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ABSTRACT

We show that differential calculus on discrete group Z_2 is equivalent to A. Connes' approach in the case of two discrete points. They are the same theory in terms of different basis and the discrete group Z_2 is the permutation group of two discrete points.

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1 Introduction

Since A. Connes introduced non-commutative geometry into particle physics[1], a lot of efforts have been made along this direction [2-6]. Soon after, Sitarz [7] proposed an attractive approach towards the construction of a pure gauge theory on arbitrary discrete groups in which Higgs appear as gauge fields on discrete groups. Though simple in its structure, it contains all the main results in [1]. But Sitarz [7] could not reach the goal towards the realistic model building in particle physics, then its results were mathematical one. Fortunately, this problem had been solved by the authors[9,10].

The main idea in these two approach is to regard Higgs field as a kind of gauge field. The first approach is to take Higgs as gauge field on discrete points and the second is on discrete group. After the physical model building, we find that the results are the same whenever either method is used. This made us think: what is the relationship between these two approaches?

In this paper, first of all, we summarize A. Connes' differential calculus on two discrete points (I) and Sitarz' differential calculus on discrete group Z_2 (II). Then we show that these two approaches are equivalent to each other and the Z_2 group in (II) is just the permutation group of two discrete points in (I).

2 Differential Calculation on two discrete points

In this section, we summarize A. Connes' non-commutative differential calculus on two discrete points. For detailed information, see [8].

Algebra $\bar{\mathcal{A}}$ is complex number $\mathcal{D}$ valued functions on two points space $X = \{a, b\}$. An element $f \in \bar{\mathcal{A}}$ is given by two complex numbers $f(a), f(b) \in \mathcal{D}$. We let $\Omega^\bullet(\bar{\mathcal{A}})$ be the universal differential graded algebra over $\bar{\mathcal{A}}$. The degree of $\bar{\mathcal{A}}$ is 0 and elements of differential one form are generated by da, $a \in \bar{\mathcal{A}}$ with the following properties:

$$\begin{aligned} & d(ab) = d(a)b + adb, \quad \forall a, b \in \bar{\mathcal{A}} \\ & (da)^* = -da^* \\ & d1 = 0. \end{aligned} \tag{1}$$

The higher order differential forms are defined by

$$d(a^0 da^1 \cdots da^n) = da^0 da^1 \cdots da^n, \quad \forall a^j \in \bar{\mathcal{A}} \tag{2}$$

and satisfy

$$\begin{aligned} d^2\omega = 0, \quad \forall \omega \in \Omega^*(\bar{\mathcal{A}}) \\ d(\omega_1\omega_2) = (d\omega_1)\omega_2 + (-1)^{\partial\omega_1}\omega_1 d\omega_2, \quad \forall \omega_j \in \Omega^*(\bar{\mathcal{A}}). \end{aligned} \tag{3}$$

A 0-dimensional K cycle (h, D, γ) over an algebra $\bar{\mathcal{A}}$ with involution $*$, is given by a representation of $\mathcal{A}$ in the Hilbert space h corresponding to a decomposition of h as a direct sum $h = h_a \oplus h_b$ with the action of $\mathcal{A}$ given by $f \in \mathcal{A} \rightarrow \begin{pmatrix} f(a) & \\ & f(b) \end{pmatrix}$ and an unbounded selfadjoint operator D with compact resolvant, such that $[D, a]$ is bounded for any $a \in \bar{\mathcal{A}}$. An involutive representation π of the universal algebra $\forall \omega \in \Omega^*\bar{\mathcal{A}}$ in h is defined as follows:

$$\pi(a^0 da^1 \cdots da^n) = a^0[D, a^1] \cdots [D, a^n], \quad \forall a^j \in \bar{\mathcal{A}}. \tag{4}$$

The operator D may be represented in h as a 2×2 matrix in the following decomposition form

$$D = \begin{pmatrix} D_{aa} & D_{ab} \\ D_{ba} & D_{bb} \end{pmatrix}. \tag{5}$$

We shall take D of the form

$$D = \begin{pmatrix} & D_{ab} \\ D_{ba} & \end{pmatrix} \tag{6}$$

because the diagonal elements commute with the action of $\bar{\mathcal{A}}$. Since D is a selfadjoint operator, so we can introduce $D_{ba} = D^*_{ab} = M$, then D may be written as

$$D = \begin{pmatrix} & M^* \\ M & \end{pmatrix}. \tag{7}$$

In the following, we will discuss the simplest case $n_a = n_b$, where n_a and n_b are the dimension of Hilbert space h_a and h_b.

First we introduce the idempotent functional basis $e_1, e_2 \in \bar{\mathcal{A}}$, which satisfy

$$\begin{aligned} & e_1(a) = e_2(b) = 1, \\ & e_1(b) = e_2(a) = 0, \\ & e_i \cdot e_j = \delta_{ij} e_i, \\ & e_1 + e_2 = 1 \end{aligned} \tag{8}$$

Then a function on discrete points may be written as $f = f_1 e_1 + f_2 e_2$ and the identity $1 = e_1 + e_2$ and we have

$$de_1 + de_2 = 0. \tag{9}$$

The space $\Omega^1(\bar{\mathcal{A}})$ is a 2 dimensional space , which has the following basis:

$$e_1 de_1 , \quad e_2 de_2. \tag{10}$$

So that every element of $\Omega^1(\bar{\mathcal{A}})$ is of the form $\lambda e_1 de_1 + \mu e_2 de_2$. The differential $d: \ \bar{\mathcal{A}} \rightarrow \Omega^1(\bar{\mathcal{A}})$ is the finite difference:

$$df = (\Delta f) e_1 de_1 - (\Delta f) e_2 de_2 , \quad \Delta f = f(a) - f(b). \tag{11}$$

A gauge potential is given by a self adjoint element of Ω^1,

$$V = \Phi^\dagger e_1 de_1 + \Phi e_2 de_2, \tag{12}$$

then its curvature is

$$\theta = dV + V^2 = (\Phi + \Phi^\dagger)de_1de_1 - (\Phi\bar{\Phi})de_1de_1. \tag{13}$$

Under the representation π one has $\pi(de_1) = \begin{pmatrix} 0 & -M^* \\ M & 0 \end{pmatrix}$ and $\pi(de_1de_1) = \begin{pmatrix} -M^*M & 0 \\ 0 & -MM^* \end{pmatrix}$. Therefore we can get the Yang-Mills action

$$\mathcal{L}_{YM} = <\theta, \theta> = (|\Phi - 1|^2 - 1)^2 <de_1de_1, de_1de_1> \tag{14}$$

The inner product on Ω^k is defined by

$$<T_1, T_2> = Tr(\pi(T_1)\pi(T_2)), \quad \forall T_i \in \Omega^k \tag{15}$$

then we have

$$<de_1, de_1> = Tr(\pi(de_1)\pi(de_1)) = 2Tr(MM^*), \quad <de_1de_1, de_1de_1> = 2Tr((MM^*)^2).$$

So we get the Lagrangian for the gauge field:

$$\mathcal{L} = 2(|\Phi - 1|^2 - 1)^2 Tr((MM^*)^2). \tag{16}$$

This is of Higgs potential type up to some coupling constants. To get the entire Lagrangian of the Higgs, we need to consider the space-time part. For detail it is discussed in [8].

3 Differential calculus on discrete group Z_2

In this section, we will outline the notion of differential calculus theory on discrete groups $Z_2 = \{e, r\}$. For details, it is referred to [7].

Let $\mathcal{A}$ be the algebra of the all complex valued functions on Z_2. The right action of Z_2 group on $\mathcal{A}$ read as

$$(R_e f)(g) = f(g), \quad (R_r f)(g) = f(g \odot r), \quad g \in Z_2, \quad \forall f \in \mathcal{A}, \tag{17}$$

where $\odot$ denotes the group multiplication. The derivative is defined as

$$\partial_g f = f - R_g f, \quad g \in Z_2. \tag{18}$$

It is easy to see $\partial_e f = 0$ and the only nontrivial derivative is ∂_r, which satisfies

$$\partial_r \cdot \partial_r = 2\partial_r. \tag{19}$$

The first order differential calculus (Ω^1, d) may be given by the definition of its dual space, one dimensional vector space on $\mathcal{A}$ with basis ∂_r as follows:

$$\chi(\partial_r) = 1, \tag{20}$$

where χ is the basis of differential one form.

The definition for higher order forms is natural and we take Ω^n to be the tensor product of n copies of Ω^1, $\Omega^n = (\Omega^1)^{\otimes n}$ and $\Omega^0 = \mathcal{A}$. To complete the construction of the differential algebra $\Omega^* = \bigoplus_n \Omega^n$, we need to define the exterior derivative d, $d: \Omega^n \to \Omega^{n+1}$ whose action on $\mathcal{A}$ is defined by

$$df = \partial_r f \chi. \tag{21}$$

It is easy to prove the following lemma [7]:

There exists exactly one linear exterior derivative operator d such that it satisfies

(*i*) $d^2 = 0$,

(*ii*) $d(fg) = df \cdot g + (-1)^{deg f} f \cdot dg, \quad \forall f, g \in \Omega^*$,

provided that χ satisfy the following two conditions

$$\begin{aligned} \chi f &= (R_r f)\chi, \quad f \in \mathcal{A}, \\ d\chi &= -2\chi\chi. \end{aligned} \tag{22}$$

The involution on the differential algebra agrees with the complex conjugation on $\mathcal{A}$ and (graded) commutes with d, i.e. $d(\omega^*) = (-1)^{deg\omega}(d\omega)^*$. Again, it is sufficient to calculate it if we set the involution to χ, the basis of one-forms, $(\chi)^* = -\chi$.

The Haar integral on discrete group Z_2 is introduced as a complex valued linear functional on $\mathcal{A}$ that remains invariant under the action of R_g,

$$\int_{Z_2} f = \frac{1}{2}(f(e) + f(r)), \tag{23}$$

which is normalized such that $\int_{Z_2} 1 = 1$.

Let us consider the case that there are Lie group transformations among the elements of the function space and those transformations also depend on the elements of the discrete group. Then the derivatives introduced above are no longer covariant. In order to get meaningful differential calculus in this case, the connection one form is needed to define the covariant exterior differential:

$$D = d_G + A, \tag{24}$$

where the connection one form A is a self adjoint element in Ω^1 and may be written as

$$A = \phi\chi. \tag{25}$$

Then the generalized curvature two form is

$$F = dA + A^2 = (-\phi - R_r\phi + \phi R_r\phi)\chi\chi. \tag{26}$$

The self adjoint character of A requires that $\phi^\dagger = R_r\phi$, then we have

$$F = (-\phi - \phi^\dagger + \phi\phi^\dagger)\chi\chi. \tag{27}$$

After introducing the metric, we can get the Lagrangian for the theory.

In the Z_2 case, we can define the metric as

$$< \chi, \chi >= \eta, \quad < \chi \otimes \chi, \chi \otimes \chi >= \eta^2, \tag{28}$$

where η is a positive number. Therefore

$$\mathcal{L} =< \overline{F}, F >= 2\eta^2(|\Phi - 1|^2 - 1)^2. \tag{29}$$

This is the same type of Higgs potential as we get in A. Connes' approach. The entire Lagrangian of the Higgs may be introduced also if we consider the space-time part. For detail it is referred to [9, 10].

It is easy to see that these two approaches are similar whatever the structures or results. Then a stimulating question may be raised as to whether we can build a bridge between these two approaches? In next section, we will find that the answer is yes.

4 Equivalence of Two Approaches

To illustrate the equivalence between these two approaches, we start from A. Connes' derivative on two discrete points. Let $f = f_1e_1 + f_2e_2$ be a function on two discrete point space X, then

$$df = (f_1 - f_2)(e_1de_1 - e_2de_2). \tag{30}$$

We know that the permutation group of two points is a Z_2 group. Then we can define the Z_2 action on the two points according to permutation as follows:

$$R_ee_1 = e_1, \ R_re_1 = e_2, \ R_re_2 = e_1. \tag{31}$$

We find that the derivative (30) may be written as

$$df = (f - R_rf)(e_1de_1 + e_2de_2). \tag{32}$$

If we introduce

$$\bar{\partial}_r f = f - R_r f, \quad \bar{\chi} = (e_1 de_1 + e_2 de_2), \tag{33}$$

we have

$$df = \bar{\partial}_r f \cdot \bar{\chi} \tag{34}$$

and every element of $\Omega^1(\bar{\mathcal{A}})$ may be in terms of $\bar{\chi}$ as,

$$V = \lambda e_1 de_1 + \mu e_2 de_2 = f \cdot \bar{\chi}$$

where $f = \lambda e_1 + \mu e_2$.

To verify (34) is just the differential calculus in the case of Z_2 in Sitarz' approach, we should check the relations (19) and (22). In doing so, it is useful to write out some relations, which result from the equations (8),

$$\begin{aligned} de_1 e_2 &= -e_1 de_2, \quad de_2 e_1 = -e_2 de_1 \\ de_1 e_1 &= e_2 de_1, \quad e_1 de_1 = de_1 e_2 \\ de_2 e_2 &= e_1 de_2, \quad e_2 de_2 = de_2 e_1 \end{aligned} \tag{35}$$

¿From direct calculation, it is easy to show that:

$$\bar{\partial}_r \cdot \bar{\partial}_r = 2\bar{\partial}_r, \bar{\chi} f = R_r f \bar{\chi}, d\bar{\chi} = -2\bar{\chi}\bar{\chi},$$

The involution on the algebra $\bar{\mathcal{A}}$ is just the complex conjugation when it valued on discrete points,

$$f^*(X) = (f(X))^*, \tag{36}$$

then, we have $e_1^* = e_1$, $e_2^* = e_2$, from which we also have $(de_1)^* = de_1, (de_2)^* = de_2$. Using these relations, we can get the equation

$$(\bar{\chi})^* = -\chi. \tag{37}$$

So far, it is shown that $\bar{\partial}_r$ and $\bar{\chi}$ satisfy all the relations in (19) and (22). Hence, we conclude that A. Connes' differential calculus on two discrete points is equivalent to Sitarz' approach in the case of Z_2.

To discuss the Yang-Mills action, first we should study the metric. From the definition of inner product on Ω^K, we have

$$< \bar{\chi}, \bar{\chi} >= Tr\pi(\bar{\chi})^2 = 2Tr(MM^*), \quad < \bar{\chi}\bar{\chi}, \bar{\chi}\bar{\chi} >= Tr\pi(\bar{\chi})^4 = 2Tr(MM^*)^2. \tag{38}$$

This metric appears a little different from (28) in Sitarz' approach. However they may be consistent with each other if we redefine the definition of inner product on Ω^k as

$$< T_1, T_2 >= \alpha Tr(\pi(T_1)\pi(T_2)), \quad \forall T_i \in \Omega^k, \tag{39}$$

where $\alpha = \frac{Tr(MM^*)^2}{(TrMM^*)^2}$. Then we get

$$< \bar{\chi}, \bar{\chi} >= 2\bar{\eta}, \quad < \bar{\chi}\bar{\chi}, \bar{\chi}\bar{\chi} >= 2\bar{\eta}^2, \tag{40}$$

where $\bar{\eta} = \frac{Tr(MM^*)^2}{(TrMM^*)^2}$ is a positive number. These results correspondence to the formulas in (28) after integrating over discrete group Z_2.

$$\int_{Z_2} < \chi, \chi >= 2\eta, \quad \int_{Z_2} < \chi\chi, \chi\bar{\chi} >= 2\eta^2. \tag{41}$$

In A. Connes' approach, a gauge potential is given by a self adjoint element of Ω^1 as

$$V = \Phi^\dagger e_1 de_1 + \Phi e_2 de_2. \tag{42}$$

In terms of $\bar{\chi}$ the gauge potential may be written as $V = f \cdot \chi$, where $f = \Phi^\dagger e_1 + \Phi e_2$. Using the formulas in Sitarz' approach, we obtain the curvature

$$\theta = dV + V^2 = (-\Phi - \Phi^* + \Phi\Phi^*)\bar{\chi}\bar{\chi}. \tag{43}$$

Therefore, the Yang-Mills action is

$$\mathcal{L} = <\theta, \theta> = 2\bar{\eta}^2(|\Phi - 1|^2 - 1). \tag{44}$$

It is easy to see that whenever either way is used, we get the same results.

If starting from differential calculus on discrete group Z_2, we can get A. Connes's approach also. For this purpose, we should introduce the idempotent basis $e_e, e_r \in \mathcal{A}$, which satisfy

$$e_e(e) = e_r(r) = 1$$

$$e_e(r) = e_r(e) = 0$$

$$e_g \cdot e_h = \delta_g^h e_h, \quad g, h \in Z_2$$

$$e_e + e_r = 1$$

Then a function on discrete group Z_2 may be written as $f = f_e e_e + f_r e_r$ and the right action of the group may be introduced as $R_r \cdot e_e = e_r, R_r \cdot e_r = e_e$. It is easy to show that $(R_r \cdot f)(g) = f(g \cdot r)$. The more detailed calculation is similar as the previous part of this section.

From a mathematical point of view, We have shown that A. Connes' differential calculus on two discrete points is equivalent to Sitarz' approach in the case of Z_2. The basis of differential one form in the second approach is a recombination of the first one, and the discrete group is nothing but the permutation group of the discrete points. In [11], we have shown that another known non-commutative geometry approach proposed by R. Coquereaux et.al.[3] is the matrix representation of the gauge theory on discrete group Z_2. Therefore, these three approaches of non-commutative geometry are consistent with each other.

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