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FERROMAGNETIC POTTS MODEL IN A MAGNETIC FIELD: FINITE LATTICE EXTRAPOLATION OF THE (1+1)-DIMENSIONAL HAMILTONIAN

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ABSTRACT

The $(1+1)$-dimensional Hamiltonian q-state ferromagnetic Potts model in an external magnetic field H is studied using finite lattice extrapolation techniques. The possible phases and their boundaries are determined from the ground state energy and the gap in the excitation energy, for arbitrary values of q and both for positive and negative field.

We found that, for H positive, there is a critical value q_c, where a first order transition line appears for $q > q_c$, starting at the zero field transition point and terminating at a critical point in $(h/\lambda, H/\lambda)$ plane, which separates the disordered phase and the ordered phase in the direction of H. For h negative a critical transition line separating the disordered phase and the others $(q-1)$ ordered phases becomes first order for $q > q_c$.

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I Introduction

There has been considerable recent work on the phase transitions of the q-state Potts model [1], which exhibits interesting properties depending on the number of states q. Indeed there exists a critical value of q, $q_c(D)$, D is the space dimension, where the phase transition is of first order for $q > q_c(D)$, otherwise it is continuous.

On the other hand, in the present paper, we are interested in the Hamiltonian version of 2D Ferromagnetic Potts Model (FPM) in a magnetic field H. By taking the time continuum limit, the classical 2D model [2–7] can be related by the transfer matrix to one–dimensional (1D) quantum equivalent one which is the Potts chain with a transverse field h [8]. However, we investigate the phase diagrams of the quantum model in $(1+1)$–dimension, using finite lattice extrapolation techniques [9]. The method is used to calculate the energy gap from the ground state and low lying excited state; for a chain of N sites with periodic boundary conditions. This enables us to determine the boundaries of the phases for arbitrary values of q and both for positive and negative field.

We concentrated mainly on locating the critical magnetic field as a function of the transverse field, $H_c(h)$. We obtain two kinds of phase diagrams depending on q. For $q > q_c$, we have two first order transition lines, one for H positive and the other for H negative, which meet at the critical transverse field $h_c(H = 0)$. The upper line $(H > 0)$ terminates at a critical point while the lower one $(H < 0)$ does not.

The content of the present paper is as follows. In Section II the formulation of the Hamiltonian version of the q–state FPM in an external magnetic field, as well as the results of the finite lattice extrapolation are presented. Section III contains our conclusion.

II Hamiltonian formulation and results

The Hamiltonian version of the q–state Potts model has been derived by several authors using the time–continuum limit of the transfer matrix. Here we use the form given by Solyom and Pfeuty [10]. However, in the presence of an external magnetic field the Hamiltonian can be written in the following form:

$$H = H_{\text{Potts}} + H_{\text{ext}} + H_{\text{field}} \ , \tag{1}$$

where H_{Potts} is the usual Potts coupling between the neighbouring spins.

$$H_{\text{Potts}} = -\lambda \sum_i \left(\delta_{s_i s_{i+1}} - \frac{1}{q} \right) = -\frac{\lambda}{q} \sum_i \sum_{k=1}^{q-1} \Omega_i^k \, \Omega_{i+1}^{q-k} \tag{2}$$

where $s_i = 1, 2, \ldots, q$ and Ω_i is a diagonal matrix,

$$\Omega = \begin{bmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{q-1} \end{bmatrix}, \quad \omega = \exp(2\pi i/q)$$

$H_{\rm ext}$ represents the external magnetic field applied in the direction of state 1,

$$H_{\rm ext} = -H \sum_i \left(\delta_{s_i,1} - \frac{1}{q} \right) = -\frac{H}{q} \sum_i \sum_{k=1}^{q-1} \Omega_i^k \tag{3}$$

$H_{\rm field}$ corresponds to a transverse field which flips the spins,

$$H_{\rm field} = -h \sum_i \sum_{k=1}^{q-1} M_i^k \tag{4}$$

where,

$$M = \begin{bmatrix} 0 & 1 & 0 & \ldots & \ldots & \\ 0 & 0 & 1 & \ldots & \ldots & \\ \vdots & & & & & \\ \vdots & & & & \ldots & 1 \\ 1 & 0 & 0 & & \ldots & 0 \end{bmatrix}$$

It is sometimes more convenient to use a different representation, namely one in which the $H_{\rm field}$ is diagonal. Denoting the states at site i in a representation where the Potts coupling is diagonal by $|k\rangle_i, k = 1, 2, \ldots q$, the states

$$|\ell'\rangle_i = \frac{1}{\sqrt{q}} \sum_{k=1}^{q} \omega^{(\ell-1)(k-1)} \; |k\rangle_i \tag{5}$$

$\ell' = 1, 2, \ldots, q$ are eigenstates of the transverse field.

In this representation the Hamiltonian (1) takes the form,

$$H = -\frac{\lambda}{q} \sum_{i=1} \sum_{k=1}^{q-1} M_i^k \; M_{i+1}^{q-k} - \frac{H}{q} \sum_i \sum_{k=1}^{q-1} M_i^k - h \sum_i R_i \tag{6}$$

with

$$R = \begin{bmatrix} q-1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

On the other hand, it is known [11, 12] that the ground state energy and the energy gap (i.e. the energy difference between the ground state and the first excited state) of the 1D quantum model are proportional to the free energy and the inverse of the correlation length of the 2D classical model respectively. Therefore, it suffices to calculate the two–lowest lying energy levels of the quantum Hamiltonian to get all the interesting information on the classical model. However, our approach consists in calculating these energies exactly on a chain of N sites with periodic boundary conditions. The numerical task thus consists in obtaining the two largest eigenvalues of $q^N \times q^N$ matrices.

If the energy gap is $G_N(K)$, where N is the size of the chain, and $K(K = \lambda/h)$ is the coupling, then the location of a conventional phase transition point is the limit of K_N^c values, at which the scaled gap ratio is unity.

$$(N-1)\ G_{N-1}(K_N^c)/N\ G_N(K_N^c) = 1 \tag{7}$$

Two kinds of phase diagrams are obtained. The first for $2 < q < q_c$ and the second for $q > q_c$, where the critical value of q, $q_c = 4$. For the former, there is no phase transition for positive field, as for the Ising model because the magnetic field destroys the transition, while for the negative field a second order transition line separates the disordered "(D)" and "$(q-1)$" ordered phases. The phase diagram is shown in Fig.1.

For the latter case, and for positive values of the field, a first order transition line appears, starting at zero field transition point $h_c(H = 0)$ and terminating at a critical point in the $(h/\lambda, H/\lambda)$ plane. This line separates the ordered phase ("0": in the direction of H) and the disordered phase. For negative field H, the critical line obtained in the first case $(2 < q < q_c)$ becomes first order (Fig.2).

III Conclusion

In this paper we have studied the ground state and low–lying excited state of the $(1+1)$–dimensional Hamiltonian q–state FPM in an external magnetic field to get information on the free energy and correlation length of the classical 2D model. We have shown by using a finite lattice extrapolation technique, that two distinct kinds of phase boundaries separate the ordered and disordered phases, depending on the values of q. More precisely:

(i) For $2 < q < q_c$, $q_c = 4$, there is no phase transition for positive field $(H > 0)$, while for the negative field a second order transition line separates the disordered "(D)" and "$(q-1)$" ordered phases.

(ii) For $q > q_c$, we have two first order transition lines, one for H positive and the other for H negative, which meet at the critical transverse field $h_c(H = 0)$. The upper line $(H > 0)$ terminates at a critical point while the lower one $(H < 0)$ does not.

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Phase diagram of the ferromagnetic Potts model in an external magnetic field for $q = 3, 4$ as determined from sizes 3/4.
"D" is the disordered phase and "$(q-1)$" is the ordered one.

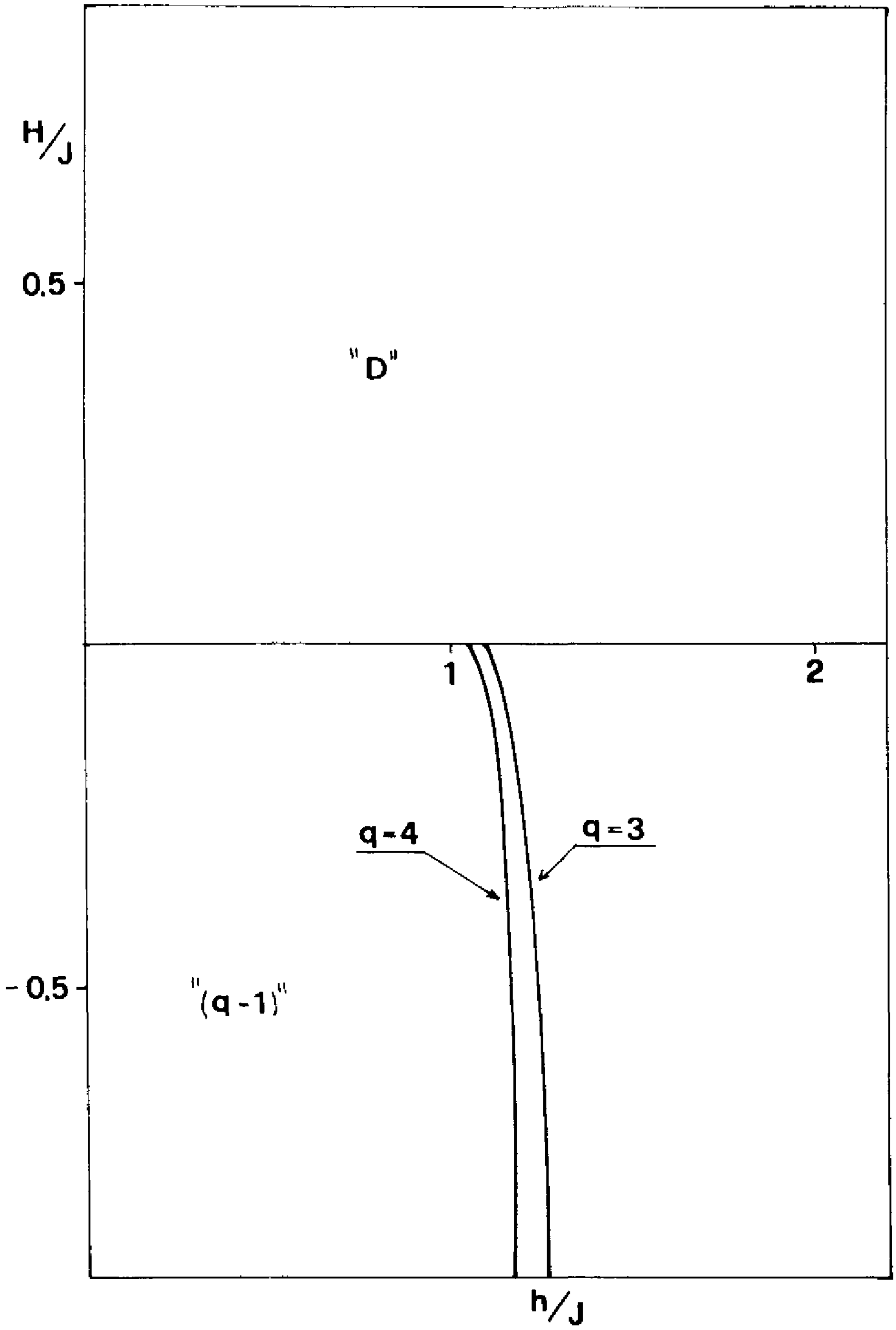

Fig.1

The same as Fig.1, for $q = 5, 6$, determined with sizes 3/4. "0" is the ordered phase in the field direction.

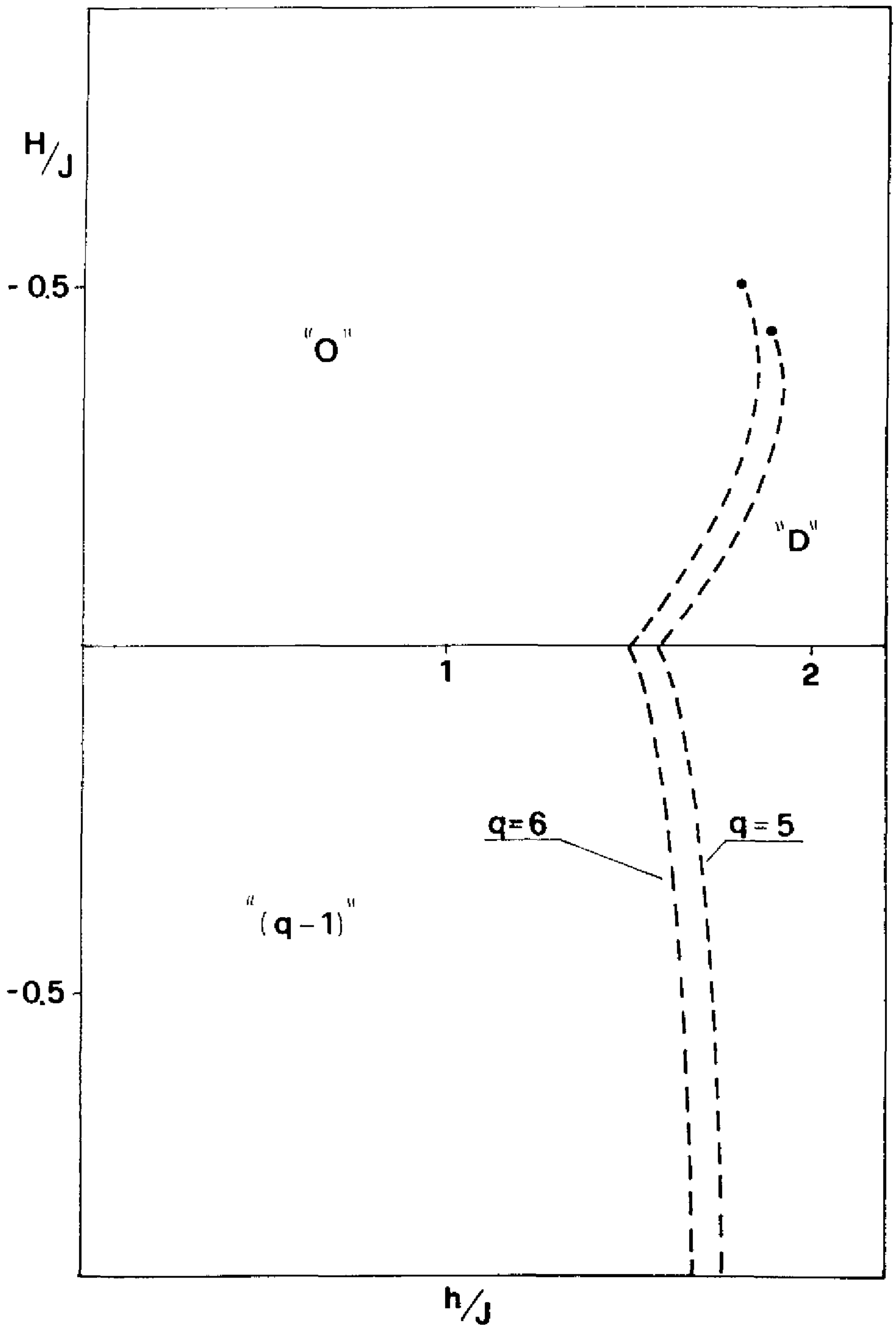

Fig.2

[illegible]
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