, which is defined by Krichever and Novikov approach [14].)

$$\begin{aligned}
\{\xi^m(x) \otimes \xi^m(x')\} &= \xi^m(x) \otimes \xi^m(x')\ \ r_\pm \\
&\quad + \xi^m(x) \otimes \xi^g(x')\ \ r_+\ \ (I \otimes T^g) \\
&\quad + \xi^g(x) \otimes \xi^m(x')\ \ r_-\ \ (T^g \otimes I) \\
\{\bar{\xi}^m(x) \otimes \bar{\xi}^m(x')\} &= r_\pm\ \ \bar{\xi}^m(x) \otimes \bar{\xi}^m(x') \\
&\quad + (T^g)^{-1} \otimes I\ \ r_+\ \ \bar{\xi}^g(x) \otimes \bar{\xi}^m(x') \\
&\quad + I \otimes (T^g)^{-1}\ \ r_-\ \ \bar{\xi}^m(x) \otimes \bar{\xi}^g(x') \\
\{\xi^m(x) \otimes \bar{\xi}^m(x')\} &= -\xi^g(x) \otimes I\ \ r_-\ \ T^g \otimes \bar{\xi}^m(x') \\
&\quad + \xi^m(x) \otimes T^g\ \ r_-\ \ I \otimes \bar{\xi}^g(x') \\
&\quad - \xi^m(x) \otimes I\ \ r_\pm\ \ I \otimes \bar{\xi}^m(x')
\end{aligned} \tag{26}$$

From the formula (26), we are able to find the Poisson brackets among the chiral fields ξ , $\bar{\xi}$ and the monodromy transformations $m, \bar{m}$.

$$\begin{aligned}
\{\xi \otimes m\} &= \xi \otimes m\ \ [\ g_-^{-1} \otimes I\ r_-\ g_- \otimes I + \tfrac{1}{2} m \otimes I\ r_-\ m^{-1} \otimes I\ ] \\
\{\bar{\xi} \otimes \bar{m}^{-1}\} &= [\ g_+^{-1} \otimes I\ r_+\ g_+ \otimes I + \tfrac{1}{2}\bar{m} \otimes I\ r_+\ \bar{m}^{-1} \otimes I\ ]\ \ \bar{\xi} \otimes \bar{m}^{-1} \\
\{\xi \otimes \bar{m}^{-1}\} &= -\xi \otimes \bar{m}^{-1}[g_-^{-1} \otimes \bar{m}\ r_-\ g_- \otimes \bar{m}^{-1} + \tfrac{1}{2} m \otimes \bar{m}\ r_-\ m^{-1} \otimes \bar{m}^{-1}] \\
\{\bar{\xi} \otimes m\} &= -[g_+^{-1} \otimes m\ r_+\ g_+ \otimes m^{-1} + \tfrac{1}{2}\bar{m} \otimes m\ r_+\ \bar{m}^{-1} \otimes m^{-1}]\bar{\xi} \otimes m
\end{aligned} \tag{27}$$

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