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What's wrong with anomalous chiral gauge theory?[1]

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Abstract

It is argued on general ground and demonstrated in the particular example of the Chiral Schwinger Model that there is nothing wrong with apparently anomalous chiral gauge theory. If quantised correctly, there should be no gauge anomaly and chiral gauge theory should be renormalisable and unitary, even in higher dimensions and with non-abelian gauge groups. Furthermore, mass terms for gauge bosons and chiral fermions can be generated without spoiling the gauge invariance.

Introduction

Symmetry has long been an important concept in physics and even broken symmetry is neither less important nor less useful. In particle physics, symmetry is the underlying principle of gauge field theory wherein all fundamental interactions are described.

In addition to explicit or spontaneous breaking, quantum fluctuations can also destroy a symmetry which is exact at the classical level. Such breaking was a surprise and hence named anomaly. The well-known anomaly of global axial symmetry is due to the need to regulate short distance singularity of the kind of quantum fluctuations associated with all Feynman loop diagrams [1]. This kind of anomaly has found applications in the phenomenology of neutral pion decays, in the $U(1)$ problem and possibly in the proton spin structure.

On the other hand, the anomalous breaking of the local symmetry of a gauge theory is undesirable. It is logically inconsistent if the local gauge invariance which is the underlying principle and also the definition of a gauge theory is lost upon quantisation. Indeed, the inconsistency does show up in the mathematical formulation of anomalous gauge theory.

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Take the example of an abelian chiral gauge theory where only the left-handed fermion current is coupled to the gauge fields,

$$\mathcal{L} \;=\; -\frac{1}{4}F^2 + \bar{\psi}(i\,\partial\!\!\!/ + g\,A\!\!\!/ P_L)\psi. \tag{1}$$

$P_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$ are the chirality projectors. This Lagrangian density is invariant under the gauge transformations

$$\begin{aligned} gA_\mu(x) &\rightarrow gA_\mu(x) + \partial_\mu\theta(x), \\ \psi(x) &\rightarrow [\exp(i\theta(x))P_L + P_R]\psi(x), \\ \bar{\psi}(x) &\rightarrow \bar{\psi}(x)[P_L + \exp(-i\theta(x))P_R]. \end{aligned} \tag{2}$$

Upon quantisation along the conventional route, it can be shown, via either Feynman diagrams [1] or the path-integral jacobean [2], that the gauged current

$$J_L^\mu \;=\; \bar{\psi}\gamma^\mu P_L\psi \tag{3}$$

is not conserved, as in two dimensions (with $\epsilon^{01} = 1$)

$$\partial_\mu J_L^\mu \;=\; \frac{\hbar g}{4\pi}\epsilon^{\mu\nu}\partial_\mu A_\nu. \tag{4}$$

Explicit $\hbar$ is inserted to remind us that this is a quantum effect.

The gauge symmetry is then lost, destroying the Ward-Takahashi identities which are instrumental in the consideration of perturbative renormalisability and unitarity.

Furthermore, the anomaly (4) is not compatible with the equation of motion, $\partial_\mu F^{\mu\nu} = gJ_L^\nu$,

$$0 = \partial_\mu\partial_\nu F^{\mu\nu} \;=\; g\partial_\mu J_L^\mu \neq 0. \tag{5}$$

The Gauss-law constraint also changes its nature from first-class to second-class when the commutator of Gauss-law operators is no longer non-vanishing. This transmutation takes place *after* quantisation and consequently entails contradictions both in the quantisation procedure, which requires different treatments for different types of constraints, and in the imposition of the constraints on the Hilbert space. To make it even harder to be enforced, the constraint is now time dependent [3] since it is no longer the generator of a symmetry and thus does not commute with the Hamiltonian.

Last but neither least nor separately, chiral gauge theory, anomalous or non-anomalous, does not seem to admit any non-perturbative definition. In Lattice Gauge Theory, in particular, there are un-surmountable difficulties to implement chiral fermions.

All the above, except the last difficulty of the last paragraph, leads to the requirement that gauge anomaly is to be cancelled. The accepted wisdom has the cancellation in appropriate choice of gauge groups or fermion representation, as with the conspiracy of quarks and leptons in the standard electroweak model.

Notwithstanding this, I take the approach that gauge anomaly and its problems are perhaps due to our ignorance. I have advocated elsewhere [4, 5, 6] and will substantiate in an example below that the conventional quantisation of chiral gauge theory is wrong. The proposition is that if the quantisation is done correctly, a chiral gauge theory should:

- have no gauge anomaly, but anomaly of *global* symmetry is still admissible;
- be perturbative renormalisable;
- be unitary;
- lead to a solution of the problem of lattice chiral fermions;
- and as a bonus, be able to admit gauge-invariant mass for both gauge bosons and chiral fermions without the need for scalar field.

In the next section, I review a classic example of how a theory might be quantised incorrectly. Following this is a section of the pitfalls to be aware of in the second quantisation. The example of two-dimensional abelian chiral gauge theory, known as the Chiral Schwinger Model, in the following section illustrates some of the consistency claims above for chiral gauge theory. The last section of the paper is for some concluding remarks.

Correct quantisation

Correct quantisation is synonymous with correct Feynman rules in perturbation context or correct path-integral action in general. Richard Feynman himself was probably the first to run into inconsistency with incomplete set of Feynman rules in a non-abelian gauge theory [7]. If loop diagrams involving two or more gluons in a given gauge are constructed according to the simple rules for tree diagrams, that is, from classical action as opposed to path-integral action, unitarity will be violated. Feynman then recognised that ghosts, scalar particles with wrong statistics but not showing up in asymptotic states, were necessary to restore unitarity. However, this *ad hoc* treatment only works at one-loop level because unitarity is not a sufficient constraint to treat more than one gluon loop.

These ghosts are now known as Faddeev-Popov ghosts and are shown in the path-integral framework to be necessary to restore unitarity to all orders in perturbation theory. The introduction of ghosts brings some degrees of non-locality to the path integral. But even so, the theory is still considered to be local in the sense that space-like commutators of physical fields vanish. I will come back to this *apparent* non-locality later on in the treatment of chiral gauge theory.

The lesson we have learned from this example is that apparent inconsistency might be due to the use of erroneous or incomplete Feynman rules. And correct Feynman rules are not those naively derived from the classical action. This is the central theme of the paper as I will argue that similar ignorance could lead to problems in a general chiral gauge theory.

Review of second quantisation

In the interaction picture, one introduces an evolution operator, $U(t,t_0)$, to carry a state from time t_0 to time t. On the other hand, all the field operators are free-field operators, that is, they are evolving in time according to the non-interaction part of the Hamiltonian. And the S matrix can be obtained in the limit

$$S = \lim_{t\to\infty}\lim_{t_0\to-\infty} U(t,t_0). \tag{6}$$

The Hilbert-space state evolution operators satisfies the following Tomonaga-Schwinger equation

$$\begin{aligned} \partial_t U(t,t_0) &= -iH_{int}(t)U(t,t_0), \\ U(t_0,t_0) &= 1, \end{aligned} \tag{7}$$

where $H_{int}(t)$ is the interaction part of the Hamiltonian. Usually, the solution for the evolution operator is given in the form,

$$\begin{aligned} \tilde{U}(t,t_0) &= \mathcal{T}\exp\left\{-i\int_{t0}^{t} H_{int}(\tau)d\tau\right\}, \\ &= 1 + (-i)\int_{t_0}^{t} H_{int}(\tau)d\tau + \frac{(-i)^2}{2!}\int_{t_0}^{t}\int_{t_0}^{t}\mathcal{T}[H_{int}(\tau)H_{int}(\tau')]d\tau d\tau' + \cdots \end{aligned} \tag{8}$$

$\mathcal{T}$ is the Dyson time ordered product,

$$\begin{aligned} \mathcal{T}[A(\tau)B(\tau')] &= \theta(\tau-\tau')A(\tau)B(\tau') + \theta(\tau'-\tau)B(\tau')A(\tau), \\ &= \frac{1}{2}\{A(\tau),B(\tau')\} + \frac{1}{2}\mathrm{sign}(\tau-\tau')\,[\mathrm{A}(\tau),\mathrm{B}(\tau')]\,. \end{aligned} \tag{9}$$

Expression (8) is the form from which one derives the usual Feynman rules or path integral which lead to anomaly of gauge symmetry and its associated inconsistencies.

I have put a tilde on U because I wish to claim that $\tilde{U}$ of (8) is not the solution for the Tomonaga-Schwinger equation in the case of chiral gauge theory. It would have been the solution only if when considering the time derivative

$$\partial_t\tilde{U}(t,t_0) = \lim_{\epsilon\to 0}\frac{\tilde{U}(t+\epsilon,t_0) - \tilde{U}(t,t_0)}{\epsilon} \tag{10}$$

one had ignored, as one often does, terms of *apparent* order $O(\epsilon)$ or higher like

$$\lim_{\epsilon\to 0}\frac{(-i)^2}{2!\epsilon}\int_t^{t+\epsilon}\int_t^{t+\epsilon}\mathcal{T}[H_{int}(\tau)H_{int}(\tau')]d\tau d\tau'. \tag{11}$$

However, the time-ordered product is highly singular at equal time arguments and may be able to reduce the term above to order $O(\epsilon^0)$. This is feasible, and is in fact the case, if the singularity behaves as a delta function killing of one integration of (11). It is worth stressing that such singularity is more than the usual statement of the indefiniteness of the time-ordered product at equal times.

Wolfgang Pauli gave the example of a scalar field theory with dervivative interactions wherein expression (11) does not vanish [8]. I now consider the example of chiral gauge interactions as typified in the Chiral Schwinger Model.

Chiral Schwinger Model

The Chiral Schwinger Model is a two dimensional abelian chiral gauge theory. It has the lagrangian density of (1), from which follows the interaction Hamiltonian

$$\begin{aligned} H_{int}(t) &= g\int A_\mu J_L^\mu(x,t)dx, \\ &= g\int (A_0 + A_1)J_{0L}(x,t)dx. \end{aligned} \tag{12}$$

We have used the metric $(1,-1)$ with $(t = x_0, x = x_1)$ and $\gamma_5 = \gamma_0\gamma_1$ to get the last equation from

$$J_{0L} = -J_{1L}. \tag{13}$$

For this model only expression of (11) is non-vanishing, as we shall see. All other terms of apparent order $O(\epsilon^2)$ or higher indeed vanish in the limit $\epsilon \to 0$.

The first term of the time-ordered product (9), the anti-commutator, does not contribute to (11) but the other term involving commutator does. Of this commutator term, the contribution comes from the commutator of fermion currents at different times [9]

$$[J_{0L}(x,\tau), J_{0L}(x',\tau')] = \frac{i\hbar}{2\pi}\partial_x\delta(x - x' + \tau - \tau'). \tag{14}$$

To evaluate (11) it suffices to consider

$$K(t) = \lim_{\epsilon\to 0}\frac{i\hbar g^2}{2\pi}\frac{(-i)^2}{2!\epsilon}\int dx\int_t^{t+\epsilon} A_+(x,\tau)\partial_x Q(x,\tau,t)d\tau, \tag{15}$$

where $A_+ = A_0 + A_1$ and

$$Q(x,\tau,t) = \frac{1}{2}\int dx'\int_t^{t+\epsilon} \mathrm{sign}(\tau - \tau')\delta(\mathrm{x} - \mathrm{x}' + \tau' - \tau)\mathrm{A}_+(\mathrm{x}',\tau')\mathrm{d}\tau'. \tag{16}$$

From the identity

$$(\partial_\tau - \partial_x)\left(\frac{1}{2}\mathrm{sign}(\tau - \tau')\delta(\mathrm{x} - \mathrm{x}' + \tau - \tau')\right) = \delta(x - x')\delta(\tau' - \tau), \tag{17}$$

we have

$$(\partial_\tau - \partial_x)Q(x,\tau,t) = A_+(x,\tau), \tag{18}$$

as the delta functions have eliminated the integrations over x' and τ'.

Thus $K(t)$ survives the $\epsilon \to 0$ limit,

$$K(t) = \frac{i\hbar g^2}{4\pi}\int A_+(x,t)\frac{\partial_1}{\partial_0 + \partial_1}A_+(x,t)dx. \tag{19}$$

(In our convention, $\partial_x = -\partial_1$.) $K(t)$ spoils the Tomonaga-Schwinger equation for $\tilde{U}$ as claimed.

It is a simple combinatoric exercise to show that the true solution of the Tomonaga-Schwinger equation is

$$U(t,t_0) = T\exp\left\{-i\int_{t_0}^{t}(H_{int}(\tau) - iK(\tau))d\tau\right\}. \tag{20}$$

$K(t)$ leads to an extra term in the path-integral action in comparison to the classical action (with appropriate gauge-fixing term),

$$\begin{aligned} S_{extra} &= -i\int K(t)dt, \\ &= \frac{\hbar g^2}{4\pi}\int\left\{\left(\frac{\partial A}{\Box}\right)\epsilon^{\mu\nu}\partial_\mu A_\nu + A_\mu\left(g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\Box}\right)A_\nu \right. \\ &\quad \left. + A_1(A_0 + A_1)\right\} dx dt. \end{aligned} \tag{21}$$

Local counterterms are then needed to restore Lorentz invariance.

The Feynman rules for chiral gauge theory are clearly not as naively expected. From the gauge transformation of the gauge fields, the first term on the right hand side of the last equation has the right coefficient to cancel out the anomaly (4). The second term is the mass term for the tranverse gauge fields. It is proportional to the coupling constant squared.

The gauge anomaly cancellation is exact to all orders in perturbation theory thanks to the non-renormalisation property of the coefficient of chiral anomaly [10]. Gauge invariance is never lost.

To demonstrate that mass term for chiral fermions is admissible, let me change the variables in the path integral as follows

$$\begin{aligned} \psi' &= \left[\exp\left(-ig\frac{\partial A}{\Box}\right)P_L + P_R\right]\psi, \\ \bar{\psi}' &= \bar{\psi}\left[P_L + \exp\left(ig\frac{\partial A}{\Box}\right)P_R\right]. \end{aligned} \tag{22}$$

Associated with this change of variables is the Fujikawa jacobean [2] which cancels out the the first term of the right hand side of (21), modulo local counterterms. The path-integral action can be written in the form of a space-time integral over

$$-\frac{1}{4}F^2 + \frac{\hbar g^2}{4\pi}A_\mu^T A^{\mu T} + \bar{\psi}'(i\,\partial\!\!\!/ + g\,A\!\!\!/^T P_L)\psi' + m\bar{\psi}'\psi', \tag{23}$$

where $A_\mu^T = P_{\mu\nu}A^\nu$ is the tranverse part of A_μ projected out by the tranversality projector $P_{\mu\nu}$. As the new fermionic fields (22) are *gauge invariant*, a mass term $m\bar{\psi}'\psi'$ can be added to the path-integral action without spoiling its gauge invariance. Such a mass term, not being protected by any symmetry, will indeed be generated by radiative corrections in general. We thus need no scalar field to generate masses for both gauge and fermionic fields with chiral interactions.

Also from (23) it can be seen that the gauge current that is coupled to the gauge fields is

$$C_\mu \quad = \quad P_{\mu\nu} J_L^\nu. \tag{24}$$

The tranversality projector $P_{\mu\nu}$ renders the gauge current conserved automatically, in compatibility with the equation of motion,

$$\partial_\mu F^{\mu\nu} + \frac{\hbar g^2}{2\pi} A^{\nu T} \quad = \quad g C^\nu. \tag{25}$$

One can then exploit the current conservation to prove the renormalisability and unitarity for abelian gauge group even in dimensions higher than two [6].

If one chooses the Lorentz gauge in (23),

$$\partial A \quad = \quad 0, \tag{26}$$

then the usual, anomalous formulation of the Chiral Schwinger Model is recovered but with mass terms for the fields. From this we can now see that the conventional formulation although apparently anomalous is consistent because it is the gauge-fixed form of a gauge-invariant theory and because there is no ghost for abelian gauge group. Such consistency has been explicitly verified elsewhere [11] (without fermion mass term) and [12] (with explicit fermion mass term).

Concluding remarks

In four dimensions, one will have to consider, in addition to (11), term of of order $O(\epsilon^2)$

$$\frac{1}{3!\epsilon} \int_t^{t+\epsilon} \int_t^{t+\epsilon} \int_t^{t+\epsilon} T[H_{int}(\tau) H_{int}(\tau') H_{int}(\tau'')] d\tau d\tau' d\tau'' \tag{27}$$

for abelian gauge group and even higher-order terms for non-abelian groups. With the former group, the gauge current can be written in the form (24) which is obviously divergenceless, facilitating the proof of renormalisability and unitarity [6]. With non-abelian groups, proof of such desirable properties is much more involving but can be constructed. I will present it somewhere else.

Expressions (11) and (27) are connected directly to the one-loop anomalous Feynman graphs. These expressions cancel the current anomalies exactly to all orders thanks to the non-renormalisation theorem of chiral anomalies. Furthermore, the non-vanishing of these terms highlights and clarifies the relationship between chiral anomalies and the non-closure of current commutators, and of double and triple commutators (corresponding with time ordering of three and four currents respectively). The violation of the abelian-current Jacobi identity [13], being a special case of the non-closure of double commutators at different times, can now be seen as necessary to preserve chiral gauge invariance. I suspect that the Malcev identity is likewise violated for non-abelian chiral currents. Consequently, we may appreciate more the relevance of two and three cocycles [14].

It should be emphasised that even though the above treatment prevents chiral gauge invariance from going anomalous, it has no consequence on the anomalies of global symmetries. This is simply because the Noether current of a global continuous symmetry does not appear in the lagrangian density at all and hence cannot modify the Feynman rules that render it anomalous in the first place. Anomalies do exist and have important relevance to nature through the breaking of global invariances.

The path-integral action of the Chiral Schwinger Model contains non-local terms. This is also a general feature for higher dimensions. However, like the non-locality associated with the introduction of Faddeev-Popov ghosts, this is only in appearance. The micro-causality from vanishing space-like commutators of physical fields are untouched. In fact, similar to the existence of ghost-free gauges in non-abelian theory, one can fix a gauge such that all the non-local terms disappear.

On the other hand, with this new degree of non-locality we might be able to avoid the theorem of [15]. The theorem is a kind of a no-go theorem stating that the only (non-abelian) vector-meson theories with tree graphs with good high-energy behaviour are Higgs theories. Work in this direction will be reported when it is completed.

Let me move to another no-go theorem, that of lattice chiral fermions [16]. The quantisation of this paper propoces a way around the theorem in that global axial can be broken while chiral gauge symmetry is preserved [18]. This solution for lattice chiral fermions should also provide an answer to the question raised in [17]. Numerical simulations of a chiral gauge theory is in progress.

The existence of new Feynman rules can also be seen through the Berry's phase of a path-integral quantisation of chiral gauge theory [4]. Based on that work and the work reported here, I also suspect that there is no anomaly associated with homotopically non-trivial gauge transformations [19].

The answer to the title question is that there is nothing wrong with "anomalous" chiral gauge theory simply because it is not anomalous at all. Other questions can be raised here: What are the implications on the electroweak interactions? Can the Higgsless mass generation be realised?

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