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ABSTRACT

In this paper, we shall deal with a linear control system Σ defined on a Lie group G with Lie algebra g. The dynamic of Σ is determined by the drift vector field which is an element in the normalizer of g in the Lie algebra of all smooth vector fields on G and by the control vectors which are elements in g considered as left-invariant vector fields.

We characterize the normalizer of g identifying vector fields on G with C^{∞}-functions defined on G into g. For this class of control systems we study algebraic conditions for the controllability problem. Indeed, we prove that if the drift vector field has a singularity then the Lie algebra rank condition is necessary for the controllability property, but in general this condition does not determine this property. On the other hand, we show that the rank (ad-rank) condition is sufficient for the controllability of Σ. In particular, we extend the fundamental Kalman's theorem when G is an Abelian connected Lie group. Our work is related with a paper of L. Markus and we also improve his results.

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1 Introduction

The purpose of this paper is to study algebraic conditions which give information about the controllability property for a particular class of systems, linear control systems of the form

$$\Sigma = (G, \mathcal{D})$$

for which the state space is a real finite dimensional Lie group G and the dynamic $\mathcal{D}$ is determined by the family of differential equations on G

$$\dot{x} = X(x) + \sum_{j=1}^{\kappa} \mu_j Y^j(x) \ .$$

Here, the drift vector field X is an element of the normalizer of g in the Lie algebra $X(G)$ of all C^∞ vector fields on G. The control vectors Y^j, $j = 1, 2, \ldots, \kappa$, belong to the Lie algebra g of G. We shall think of g as the set of left-invariant vector fields. The input functions $u = (u_1, u_2, \ldots, u_\kappa)$ belong to $\mathcal{U}$, the class of unrestricted admissible controls. The elements of $\mathcal{U}$ are piecewise constant functions of the form

$$u : [0, \infty) \to \mathbb{R}^\kappa \ .$$

$\mathcal{D}$ is the family of vector fields associated with Σ, i.e.

$$\mathcal{D} = \{X + \sum_{j=1}^{\kappa} u_j Y^j | u \in \mathbb{R}^\kappa\} \ .$$

First we study in details linear vector fields on Lie groups, i.e., the elements in the normalizer

$$\eta = norm_{X(G)}(g).$$

We find convenient to identify vector fields on G with C^∞-functions defined on G into the Lie algebra g. We consider the Lie group $Aut(G)$ of G-automorphisms, the Lie algebra $\partial(g)$ of g-derivations and the semidirect product of g with the Lie algebra $gl(g)$ of g, i.e., the Lie algebra

$$g \oslash gl(g).$$

Via the identification $\tilde{F} \in X(G)$ with $F \in C^\infty(G, g)$ defined by

$$\tilde{F}_x = dL_x(F(x)), \ \ x \in G$$

where as usual L_x denotes the left translation by x, we define the Lie algebra homomorphism

$$\Phi : \eta \to g \oslash gl(g)$$

by

$$\Phi(\tilde{F}) = (F(e), -dF(e))$$

where e is the neutral element of G and we prove :
Theorem 2.12. If G is connected and simply connected, then

$$\Phi : \eta \to g \oslash \partial g$$

is a surjective isomorphism. □

Theorem 2.15. If G is connected, then Φ maps η isomorphically onto $g \otimes aut(G)$. □

Also we shall obtain in Theorem 2.17 an expression for the integrable curves of $\tilde{F} \in \eta$.

Let $\Sigma = (G, \mathcal{D})$ be a linear control system. This class of systems generalizes :

Linear control systems on $\mathbb{R}^n$, [4] and linear control systems on a matrix Lie group, [5]. We study the controllability property of Σ when the drift vector field X has a singularity and we show that it is possible to reduce the study to the case when $X_e = 0$, equivalently the 1-parameter group T of G-diffeomorphisms induced by X is a subgroup of $Aut(G)$. We consider the Lie algebra generated by the control vectors

$$\mathcal{H} = Span_{\mathcal{L}.A.}\{Y^1, \ldots, Y^\kappa\}$$

and

$\langle X|\mathcal{H}\rangle$ = the smallest $ad(X)$-invariant subalgebra of g containing $\mathcal{H}$.

For this class of control systems, we prove :

Theorem 3.4. $Span_{\mathcal{L}.A.}(\mathcal{D}) \cong \langle X|\mathcal{H}\rangle \otimes L(T)$. □

In particular, we obtain that the Lie algebra rank condition is necessary for the controllability, in fact we show :

Theorem 3.6. If Σ is transitive, then

$$dim.Span_{\mathcal{L}.A.}\{Y^j, ad^i(X)(Y^j)|0 \leq i \leq p, \;\; 1 \leq j \leq \kappa\} = dim(G) \; ,$$

where the integer p is determined by the $ad(X)(\mathcal{H})$-sequence. □

Unfortunately, this condition does not characterize controllability. To show this, we give an example in the Heisenberg Lie group. On the other hand, we have :

Theorem 3.8. Let G be a connected Lie group. If

$$dim.Span\{Y^j, ad^i(X)(Y^j)|0 \leq i \leq p, \;\; 1 \leq j \leq \kappa\} = dim(G)$$

then Σ is controllable. □

In particular, we generalize in Corollary 3.9 the Kalman's theorem, [4] for Abelian connected Lie groups. Our work is related with a paper of L. Markus, [5] and we also improve such results.

This paper is organized as follows. Section 2. contains a characterization of the normalizer of g. Section 3 contains the controllability results. In Section 4 we give some examples.

2 Linear Vector Fields

Let $\Sigma = (G, \mathcal{D})$ be a linear control system. By definition the drift vector field X of Σ is an element in the normalizer of g in the Lie algebra of all C^∞ vector fields on G. In this section we wish to characterize this kind of dynamic that we have called linear vector fields on G. A vector field X on $\mathbb{R}^n$ is called linear if $X_p = A(p) + u$ for all $p \in \mathbb{R}^n$, where A is a linear map of $\mathbb{R}^n \to \mathbb{R}^n$ and u is a fixed vector of $\mathbb{R}^n$. This is the notion we want to generalize to any Lie group G.

Linear vector fields on $\mathbb{R}^n$ can be characterized in the following way: a C^∞ vector field X on $\mathbb{R}^n$ is linear if and only if $[X, U]$ is a constant vector field for every constant vector field U on $\mathbb{R}^n$ (see Corollary 2.5). Therefore the set of all linear vector fields on $\mathbb{R}^n$ is the normalizer in the Lie algebra $X(\mathbb{R}^n)$, of all C^∞ vector fields on $\mathbb{R}^n$, of the Lie algebra of $\mathbb{R}^n$.

If G is a Lie group the Lie algebra g of G will be identified with the tangent space G_e of G at the identity e. Also $X(G)$ will be the Lie algebra of all C^∞ vector fields on G. Moreover in this section for any $Y \in g$, $\tilde{Y} \in X(G)$ will denote the left invariant vector on G such that $\tilde{Y}_e = Y$. Let

$$\mathrm{norm}_{X(G)}(g) = \{X \in X(G) \mid [X, \tilde{Y}] \in g, \text{ for all } Y \in g\}.$$

Definition 2.1. The elements in $\mathrm{norm}_{X(G)}(g)$ will be called linear vector fields on G. $\square$

We find convenient to identify vector fields on G with functions of G into g: If $F : G \to g$ let

$$\tilde{F}_x = dL_x(F(x)) = \widetilde{F(x)}_x, \quad x \in G.$$

Clearly the map $F \to \tilde{F}$ is a linear isomorphism of $C^\infty(G, g)$ onto $X(G)$. Just observe that the constant function $F(x) = Y$ correspond to the left invariant vector field $\tilde{Y}$.

Lemma 2.2. Given $Y \in g$ and $F \in C^\infty(G, g)$ let $H \in C^\infty(G, g)$ be defined by

$$(1) \qquad H(x) = \tilde{Y}_x(F) + [Y, F(x)], \quad x \in G.$$

Then $\tilde{H} = [\tilde{Y}, \tilde{F}]$.

Proof.

$$([\tilde{Y}, \tilde{F}]f)(x) = (\tilde{Y}(\tilde{F}f))(x) - (\tilde{F}(\tilde{Y}f))(x), \quad f \in C^\infty(G).$$

$$(\tilde{Y}(\tilde{F}f))(x) = \left(\frac{d}{dt}\right)_{t=0}(\tilde{F}f)(x\exp tY) = \left(\frac{d}{dt}\right)_{t=0}(\widetilde{F(x\exp tY)}f)(x\exp tY)$$

$$= ((\widetilde{\tilde{Y}F})(x)f)(x) + (\tilde{Y}(\widetilde{F(x)}f))(x).$$

$$(\tilde{F}(\tilde{Y}f))(x) = \tilde{F}_x(\tilde{Y}f) = (\widetilde{F(x)}(\tilde{Y}f))(x).$$

$$([\tilde{Y}, \tilde{F}]f)(x) = ((\widetilde{\tilde{Y}F})(x)f)(x) + ([\tilde{Y}, \widetilde{F(x)}]f)(x) = ((\widetilde{\tilde{Y}F})(x)f)(x) + ([Y, \widetilde{F(x)}]f)(x).$$

Therefore

$$[\tilde{Y}, \tilde{F}]_x = \widetilde{\tilde{Y}_x(F)}_x + [Y, \widetilde{F(x)}]_x = (\tilde{Y}_x(F) + \widetilde{[Y, F(x)]})_x.$$

□

Theorem 2.3. If $F \in C^\infty(G,g)$ then $\tilde{F} \in \text{norm}_{X(G)}(g)$ if and only if for each $Y \in g$

$$F(x\exp Y) = (e^{-\text{ad}Y})F(x) + \Big(\frac{1-e^{-\text{ad}Y}}{\text{ad}Y}\Big)(Y(F) + [Y, F(e)]). \tag{2}$$

In particular all vector fields in $\text{norm}_{X(G)}(g)$ are analytic.

Proof. If $F \in C^\infty(G,g)$ let $B(Y) = Y(F) + [Y, F(e)]$. Then from (1) we get $\tilde{F} \in \text{norm}_{X(G)}(g)$ if and only if

$$\tilde{Y}_x(F) + [Y, F(x)] = B(Y) \quad \text{for all } x \in G,$$

if and only if

$$\Big(\frac{d}{dt}\Big)F(x\exp tY) = -[Y, F(x\exp tY)] + B(Y) \quad \text{for all } x, t.$$

Let $f(t) = F(x\exp tY)$, then $f^{(n)}(t) = (-\text{ad}Y)^n f(t) + (-\text{ad}Y)^{n-1}B(Y)$. Hence

$$f(t) = \sum_{n\geq 0}\frac{(-\text{ad}Y)^n}{n!}F(x)t^n + \sum_{n\geq 1}\frac{(-\text{ad}Y)^{n-1}}{n!}B(Y)t^n = (e^{-t\text{ad}Y})F(x) + \Big(\frac{1-e^{-t\text{ad}Y}}{\text{ad}Y}\Big)B(Y).$$

□

Corollary 2.4. Let G be a connected Lie group. Then an element $\tilde{F} \in \text{norm}_{X(G)}(g)$ is characterized by the pair $(F(e), dF_e)$.

Proof. Let $U = \exp(G)$. By induction on n we shall prove that the function F is determined by the pair $(F(e), dF_e)$ on U^n. Since $G = \bigcup_n U^n$ the corollary will follow. From (2):

$$F(\exp Y) = (e^{-\text{ad}Y})F(e) + \Big(\frac{1-e^{-\text{ad}Y}}{\text{ad}Y}\Big)(dF_e(Y) + [Y, F(e)]),$$

which shows that F is determined on U. Now assume that F is determined on U^n $n \geq 1$ and take $x \in U^n$. Then

$$F(x\exp Y) = (e^{-\text{ad}Y})F(x) + \Big(\frac{1-e^{-\text{ad}Y}}{\text{ad}Y}\Big)(dF_e(Y) + [Y, F(e)]),$$

which shows that F is determined on U^{n+1}. □

Corollary 2.5. Let $G = \mathbb{R}^n$. Then $X \in \text{norm}_{X(G)}(g)$ if and only if $X_p = A(p) + u$ for all $p \in \mathbb{R}^n$, where A is a linear map of $\mathbb{R}^n \to \mathbb{R}^n$ and $u \in \mathbb{R}^n$.

Proof. Under the usual identification of the tangent space $\mathbb{R}^n_p$ with $\mathbb{R}^n$ the vector field $\tilde{F}$ becomes the same as the function $F \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Thus if we replace x by 0 and Y by p in (2) we get $X_p = dF_0(p) + X(0)$. □

Theorem 2.6. If $\tilde{F} \in \text{norm}_{X(G)}(g)$ then for each $X, Y \in g$

$$e^{\text{ad}Y}dF_e(e^{-\text{ad}Y}X) = \Big[X, \Big(\frac{e^{\text{ad}Y}-1}{\text{ad}Y}\Big)dF_e(Y)\Big] + dF_e(X).$$

Proof. From (2):

$$e^{\text{ad}Y}F(\exp(-Y)\exp tX\exp Y) = F(\exp(-Y)\exp tX) + \Big(\frac{e^{\text{ad}Y}-1}{\text{ad}Y}\Big)(Y(F) + [Y, F(e)]).$$

$$e^{\mathrm{ad}Y} dF_e(e^{-\mathrm{ad}Y}X) = \Big(\frac{d}{dt}\Big)_{t=0}\Big((e^{-t\mathrm{ad}X})F(\exp(-Y)) + \Big(\frac{1-e^{-t\mathrm{ad}X}}{\mathrm{ad}X}\Big)(X(F)+[X,F(e)])\Big)$$

$$= -\mathrm{ad}X F(\exp(-Y)) + X(F) + [X, F(e)]$$

$$= -\mathrm{ad}X\Big((e^{\mathrm{ad}Y})F(e) + \Big(\frac{1-e^{\mathrm{ad}Y}}{\mathrm{ad}Y}\Big)(Y(F)+[Y,F(e)])\Big) + X(F) + [X,F(e)]$$

$$= \Big[X, \Big(\frac{e^{\mathrm{ad}Y}-1}{\mathrm{ad}Y}\Big)dF_e(Y)\Big] + dF_e(X)\ .$$

□

Theorem 2.7. For any linear map $A : g \to g$ the following conditions are equivalent:

$$(i) \qquad e^{\mathrm{ad}Y} A(e^{-\mathrm{ad}Y}X) = \Big[X, \Big(\frac{e^{\mathrm{ad}Y}-1}{\mathrm{ad}Y}\Big)A(Y)\Big] + A(X) \quad (X, Y \in g),$$

(ii) A is a derivation of g.

Proof. (ii) follows from (i) replacing Y by tY and differentiating with respect to t at $t = 0$.

Now assume that A is a derivation of g. Let $\phi(t) = A(e^{t\mathrm{ad}Y}X)$:

$$\frac{d\phi}{dt} = A([Y, e^{t\mathrm{ad}Y}X]) = [A(Y), e^{t\mathrm{ad}Y}X] + [Y, \phi(t)].$$

Let $\psi(t) = e^{t\mathrm{ad}Y}\Big[X, \Big(\frac{e^{-t\mathrm{ad}Y}-1}{\mathrm{ad}Y}\Big)A(Y)\Big] + e^{t\mathrm{ad}Y}A(X)$:

$$\frac{d\psi}{dt} = \mathrm{ad}Y e^{t\mathrm{ad}Y}\Big[X, \Big(\frac{e^{-t\mathrm{ad}Y}-1}{\mathrm{ad}Y}\Big)A(Y)\Big] + e^{t\mathrm{ad}Y}\Big[X, -e^{-t\mathrm{ad}Y}A(Y)\Big] + \mathrm{ad}Y e^{t\mathrm{ad}Y}A(X)$$

$$= [Y, \psi(t)] + [A(Y), e^{t\mathrm{ad}Y}X]$$

Therefore both functions ϕ and ψ satisfy the same first order linear differential equation. Since $\phi(0) = A(X) = \psi(0)$ both functions are identical. Evaluating at $t = -1$ we get (i).
□

Lemma 2.8. Given $D \in g$ let us consider $\mathrm{ad}D$ as the vector field defined on g by $(\mathrm{ad}D)_X = [D, X]$. Also let us identify D with the constant vector field defined by D: $D_X = D$. Let $D, E \in g$ then
(i) $[E, \mathrm{ad}D] = [D, E]$,
(ii) $[\mathrm{ad}E, \mathrm{ad}D] = \mathrm{ad}[D, E]$.

Proof. Let $\{x_i\}_{i=1}^n$ be a linear system of coordinates in g and let $f \in C^\infty(g)$. Then

$$(\mathrm{ad}Df)(X) = \Big(\frac{d}{dt}\Big)_{t=0} f(X + t[D, X]),$$

$$(E(\mathrm{ad}Df))(X) = \Big(\frac{d}{ds}\Big)_{s=0}(\mathrm{ad}Df)(X+sE) = \frac{\partial^2}{\partial s \partial t} f(X + sE + t[D,X] + ts[D,E])(0,0)$$

$$= \sum_{j,i} \frac{\partial^2 f}{\partial x_j \partial x_i}(X) E_j [D, X]_i + \sum_i \frac{\partial f}{\partial x_i}(X)[D, E]_i$$

$$(Ef)(X) = \left(\frac{d}{dt}\right)_{t=0} f(X + tE),$$

$$(\mathrm{ad} D(Ef))(X) = \left(\frac{d}{ds}\right)_{s=0} (Ef)(X + s[D, X]) = \frac{\partial^2}{\partial s \partial t} f(X + s[D, X] + tE)(0, 0)$$

$$= \sum_{j,i} \frac{\partial^2 f}{\partial x_j \partial x_i}(X)[D, X]_j E_i.$$

Hence

$$([E, \mathrm{ad} D]f)(X) = \sum_i \frac{\partial f}{\partial x_i}(X)[D, E]_i = ([D, E]f)(X).$$

This proves (i). To prove (ii) we compute

$$(\mathrm{ad} E(\mathrm{ad} Df))(X) = \left(\frac{d}{ds}\right)_{s=0} (\mathrm{ad} Df)(X + s[E, X])$$

$$= \frac{\partial^2}{\partial s \partial t} f(X + s[E, X] + t[D, X] + ts[D[E, X]])(0, 0)$$

$$= \sum_{j,i} \frac{\partial^2 f}{\partial x_j \partial x_i}(X)[E, X]_j [D, X]_i + \sum_i \frac{\partial f}{\partial x_i}(X)[D, [E, X]]_i.$$

Therefore

$$([\mathrm{ad} E, \mathrm{ad} D]f)(X) = \sum_i \frac{\partial f}{\partial x_i}(X)([D, [E, X]]_i - [E, [D, X]]_i).$$

Hence

$$[\mathrm{ad} E, \mathrm{ad} D]_X = [D, [E, X]] - [E, [D, X]] = [[D, E], X].$$

This completes the proof of the lemma. □

Lemma 2.9. If $\tilde{F} \in \mathrm{norm}_{X(G)}(g)$, then for all $Y \in g$

$$dF_x(\tilde{Y}_x) = [Y, F(e) - F(x)] + dF_e(Y).$$

Proof. This follows immediately from (2) replacing Y by tY an differentiating with respect to t at $t = 0$. □

Theorem 2.10. Let $\phi \in \partial(g)$ be a derivation of g and let $E \in g$. Then the distribution Δ in $G \times g$ defined by

$$\Delta_{(x,X)} = \{(\tilde{Y}_x, [Y, E - X] + \phi(Y)) : Y \in g\}$$

is C^∞, of dimension $n = dim.(g)$ and involutive.

Proof. The first two assertions are quite clear. To prove that Δ is involutive it is enough to consider the bracket of two vector fields in g of the form

$$D_j = [Y_j, E] - \mathrm{ad} Y_j + \phi(Y_j) \quad j = 1, 2.$$

Then using Lemma 2.8 we have:

$$[D_1, D_2] = [[Y_1, E] - \mathrm{ad} Y_1 + \phi(Y_1), [Y_2, E] - \mathrm{ad} Y_2 + \phi(Y_2)]$$

$$= -[Y_2, [Y_1, E]] + [Y_1, [Y_2, E]] + \mathrm{ad}[Y_2, Y_1] + [Y_1, \phi(Y_2)] - [Y_2, \phi(Y_1)]$$

$$= [[Y_1, Y_2], E] - \mathrm{ad}[Y_1, Y_2] + \phi([Y_1, Y_2]).$$

The theorem is proved. □

Lemma 2.11. Given $\tilde{F}_j \in \mathrm{norm}_{X(G)}(g)$, $j = 1, 2$ let $F \in C^\infty(G, g)$ be defined by

(3) $\quad F(x) = -[F_1(x), F_2(x)] + F_1(x)(F_2) - F_2(x)(F_1) + [F_1(x), F_2(e)] - [F_2(x), F_1(e)].$

Then $[\tilde{F}_1, \tilde{F}_2] = \tilde{F}$.

Proof.

$$([\tilde{F}_1, \tilde{F}_2]f)(x) = (\tilde{F}_1(\tilde{F}_2 f))(x) - (\tilde{F}_2(\tilde{F}_1 f))(x), \quad f \in C^\infty(G).$$

$$(\tilde{F}_1(\tilde{F}_2 f))(x) = \Big(\frac{d}{ds}\Big)_{s=0}(\tilde{F}_2 f)(x \exp sF_1(x))$$

$$= \Big(\frac{d}{ds}\Big)_{t=0}(\widetilde{F_2(x \exp sF_1(x)} f)(x \exp sF_1(x))$$

$$= \Big(\frac{d}{ds}\Big)_{s=0}(\widetilde{F_2(x \exp sF_1(x))} f)(x) + \Big(\frac{d}{ds}\Big)_{s=0}(\widetilde{F(x)} f)(x \exp sF_1(x)).$$

From (2):

$$\Big(\frac{d}{ds}\Big)_{s=0} F_2(x \exp sF_1(x)) = \Big(\frac{d}{ds}\Big)_{s=0}\Big((e^{-s\mathrm{ad}F_1(x)})F_2(x) + \Big(\frac{1 - e^{-s\mathrm{ad}F_1(x)}}{\mathrm{ad}F_1(x)}\Big)(F_1(x)(F_2) + [F_1(x), F_2(e)])\Big)$$

$$= -[F_1(x), F_2(x)] + F_1(x)(F_2) + [F_1(x), F_2(e)].$$

Therefore

$$(\tilde{F}_1(\tilde{F}_2 f))(x) = -(\widetilde{[F_1(x), F_2(x)]} f)(x) + (\widetilde{F_1(x)(F_2)} f)(x) + (\widetilde{[F_1(x), F_2(e)]} f)(x) + (\widetilde{[F_1(x)}(\widetilde{F_2(x)} f))(x).$$

Then

$$([\tilde{F}_1, \tilde{F}_2]f)(x) = -(\widetilde{[F_1(x), F_2(x)]} f)(x) + \Big((\widetilde{F_1(x)(F_2)} - \widetilde{F_2(x)(F_1)}) f\Big)(x)$$

$$+ \Big((\widetilde{[F_1(x), F_2(e)]} - \widetilde{[F_2(x), F_1(e)]}) f\Big)(x).$$

This completes the proof. □

Theorem 2.12. Let $\Phi : \mathrm{norm}_{X(G)}(g) \to g \otimes gl(g)$ be the linear map defined by $\Phi(\tilde{F}) = (F(e), -dF_e)$. Then

(i)Φ is a Lie algebra homomorphism of $\mathrm{norm}_{X(G)}(g)$ into the semidirect product $g \otimes \partial(g)$.

(ii) If G is connected then $\Phi : \mathrm{norm}_{X(G)}(g) \to g \otimes \partial(g)$ is an injective homomorphism.

(iii) If G is connected and simply connected then $\Phi : \mathrm{norm}_{X(G)}(g) \to g \otimes \partial g$ is a surjective isomorphism.

Proof. From Theorem 2.6 and 2.7 we know that Φ maps $\mathrm{norm}_{X(G)}(g)$ into $g \otimes \partial(g)$. Now given $\tilde{F}_j \in \mathrm{norm}_{X(G)}(g)$, $j = 1, 2$ let $F \in C^\infty(G, g)$ be defined by (3). Then

$$dF_e(Y) = -[(dF_1)_e(Y), F_2(e)] - [F_1(e), (dF_2)_e(Y)] + (dF_1)_e(Y)(F_2) - (dF_2)_e(Y)(F_1)$$

$$+[(dF_1)_e(Y), F_2(e)] - [(dF_2)_e(Y), F_1(e)]$$

$$= (dF_1)_e(Y)(F_2) - (dF_2)_e(Y)(F_1)$$

$$= (dF_2)_e(dF_1)_e(Y) - (dF_1)_e(dF_2)_e(Y)$$

$$= -[(dF_1)_e, (dF_2)_e](Y).$$

Also

$$F(e) = F_1(e)(F_2) - F_2(e)(F_1) + [F_1(e), F_2(e)] = (dF_2)_e(F_1(e)) - (dF_1)_e(F_2(e)) + [F_1(e), F_2(e)].$$

Let $\phi_j = -(dF_j)_e,\ E_j = F_j(e),\ j = 1, 2$. Then from Lemma 2.11:

$$\Phi([\tilde{F}_1, \tilde{F}_2]) = (F(e), -dF_e) = ([E_1, E_2] + \phi_1(E_2) - \phi_2(E_1), [\phi_1, \phi_2]),$$

which proves (i).

The second statement is a direct consequence of Corollary 2.4.

To prove (iii) let us start with a pair $(E, \phi) \in g \times \partial(g)$. Then let us consider the involutive distribution Δ on $G \times g$ defined in Theorem 2.10. Let $\iota : M_{(x,X)} \to G \times g$ be the maximal connected integral submanifold of Δ through the point (x, X).

Let

$$\psi(t) = (e^{-t\mathrm{ad}Y})X + \Big(\frac{1 - e^{-t\mathrm{ad}Y}}{\mathrm{ad}Y}\Big)(\phi(Y) + [Y, E])$$

Then $\psi(0) = X$ and

$$\psi'(t) = [Y, E - \psi(t)] + \phi(Y).$$

Therefore $\psi(t)$ is the integrable curve of $D \in X(g)$ defined by $D_Z = [Y, E - Z] + \phi(Y)$ through the point X. Now if we put $\gamma(t) = (x\exp(tY), \psi(t))$ then $\gamma(0) = (x, X)$ and $\gamma'(t) = (\tilde{Y}_{x\exp(tY)}, \psi'(t))$. Hence $\gamma(t)$ is the integrable curve of $(\tilde{Y}, D) \in X(G \times g)$, defined by $(\tilde{Y}, D)_{(z,Z)} = (\tilde{Y}_z, D_Z)$, through the point (x, X). Since $(\tilde{Y}, D) \in \Delta$ we have that $\gamma(t) \in M_{(x,X)}$ for all t. Thus for all $Y \in g$ we have

$$(x\exp Y, (e^{-\mathrm{ad}Y})X + \Big(\frac{1 - e^{-\mathrm{ad}Y}}{\mathrm{ad}Y}\Big)(\phi(Y) + [Y, E]) \in M_{(x,X)}.$$

Now let $\iota : M \to G \times g$ be the maximal connected integral submanifold of Δ through the point (e, E). Let π_j be the projection map of $G \times g$ into the j-factor $j = 1, 2$. Let $p = \pi_1 \cdot \iota$. Then

$$dp_{(x,X)}(\tilde{Y}_x, [Y, E - X] + \phi(Y)) = \tilde{Y}_x$$

which shows that $p : M \to G$ is a regular map. Moreover we shall prove that p is a covering. Let V and U be respectively symmetric open neighborhoods of 0 in g and of e in G, such that $\exp : V \to U$ is a diffeomorphism. Given $(x, X) \in M$ let $s : xU \to M$ be defined by

$$s(x\exp Y) = \Big(x\exp Y, (e^{-\mathrm{ad}Y})X + \Big(\frac{1 - e^{-\mathrm{ad}Y}}{\mathrm{ad}Y}\Big)(\phi(Y) + [Y, E])\Big) \in M_{(x,X)}, \quad Y \in V.$$

Then s is a local section of $p : M \to G$ through (x, X) defined on xU.

Now let us prove that $p(M) = G$. If $x \in p(M)$ then the existence of s implies that $xU \subset p(M)$. Then by induction on n it follows that $U^n \subset p(M)$. Hence $p(M) = G$ because G is connected.

For $x \in G$ let $p^{-1}(x) = \{X_\alpha\}_{\alpha \in A}$. Set s_α be the local section on xU through (x, X_α). Then

$$p^{-1}(xU) = \bigcup_{\alpha \in A} s_\alpha(xU).$$

In fact if $(x\exp Y, Z) \in M$, $Y \in V$ let σ be the local section defined on $(x \exp Y)U$ through $(x\exp Y, Z)$. Then if $(x, X_\alpha) = \sigma(x)$ we have $(x\exp Y, Z) = s_\alpha(x\exp Y)$.

On the other hand if $\alpha \neq \beta$ then $s_\alpha(xU)$ and $s_\beta(xU)$ are disjoint. In fact $s_\alpha(x\exp Y) = s_\beta(x\exp Z)$ $(Y, Z \in V)$ implies $Y = Z$ and $(e^{-\mathrm{ad}Y})X_\alpha = (e^{-\mathrm{ad}Y})X_\beta$ which is impossible because $X\alpha \neq X_\beta$. Since $p : s_\alpha(xU) \to xU$ is a diffeomorphism, $p : M \to G$ is a covering map.

Since G is simply connected $p : M \to G$ is a diffeomorphism. Moreover, let $X = \pi_2(p^{-1}(x))$ then

$$(dp^{-1})_x(\tilde{Y}_x) = (\tilde{Y}_x, [Y, E - X] + \phi(Y)).$$

Let $F = \pi_2 \cdot p^{-1}$. Then $F(e) = E$ and

$$dF_x(\tilde{Y}_x) = d\pi_2(\tilde{Y}_x, [E - X] + \phi(Y)) = [Y, F(e) - F(x)] + \phi(Y).$$

In particular $dF_e = \phi$.

Let $\lambda(t) = F(x\exp tY)$ and

$$\psi(t) = (e^{-t\mathrm{ad}Y})F(x) + \Big(\frac{1 - e^{-t\mathrm{ad}Y}}{\mathrm{ad}Y}\Big)(Y(F) + [Y, F(e)])$$

Then

$$\lambda'(t) = [Y, F(e) - \lambda(t)] + \phi(Y)$$
$$\psi'(t) = [Y, F(e) - \psi(t)] + \phi(Y).$$

Since $\lambda(0) = F(x) = \psi(0)$ it follows that $\lambda(t) = \psi(t)$. Hence applying Theorem 2.3 $\tilde{F} \in \mathrm{norm}_{X(G)}(g)$, completing the proof of the theorem. □

Theorem 2.13. Let $\pi : G_1 \to G$ be a covering homomorphism of connected Lie groups and let $K = \ker \pi$. Then $\tilde{F}_1 \in \mathrm{norm}_{X(G_1)}(g)$ is π- related to some $\tilde{F} \in X(G)$ if and only if $F_1(k) = F_1(e)$ for all $k \in K$. In this case $\tilde{F} \in \mathrm{norm}_{X(G)}(g)$ and $\Phi(\tilde{F}_1) = \Phi(\tilde{F})$.

Proof. First of all $\tilde{F}_1 \in X(G_1)$ and $\tilde{F} \in X(G)$ are π-related if and only if $F_1 = F \cdot \pi$. In fact:

$$d\pi_x(\tilde{F}_1)_x = d\pi_x dL_x(F_1(x)) = dL_{\pi(x)}(F_1(x)),$$
$$= dL_{\pi(x)}(F(\pi(x))).$$

Therefore, $d\pi_x(\tilde{F}_1)_x = \tilde{F}_{\pi(x)}$ if and only if $F_1(x) = F(\pi(x))$.

Now we shall prove $F_1 = F \cdot \pi$ if and only if $F_1(k) = F_1(e)$ for all $k \in K$. It is enough to prove that $F_1(k) = F_1(e)$ for all $k \in K$ implies that $F_1 = F \cdot \pi$, since the other implication is obvious. From (2):

$$F_1(k\exp Y) = (e^{-\mathrm{ad}Y})F_1(k) + \Big(\frac{1 - e^{-\mathrm{ad}Y}}{\mathrm{ad}Y}\Big)(Y(F) + [Y, F(e)]) = F_1(\exp Y).$$

Therefore $F_1^{l(k)}$ and F_1 coincide in a neighborhood of $e \in G_1$, since both are analytic they coincide everywhere. Hence if $x, y \in G_1$ and $\pi(x) = \pi(y)$ then $y = kx$ for some $k \in K$. Thus $F_1(y) = F_1(x)$ which implies that there is $F \in C^\infty(G)$ such that $F_1 = F \cdot \pi$.

The last assertion follows from Theorem 2.3 or from the fact that the left-invariant vector field $\tilde{X}_1 \in X(G_1)$ and $\tilde{X} \in X(G)$ generated by the same $X \in g$ are π-related. □

Let G be a connected and simply connected Lie group. Then the map $\delta : \mathrm{Aut}(G) \to \mathrm{Aut}(g)$ given by $\delta(\varphi) = (d\varphi)_e$ is a surjective isomorphism. Since $\mathrm{Aut}(g)$ is a closed subgroup of $GL(g)$ it has a unique structure of a topological Lie subgroup of $GL(g)$. Moreover its Lie algebra is $\partial(g)$. Hence $\mathrm{Aut}(G)$ inherits a unique Lie group structure making δ and isomorphism of Lie groups. We shall identify the Lie algebra of $\mathrm{Aut}(G)$ with $\partial(g)$ via $(d\delta)_e$, and we shall denote by $\mathrm{Exp} : \partial(g) \to \mathrm{Aut}(G)$ the corresponding exponential map. Now let $\pi : G_1 \to G$ be a universal covering homomorphism of a connected Lie group G, and let $K = \ker \pi$. Then the group $\mathrm{Aut}(G)$ of all Lie automorphisms of G can be identified with $\{\varphi \in \mathrm{Aut}(G_1) : \varphi(K) \subset K\}$. In this way $\mathrm{Aut}(G)$ inherits a Lie structure which makes it isomorphic to $\{\varphi \in \mathrm{Aut}(G_1) : \varphi(K) \subset K\}$ with its unique structure of a topological Lie subgroup of $\mathrm{Aut}(G_1)$. We shall identify the Lie algebra $\mathrm{aut}(G)$ of $\mathrm{Aut}(G)$ with a Lie subalgebra of $\partial(g)$ and we shall also denote by $\mathrm{Exp} : \mathrm{aut}(G) \to \mathrm{Aut}(G)$ the corresponding exponential map. Then for any $\phi \in \mathrm{aut}(G)$ and $X \in g$ we have

$$(d\mathrm{Exp}\phi)_e = e^{\phi},$$

and

$$(\mathrm{Exp}\phi)(\exp X) = \exp((d\mathrm{Exp}\phi)_e X) = \exp(e^{\phi}(X)). \tag{4}$$

Theorem 2.14. Suppose G is a connected Lie group with Lie algebra g. If $X \in g$ and $\phi \in \mathrm{aut}(G)$ then there are vector fields $\tilde{X}, \tilde{F} \in \mathrm{norm}_{X(G)}(g)$ such that $\Phi(\tilde{X}) = (X, 0)$ and $\Phi(\tilde{F}) = (0, -\phi)$. Then the corresponding one parameter groups $\tilde{X}_t, \tilde{F}_t$ of diffeomorphisms of G are given by

$$\tilde{X}_t(x) = x \exp tX$$
$$\tilde{F}_t(x) = (\mathrm{Exp}t\phi)(x),$$

for all $x \in G$ and $t \in \mathbb{R}$.

Proof. Since $R_{\exp tX}$ and $\mathrm{Exp}t\phi$ are one parameter groups of diffeomorphisms of G (or G_1) it is enough to compute the associated vector fields. Let us first take $x(t) = x \exp tX$. Then $\dot{x}(0) = dL_x(X)$. Since the constant function $X(x) = X$ satisfies (2) and $\Phi(X) = (X, 0)$ it follows that $\tilde{X}_x = dL_x(X)$ is the left invariant vector field such that $\tilde{X}_e = X$ and $\dot{x}(0) = \tilde{X}_x$, which proves the first assertion.

Let us now consider the vector fields $\tilde{H}_1 \in X(G_1)$ and $\tilde{H} \in X(G)$ corresponding to $\mathrm{Exp}t\phi$. Take $x_1 \in G_1$ and consider $x_1(t) = (\mathrm{Exp}t\phi)(x_1)$ for $x_1 = \exp X$. Then using (4) and Theorem 1.7 of [2]:

$$\dot{x}_1(0) = \left(\frac{d}{dt}\right)_{t=0}(\mathrm{Exp}t\phi)(x_1) = \left(\frac{d}{dt}\right)_{t=0}\exp(e^{t\phi}(X))$$

$$= d\exp_X(\phi(X)) = (dL_{x_1})_e\left(\frac{1 - e^{-\mathrm{ad}X}}{\mathrm{ad}X}\right)(\phi(X)).$$

Then the analytic function $H_1 \in C^\infty(G_1)$ which corresponds to the vector field $\tilde{H}_1$ satisfies

$$H_1(\exp X) = \left(\frac{1 - e^{-\mathrm{ad}X}}{\mathrm{ad}X}\right)(\phi(X)), \quad X \in g.$$

From (2) the analytic function F_1 corresponding to $\tilde{F}_1 = \Phi^{-1}(0, -\phi)$ also satisfies

$$F_1(\exp X) = \left(\frac{1 - e^{-\mathrm{ad}X}}{\mathrm{ad}X}\right)(\phi(X)), \quad X \in g.$$

Therefore $F_1(x_1) = H(x_1)$ for all $x_1 \in G$. But

$$(5) \qquad d\pi_{x_1}(\tilde{F}_1)(x_1) = \left(\frac{d}{dt}\right)_{t=0} \pi(\text{Exp}t\phi(x_1)) = \left(\frac{d}{dt}\right)_{t=0} (\text{Exp}t\phi(\pi(x_1))).$$

Therefore (Theorem 2.13) there exists $\tilde{F} \in \text{norm}_{X(G)}(g)$ π-related to $\tilde{F}_1$ and $\Phi(\tilde{F}_1) = \Phi(\tilde{F}) = (0, -\phi)$. Moreover from (5) it follows that $\text{Exp}t\phi$ is the one parameter group of diffeomorphisms of G associated to $\tilde{F}$, as we wanted to prove. □

Let $\iota : \text{norm}_{X(G)}(g) \to \text{norm}_{X(G_1)}(g)$ be defined by $\iota(\tilde{F}) = \widetilde{F \cdot \pi}$. Then from Theorem 2.12 it follows that ι is an injective homomorphism of Lie algebras and that the following diagram commutes:

$$\begin{array}{ccc} \text{norm}_{X(G_1)}(g) & \longrightarrow & g \otimes \partial(g) \\ \iota \uparrow & & \uparrow \text{id} \\ \text{norm}_{X(G)}(g) & \longrightarrow & g \otimes \partial(g) \end{array}$$

Theorem 2.15. Let $\pi : G_1 \to G$ be a universal covering homomorphism of a connected Lie group G. Then Φ maps $\text{norm}_{X(G)}(g)$ isomorphically onto $g \otimes \text{aut}(G)$. Moreover if $i : g \otimes \text{aut}(G) \to g \otimes \partial(g)$ denotes the inclusion map then we have the following commutative diagram:

$$\begin{array}{ccc} \text{norm}_{X(G_1)}(g) & \longrightarrow & g \otimes \partial(g) \\ \iota \uparrow & & \uparrow i \\ \text{norm}_{X(G)}(g) & \longrightarrow & g \otimes \text{aut}(G) \end{array}$$

Proof. The only thing that remains to be proved is that $\Phi(\text{norm}_{X(G)}(g)) \subset g \otimes \text{aut}(G)$. Let $\tilde{F} \in \text{norm}_{X(G)}(g)$ and let $\Phi(\tilde{F}) = (E, \phi)$. Since $g \otimes \text{aut}(G) \subset \Phi(\text{norm}_{X(G)}(g))$ (Theorem 2.14) we may assume that $\Phi(\tilde{F}) = (0, \phi)$. Now let $\tilde{F}_1 = \iota(\tilde{F})$. Given $k \in K$ by the same Theorem 2.14 we know that $x_1(t) = (\text{Exp}t\phi)(k)$ is the integral curve of $\tilde{F}_1$ through the point k. But $(\tilde{F}_1)_k = (dL_k)_e F_1(e) = 0$. Therefore $x_1(t) = k$ for all $t \in \mathbb{R}$. Hence

$$e = \pi(x_1(t)) = \pi((\text{Exp}t\phi)(k))$$

for all $t \in \mathbb{R}$. In other words $\text{Exp}(t\phi) \in \text{Aut}(G)$ for all $t \in \mathbb{R}$; thus $\phi \in \text{aut}(G)$. The theorem is proved. □

Let $Dif(G)$ denote the group of all diffeomorphisms of G. Let us now consider the map $\alpha : G \oslash \text{Aut}(G) \to Dif(G)$ defined by $\alpha(x, \Gamma) = R_{x^{-1}} \cdot \Gamma$, and let $\text{Exp} : \text{norm}_{X(G)}(g) \to Dif(G)$ be defined by requiring the commutativity of the following diagram:

$$\begin{array}{ccc} \text{norm}_{X(G)}(g) & \longrightarrow & g \oslash \text{aut(G)} \\ \text{Exp} \downarrow & & \downarrow \text{Exp} \\ Dif(G) & \longleftarrow & G \oslash \text{Aut}(G) \end{array}$$

Theorem 2.16. (i) The map $\alpha : G \oslash \text{Aut}(G) \to Dif(G)$ is an injective group homomorphism.
(ii) The one parameter group $\tilde{F}_t$ of diffeomorphisms of G associated to $\tilde{F} \in \text{norm}_{X(G)}(g)$ is given by

$$\tilde{F}_t = \text{Exp}(-t\tilde{F}).$$

In particular all vector fields in $\mathrm{norm}_{X(G)}(g)$ are complete.
Proof. Let us prove (i):

$$\alpha((x,\Gamma)(y,\Psi)) = \alpha(x\Gamma(y), \Gamma\Psi) = R_{((x\Gamma(y))^{-1}}\Gamma\Psi,$$

$$R_{(x\Gamma(y))^{-1}} = R_{x^{-1}}R_{\Gamma(y))^{-1}} = R_{x^{-1}}(\Gamma R_{y^{-1}}\Gamma^{-1}),$$

$$\alpha((x,\Gamma)(y,\Psi)) = R_{x^{-1}}(\Gamma R_{y^{-1}}\Gamma^{-1})\Gamma\Psi = \alpha(x,\Gamma)\alpha(y,\Psi).$$

Moreover α is injective, since $\alpha(x,\Gamma) = I$ implies $e = R_{x^{-1}}\Gamma(e) = x^{-1}$, and therefore $I = \alpha(e,\Gamma) = \Gamma$.

(ii) Since $\mathrm{Exp}(-t\tilde{F})$ is a one parameter group of diffeomorphisms of G it is enough to compute the associated vector field $\tilde{H} \in X(G)$. Take $x = \exp X, X \in g$ and consider $x(t) = \mathrm{Exp}(-t\tilde{F})(x)$. Let $\Phi(\tilde{F}) = (E,\phi)$ and let $\mathrm{Exp}(-t(E,\phi)) = (g(t),\Gamma(t))$. Then

$$x(t) = \mathrm{Exp}(-t(E,\phi))(x) = (R_{g(t)^{-1}}\Gamma(t))(x) = (\Gamma(t)(x))g(t)^{-1}. \tag{6}$$

Now let $\mu : G \times \mathrm{Aut}(G) \to G$ be defined by $\mu(x,\Gamma) = \Gamma(x)$. Then using (4)

$$\Big(\frac{d}{dt}\Big)_{t=0}\Gamma(t)(x) = d\mu_{(x,I)}(0,\dot{\Gamma}(0)) = d\mu_{(x,I)}(0,-\phi)$$

$$= \Big(\frac{d}{dt}\Big)_{t=0}\mu(x,\mathrm{Exp}(-t\phi)) = \Big(\frac{d}{dt}\Big)_{t=0}(\mathrm{Exp}(-t\phi))(\exp X)$$

$$= \Big(\frac{d}{dt}\Big)_{t=0}\exp(e^{-t\phi}(X)) = d\exp_X(-\phi(X))$$

Going back to (6) and applying Theorem 1.7 of [2]:

$$\dot{x}(0) = \Big(\frac{d}{dt}\Big)_{t=0}\Gamma(t)(x) + dL_x(E) = (dL_x)_e\Big(\frac{1-e^{-\mathrm{ad}X}}{\mathrm{ad}X}\Big)(-\phi(X)+E).$$

Then the analytic function $H \in C^\infty(G,g)$ which corresponds to the vector field $\tilde{H}$ satisfies

$$H(\exp X) = \Big(\frac{1-e^{-\mathrm{ad}X}}{\mathrm{ad}X}\Big)(-\phi(X)+E), \quad X \in g.$$

From (2) the analytic function F corresponding to $\tilde{F}$ also satisfies

$$F(\exp X) = \Big(\frac{1-e^{-\mathrm{ad}X}}{\mathrm{ad}X}\Big)(-\phi(X)+E), \quad X \in g.$$

Therefore $F(x) = H(x)$ for all $x \in G$. This completes the proof of the theorem. □

For $\tilde{F} \in \mathrm{norm}_{X(G)}(g)$ let $\Phi(\tilde{F}) = (E,\phi)$. Now using the Baker-Campbell-Hausdorff formula, [7] we shall obtain an explicit expression for the integrable curves of $\tilde{F}$.

To this end let $g(z) = (1-e^{-z})/z$. Then g is an entire function on **C**. Put $f(z) = g(z)^{-1} - z/2$. Then f is analytic in a neighborhood of $z = 0$, $f(0) = 1$, and f is even. We write

$$f(z) = 1 + \sum_{p=1}^{\infty} K_{2p}z^{2p}.$$

Then it is easy to verify that the K_{2p}'s are all rational numbers. Now define $d_1(E,\phi)=E$ and $d_n(E,\phi)$ $n\geq 1$ by the recursion formula

$$(n+1)d_{n+1}(E,\phi)=\frac{1}{2}[E,d_n(E,\phi)]+\phi(d_n(E,\phi))$$

$$(7)\qquad +\sum_{2\leq 2p\leq n}K_{2p}\sum_{k_1,\dots,k_{2p}>0,\ k_1+\dots+k_{2p}=n}[d_{k_1}(E,\phi),[\cdots[d_{k_{2p}}(E,\phi),E]\cdots].$$

Then for each $n\geq 1$, d_n is a polynomial map of $g\otimes\partial g$ into g homogeneous of degree n. In particular we have

$$d_2(E,\phi)=\frac{1}{2}\phi(E),\quad d_3(E,\phi)=\frac{1}{12}[E,\phi(E)]+\frac{1}{6}\phi^2(E),\quad d_4(E,\phi)=\frac{1}{24}[E,\phi^2(E)]+\frac{1}{24}\phi^3(E).$$

Moreover the series

$$\sum_{n=1}^{\infty}d_n(E,\phi)$$

converges absolutely in a neighborhood of $(0,0)\in g\otimes\partial g$.

Theorem 2.17. The integrable curve $x(t)$ of $\tilde{F}\in\mathrm{norm}_{X(G)}(g)$ through the point $x\in G$ is given in a neighborhood of $t=0$ by

$$x(t)=(\mathrm{Exp}(-t\phi)(x))\exp\Big(\sum_{n\geq 1}(-1)^{n+1}t^n d_n(E,\phi)\Big).$$

In particular if $x=\exp X$ then

$$x(t)=\exp(e^{-t\phi}(X))\exp\Big(\sum_{n\geq 1}(-1)^{n+1}t^n d_n(E,\phi)\Big).$$

Proof. Let $\mathrm{Exp}(-t(E,\phi))=(g(t),\Gamma(t))$. Then from (6) we have $x(t)=(\Gamma(t)(x))g(t)^{-1}$. Now we use the Baker-Campbell- Hausdorff formula to compute

$$\mathrm{Exp}(E,\phi)\mathrm{Exp}(0,-\phi)=\mathrm{Exp}C((E,\phi),(0,-\phi))$$

where $C((E,\phi),(0,-\phi))=\sum_{n\geq 1}c_n((E,\phi),(0,-\phi))$. From Lemma 2.15.3 of [7] we know that

$$c_1((E,\phi),(0,-\phi))=(E,0)$$

and that for $n\geq 1$

$$(n+1)c_{n+1}((E,\phi),(0,-\phi))=\frac{1}{2}[(E,2\phi),c_n((E,\phi),(0-\phi))]$$

$$+\sum_{2\leq 2p\leq n}K_{2p}\sum_{k_1,\dots,k_{2p}>0,\ k_1+\dots+k_{2p}=n}[c_{k_1}((E,\phi),(0,-\phi)),[\cdots[c_{k_{2p}}((E,\phi),(0,-\phi)),(E,0)]\cdots].$$

By induction on n it follows easily that $c_n((E,\phi),(0-\phi))=(d_n(E,\phi),0)$ with $d_n(E,\phi)\in g$. Hence $d_1(E,\phi)=E$. Moreover, using the semidirect product in $g\odot\partial g$, we get

$$(n+1)d_{n+1}(E,\phi)=\frac{1}{2}[E,d_n(E,\phi)]+\phi(d_n(E,\phi))$$

$$+\sum_{2\leq 2p\leq n} K_{2p} \sum_{k_1,\dots,k_{2p}>0,\ k_1+\cdots+k_{2p}=n} [d_{k_1}(E,\phi),[\cdots[d_{k_{2p}}(E,\phi),E]\cdots]$$

which coincides with (7). Now

$$\mathrm{Exp}(-t(E,\phi)) = \mathrm{Exp}C(-t(E,\phi),(0,t\phi))\mathrm{Exp}(0,-t\phi) = \Big(\exp\Big(\sum_{n\geq 1}(-t)^n d_n(E,\phi)\Big), I\Big)\Big(e, \mathrm{Exp}(-t\phi)\Big).$$

Therefore $g(t) = \exp\Big(\sum_{n\geq 1}(-t)^n d_n(E,\phi)\Big)$ and $\Gamma(t) = \mathrm{Exp}(-t\phi)$. This completes the proof of the theorem. □

3 Controllability

This section is devoted to the statement and proof of the controllability results for a linear control system $\Sigma = (G, \mathcal{D})$ such that the drift vector field X has a singularity. First, we will concentrate the study to linear control systems $\Sigma = (G, \mathcal{D})$ determined by :

$$\dot{x} = X(x) + \sum_{j=1}^{k} u_j Y^j(x)$$

where the control vectors $Y^j \in g$, for $j = 1, \ldots, \kappa$ are considered as left-invariant vector fields and the flow generated by X

$$T = \{X_t | t \in \mathbb{R}\}$$

is a 1-parameter subgroup of $Aut(G)$. Later we will extend these results. Of course, X has a singularity at the neutral point e of G. We call X an infinitesimal automorphism of G.

Σ induces the following objects: the group

$$G_\Sigma = \{\theta^1_{t_1} \circ \theta^2_{t_2} \circ \ldots \circ \theta^\nu_{t_\nu} | \theta^j \in \mathcal{D},\ t_j \in \mathbb{R},\ \nu \in \mathbb{N}\}$$

and the semi-group

$$S_\Sigma = \{\theta^1_{t_1} \circ \theta^2_{t_2} \circ \ldots \circ \theta^\nu_{t_\nu} | \theta^j \in \mathcal{D},\ t_j \geq 0,\ \nu \in \mathbb{N}\}$$

of global diffeomorphisms on G.

Definition 3.1. A linear control system $\Sigma = (G, \mathcal{D})$ is said to be :

a) controllable if $S_\Sigma(e) = G$.

b) transitive if $G_\Sigma(e) = G$. □

The Lie algebra induced by Σ is defined by $L(\Sigma) = \text{Span}_{\mathcal{L}.A.}(\mathcal{D})$. It is well known that for analytical control systems the Σ-orbits $G_\Sigma(x)$, $x \in G$ are the integral manifolds of the distribution generated by $L(\Sigma)$, [6].

We wish to remind that there exists a correspondence between connected Lie subgroups of G and Lie subalgebras of g. Under this correspondence normal subgroups are associated with ideals, [7]. And, if H is a connected Lie subgroup of G then each $h \in H$ is a finite product of elements of the form

$$\exp(tY)$$

where Y belongs to the Lie algebra $\mathcal{H}$ of H and $t \in \mathbb{R}$, [3].

We denote by $\mathcal{H}$ the Lie subalgebra of g generated by the control vectors of Σ, i.e.,

$$\mathcal{H} = \text{span}_{\mathcal{L}.A.}\{Y^1, \ldots, Y^\kappa\} .$$

Each subspace V of g induces by translation a left invariant distribution Δ_V defined by

$$\Delta_V(x) = (L_x)_* V, \quad x \in G .$$

We have the following basic result:

Lemma 3.2. If X is an infinitesimal automorphism, then g is $\mathrm{ad}^i(X)$-invariant for each $i \geq 0$.

Proof. It is enough to prove the lemma for $i = 1$. Of course, this comes directly from Theorem 2.15. More simple proof is the following. Let us take $Y \in g$, then

$$[X,Y](e) = -Y_e X = -\frac{d}{ds}\Big|_{s=0} X_{exp(sY)}.$$

On another hand,

$$[X,Y](x) = \frac{d}{dt}|_{t=0}(X_{-t})_*(Y_{X_t(x)})$$
$$= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0} X_{-t} \circ Y_s \circ X_t(x)$$
$$= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0} L_x \circ X_{-t}(expsY)$$
$$= -\frac{d}{ds}\Big|_{s=0}(L_x)_* X_{exp(sY)}$$
$$= (L_x)_*[X,Y](e).$$

Thus, $[X,Y]$ is a left-invariant vector field in $X(G)$, consequently $X \in norm_{X(G)}(g)$, i.e., $[X,Y] \in g$, $\forall Y \in g$ □

We denote by $< X \mid \mathcal{H} >$ the smallest $ad(X)$-invariant, subalgebra of g containing $\mathcal{H}$. Define inductively the $\mathrm{ad}(X)(\mathcal{H})$-sequence by :

$$\mathcal{H}_0 = \mathcal{H}$$

$$\mathcal{H}_i = \mathcal{H}_{i-1} + \mathrm{ad}^i(X)(\mathcal{H}), \quad i \in \mathbb{N} ,$$

where

$$\mathrm{ad}^i(X)(\mathcal{H}) = \{[X, \mathrm{ad}^{i-1}(X)(Y)] | Y \in \mathcal{H}\} .$$

For each i the distribution $\Delta_{\mathcal{H}_i}$ is regular. Indeed, Lemma 3.2 shows that $\mathcal{H}_i$ is a subspace of g. In particular, since we consider finite dimensional Lie groups and $\mathcal{H} \neq 0$, there exists the smallest integer p, $0 \leq p < \dim.(G)$ such that the $\mathrm{ad}(X)(\mathcal{H})$-sequence stabilized at p, i.e.

$$\mathcal{H}_p = \mathcal{H}_{p+q}, \ \forall q \in \mathbb{N} .$$

Since $\mathcal{H}p$ is an $ad(X)$-invariant subalgebra of g containing $\mathcal{H}$ it follows immediately that

$$\mathcal{H}p = \langle X | \mathcal{H} \rangle .$$

In particular we have

$$\langle X|\mathcal{H}\rangle = \mathcal{H}_{p-1} + \mathrm{ad}^p(X)(\mathcal{H}) .$$

Let us denote by H and $\langle X|H\rangle$ the connected Lie subgroups of G with Lie algebras $\mathcal{H}$ and $\langle X|\mathcal{H}\rangle$ respectively. It is clear that

$$\Delta = \Delta_{\langle X|\mathcal{H}\rangle}$$

is an involutive and regular distribution. From the Frobenius Theorem Δ is integrable. If $\mathrm{In}_\Delta(x)$ denotes the integral manifold of Δ trough the state x in G then the $ad(X)$-invariance of Δ implies that

$$X_t(\mathrm{In}_\Delta(x)) = \mathrm{In}_\Delta(X_t(x)),\ \forall t \in \mathbb{R}\ , [6].$$

Since Δ is a left-invariant distribution

$$\mathrm{In}_\Delta(x) = x\langle X|H\rangle, \quad \forall x \in G\ .$$

In particular, for each real number t

$$X_t(x\ \langle X|H\rangle) = X_t(x)\ \langle X|H\rangle\ .$$

But the neutral element e of G is a fix point of T, then

$$X_t(\langle X|H\rangle) = \langle X|H\rangle, \quad \forall t \in \mathbb{R}\ .$$

So we have shown:
Proposition 3.3. Let $\Sigma = (G, D)$ be a linear control system such that X is an infinitesimal automorphism. Then

$$X_t \in \mathrm{Aut}(\langle X|H\rangle), \quad \forall t \in \mathbb{R}.$$

□

Proposition 3.3 shows that the Lie group $\langle X|H\rangle$ is T-invariant, i.e. we can consider the canonic map

$$T \xrightarrow{\pi} \mathrm{Aut}(\langle X|H\rangle)$$

and the product Lie groups

$$\langle X|H\rangle \times T \subset G \times \mathrm{Aut}(G)\ .$$

It follows that π induces an analytical action

$$\langle X|H\rangle \times T \to \langle X|H\rangle\ .$$

$$(x, X_t) \rightsquigarrow X_t(x)$$

If $x_1, x_2 \in \langle X|H\rangle$ and $X_{t_1}, X_{t_2} \in T$, we can see that the product

$$(x_1, X_{t_1})(x_2, X_{t_2}) = (X_{t_1}(x_2)x_1, X_{t_1+t_2})$$

defines a structure of Lie groups on the product manifold. With this structure $\langle X|H\rangle \times T$ is called the semidirect product of $\langle X|H\rangle$ with T relative to π [7]. We denote this Lie group by

$$S =: \langle X|H\rangle \otimes T\ .$$

The formula

$$(x_1, X_t)(x_2, \mathrm{Id})(x_1, X_t)^{-1} = (x_1^{-1} \cdot X_t(x_2) \cdot x_1, \mathrm{Id})$$

shows that $\langle X|H\rangle$ is a closed normal subgroup of S. On the other hand, the derivative of the canonic map

$$T \to \mathrm{Aut}(\langle X|\mathcal{H}\rangle)$$

denoted by ρ is a representation of the Lie algebra $L(T)$ of T in $\langle X|\mathcal{H}\rangle$, i.e.

$$L(T) \xrightarrow{\rho} \mathrm{End}(\langle X|\mathcal{H}\rangle)$$

is an homomorphism of Lie algebras such that

$$\rho(L(T)) \subset \mathrm{Der}(\langle X|\mathcal{H}\rangle) \ .$$

Indeed, for each $W \in L(T)$

$$\rho(W)(Y) = [W,Y], \quad \forall Y \in \langle X|\mathcal{H}\rangle \ .$$

ρ permits to construct a structure of Lie algebra on the product of Lie algebras

$$\langle X|\mathcal{H}\rangle \times L(T) \ .$$

In fact, if $Y_1, Y_2 \in \langle X|\mathcal{H}\rangle$ and $W_1, W_2 \in L(T)$, the bracket

$$[(Y_1,W_1),(Y_2,W_2)] = ([Y_1,Y_2] + \rho(W_1)(Y_2) - \rho(W_2)(Y_1), [W_1,W_2])$$

is well defined and transforms this product in a Lie algebra called the semidirect product of $\langle X|\mathcal{H}\rangle$ with $L(T)$ relative to ρ [7]. Let us denote this Lie algebra by

$$s =: \langle X|\mathcal{H}\rangle \otimes L(T) \ .$$

Next we shall characterize the Lie algebra of a linear control system.
Theorem 3.4. Let $\Sigma = (G,D)$ be a linear control system such that X is an infinitesimal automorphism. Then

$$L(\Sigma) \cong \langle X|\mathcal{H}\rangle \otimes L(T) \ .$$

Proof. Since $L(T)$ is an Abelian algebra we have $[s,s] \subset \langle X|\mathcal{H}\rangle$. We remark that the drift vector field X of Σ induces the generator W of $L(T)$ via

$$W_{X_t} = (L_{X_t})_* W$$

where

$$W = \frac{d}{dt}\Big|_{t=0} X_t \in L(T) \ .$$

In order to verify that the canonic map

$$L(\Sigma) \to \langle X|\mathcal{H}\rangle \otimes L(T)$$

is an isomorphism of Lie algebras we take elements on $\mathcal{D}$ and then we compute the bracket. For this it is enough to consider the generators of $L(\Sigma)$, i.e.

$$L(\Sigma) = \mathrm{Span}_{\mathcal{L.A.}}\{X + \sum_{j=1}^{\kappa} u_j Y^j | j = 1,\ldots,\kappa\}$$

and two elements of the form $X + Y^i$, $X + Y^j$. The bilinear property of the bracket in $X(G)$ shows that

$$[X + Y^i, X + Y^j] = [Y^i, Y^j] + [X, Y^j] + [Y^i, X] .$$

From this and the bracket definition to s the proposition follows. □

It is well known that the Lie algebra of a semidirect product of Lie groups is isomorphic with the semidirect product of Lie algebras which are the Lie algebras of the groups [7]. In other words in our particular case we can conclude:

Corollary 3.5. Let $\Sigma = (G, \mathcal{D})$ be a linear control system such that X is an infinitesimal automorphism. Then,

$$L(\Sigma) \cong L(\langle X|H\rangle \otimes T)$$

□

We obtain the following result :

Theorem 3.6. Let $\Sigma = (G, \mathcal{D})$ be a linear control system such that the drift vector field is an infinitesimal automorphism. If Σ is transitive, then Σ satisfies the Lie algebra rank condition.

Proof : If Σ is transitive, then

$$G_\Sigma(e) = \langle X|H\rangle = G .$$

Indeed, since Σ is an analytic control system, the integral manifold of $L(\Sigma)$ through the point $x \in G$ coincides with the orbit of x, i.e., $In_{L(\Sigma)}(e) = \langle X|H\rangle$. In fact, $X_e = 0$ and Proposition 3.3 shows that for every $x \in \langle X|H\rangle$

$$\left(\frac{d}{dt}\right)_{t=0} X_t(x) \in T_x\langle X|H\rangle .$$

Consequently

$$X_x \in (L_x)_*\langle X|\mathcal{H}\rangle$$

i.e.

$$X_x \in \Delta_{\langle X|\mathcal{H}\rangle}(x), \ \ \forall x \in \langle X|H\rangle .$$

So there exists an integer p, $1 \leq p < n$, where $n = dim.G$ such that

$$\langle X|\mathcal{H}\rangle = \mathcal{H}_{p-1} + ad^p(X)(\mathcal{H}) = g .$$

It follows immediately that

$$dim.Span_{\mathcal{L}.A.}\{Y^j, ad^i(X)(Y^j)|0 \leq i \leq p, 1 \leq j \leq \kappa\} = n .$$

Thus, Σ satisfies the rank condition. □

Remark. In particular, Theorem 3.6 shows that the Lie algebra rank condition is necessary for the controllability of Σ. In fact, $S_\Sigma(e) \subset G_\Sigma(e)$.

Next we study the closure of the Σ-accessibility set of the neutral element e of G.

Theorem 3.7. Let $\Sigma = (G, \mathcal{D})$ be a linear control system such that X is an infinitesimal automorphism and G is a connected Lie group. Then,

$$\cup_{t\geq 0} X_t(H) \subset \overline{S_\Sigma(e)} .$$

Proof. Since $\mathcal{U}$ is the class of unrestricted admissible controls, we can consider for each $j = 1, \ldots, \kappa$ and $p \in \mathbb{Z}$ the control

$$u = u(j,p) = (0, ..., 0, p, 0, ..., 0)$$

with p in the j-th position.

Let us denote by ϕ the element in $aut(G)$ induced by X. Associated to the pair $(pY^j, \phi) \in g \otimes aut(G)$, there exists an unique element $\tilde{F^j} \in norm_{X(G)}(g)$ such that

$$\Phi(\tilde{F^j}) = (pY^j, \phi) .$$

According to the commutative diagram

$$\begin{array}{ccc} \text{norm}_{X(G)}(g) & \xrightarrow{\Phi} & g \otimes aut(G) \\ Exp \downarrow & & \downarrow Exp \\ Dif(G) & \xleftarrow{\alpha} & G \otimes Aut(G) \end{array}$$

and by the Theorem 2.16 we have for each $t \in \mathbb{R}$, $m \in \mathbb{N}$ and $u(j,m)$:

$$\tilde{F}^j_{\frac{t}{m}} = \alpha \circ Exp(-\frac{t}{m}(mY^j, \phi))$$

$$= \alpha \circ Exp(-t(Y^j, 0) + (-\frac{t}{m})(0, \phi)) .$$

If $t \geq 0$, we have

$$\tilde{F}^j_{\frac{t}{m}}(e) \in S_\Sigma(e), \quad \forall m \in \mathbb{N} .$$

If $t < 0$, we can consider the control $u(j, -m)$ and again

$$\tilde{F}^j_{-\frac{t}{m}}(e) \in S_\Sigma(e), \quad \forall m \in \mathbb{N} .$$

So when $m \to \infty$ we obtain

$$\alpha \circ Exp(t(Y^j, 0))(e) \in \overline{S_\Sigma(e)}, \quad \forall t \in \mathbb{R} .$$

In particular, Theorem 2.14 shows that for each $j = 1, \cdots, \kappa$

$$exp(tY^j) \in \overline{S_\Sigma(e)}, \quad \forall t \in \mathbb{R} .$$

But H is connected, thus each element of H is a finite product of elements of the form $exp(tY)$, $Y \in \mathcal{H}$. Then we have proved that $H \subset \overline{S_\Sigma(e)}$. Now, for each element in H we can consider the solution of the differential equation generated by the constant control $u \equiv 0$. By the Proposition 3.3, it follows immediately that for every $t \geq 0$

$$X_t(H) \subset \overline{S_\Sigma(e)}$$

and this completes the proof. □

Unfortunately, the Lie algebra rank condition does not characterize the controllability property to linear control systems. In the next section, we give an example on a nilpotent simply connected Lie group, the Heisenberg group, (see Example 3).

Next we want to use a new algebraic object to study controllability. Let $\Sigma = (G, \mathcal{D})$ be a linear control system, we define the vector subspace V of g by

$$V = Span\{Y^j, ad^i(X)(Y^j) | 1 \leq i \leq p, \ 1 \leq j \leq \kappa\}$$

where the integer p is determined by the $ad(X)(\mathcal{H})$-sequence.
We say that Σ satisfies the rank (ad-rank) condition, if

$$dim.(V) = dim.(G) \ .$$

We have the following result :
Theorem 3.8. Let G be a connected Lie group and $\Sigma = (G, \mathcal{D})$ be a linear control system such that the drift vector field X is an infinitesimal automorphism. If Σ satisfies the rank condition, then Σ is controllable.
Proof : We denote by $x(t,u)$ the solution of the differential equation

$$\dot{x} = X(x) + \sum_{j=1}^{\kappa} u_j Y^j(x)$$

determined by the control $u \in \mathcal{U} \subset L_\infty([0,t], \mathbb{R}^k)$ and the initial condition e. Let us consider for each $t \geq 0$ the end point map

$$E_t : \mathcal{U} \to G$$

defined by $E_t(u) = x(t,u)$. This map is smooth and its derivative $(F_t)_0$ at 0 is defined in a neighborhood B of the constant control $u \equiv 0$ and given by

$$(F_t)_0(u(\cdot)) = \int_0^t e^{(t-\tau)ad(X)}(\sum_{j=1}^{\kappa} u_j(\tau) Y_e^j) d\tau, \ [1].$$

Next we will suppose that there exists a vector w in the dual space G_e^* of the tangent space G_e such that

$$\langle w, (F_t)_0(u(\cdot))\rangle = 0, \ \forall u(\cdot) \in B \ .$$

In particular,

$$\int_0^t \sum_{j=1}^{\kappa} \langle w, e^{(t-\tau)ad(X)}(Y_e^j)\rangle u_j(\tau) d\tau = 0 \ .$$

Since the last expression is true for every piecewise constant function

$$u : [0,t] \to \mathbb{R}^k$$

we can conclude

$$\langle w, e^{(t-\tau)ad(X)} Y_e^j\rangle = 0, \ \forall \tau \in [0,t] \ .$$

By derivation we obtain : for each $i \geq 0$ and $j = 1, \ldots, \kappa$

$$\langle w, ad^i(X) Y_e^j\rangle = 0 \ .$$

This contradicts with our rank condition assumption. So, the linear map $(F_t)_0$ is onto.

By the implicit theorem, we can conclude that the map F_t is onto on a neighborhood U of the neutral element e in G. But, G is a connected Lie group, in particular

$$\cup_m U^m = G \ .$$

It follows immediately that

$$S_\Sigma(e) = G$$

and consequently Σ is controllable. □

Corollary 3.9. Let G be an Abelian connected Lie group and $\Sigma = (G, \mathcal{D})$ be a linear control system such that X is an infinitesimal automorphism. Then,

Σ is controllable $\Leftrightarrow$ Σ satisfies the rank condition.

Proof. By the assumption the Lie algebra g of G is Abelian. Since

$$\Theta = \{Y^j, ad^i(X)(Y^j) | 1 \leq i \leq p, \ \ 1 \leq j \leq \kappa\}$$

is a subset of g. It follows that

$$Span_{\mathcal{L}.A.}(\Theta) = Span(\Theta) \ .$$

From Theorem 3.6 and Theorem 3.8, we obtain the desired conclusion. □

Remark : By the Corollary 3.9, we can decide the controllability property with the rank condition for any Lie group

$$G = T^n \times \mathbb{R}^m, \ \ n, m \in \mathbb{N}$$

where $T^n = S^1 \times \ldots \times S^1$, $(n - times)$, is a Torus.

Extension : Let G be a connected Lie group and $X \in norm_{X(G)}(g)$. In this section, we have studied the controllability property for linear control systems such that the drift vector field X has a singularity at the neutral element e of G. It is possible to extend these results to the case : X has a singularity.

Let $\tilde{F} \in norm_{X(G)}(g)$ and $x_0 \in G$ with $\tilde{F}_{x_0} = 0$. We define the vector field $\tilde{F}^{L(x_0)}$ by

$$(\tilde{F}^{L(x_0)})y = dL_{x_0^{-1}}(\tilde{F}_{x_0 y}), \ \ y \in G \ .$$

The left-invariance of the Lie bracket shows that

$$\tilde{F}^{L(x_0)} \in norm_{X(G)}(g) \ .$$

Of course, $\tilde{F}^{L(x_0)}$ has a singularity at e and by the Theorem 2.15 we obtain : there exists $\phi \in aut(G)$ such that $\Phi(\tilde{F}^{L(x_0)}) = (0, -\phi)$.

Thus, if $X = \tilde{F}$ has a singularity at $x_0 \in G$, we conclude by the same Theorem 2.15 that $X^{L(x_0)}$ is an infinitesimal automorphism of G, i.e.

$$X_t^{L(x_0)} = Exp(t\phi) \in Aut(G), \forall t \in \mathbb{R} \ .$$

Let $\Sigma = (G, \mathcal{D})$ be a linear control system. This class of system generalizes :

1. Linear control systems on $\mathbb{R}^n$.

In fact, if L is an unrestricted time-invariant linear control system on $\mathbb{R}^n$ then L is determined by the dynamic

$$\dot{x} = Ax + Bu \ ,$$

where $x \in \mathbb{R}^n$, $u \in \mathcal{U}$ and A and B are constant matrices of appropriate dimensions. Denote by $b_1, b_2, \dots b_\kappa$ the columns of B, then, we shall describe L by

$$\dot{x} = Ax + \sum_{j=1}^{\kappa} u_j b_j \ .$$

It is well known that the constant vector b_j defines a left invariant vector field Y^j on $\mathbb{R}^n$, given by

$$Y^j(x) = b_j, \quad x \in \mathbb{R}^n \ .$$

Moreover, the flow of the linear vector field A given by $A_t = e^{tA}, \quad t \in \mathbb{R}$ belong to $GL_n(\mathbb{R})$, the Lie group of all $\mathbb{R}^n$-automorphisms. In this case $\eta = \mathbb{R}^n \otimes M_n(\mathbb{R})$. So $X = (0, A) \in \eta$, then $L = (G, \mathcal{D})$, where G is the commutative connected Lie group $\mathbb{R}^n$ and

$$\mathcal{D} = \{X + \sum_{j=1}^{\kappa} u_j Y^j | u \in \mathbb{R}^\kappa\} \ .$$

So, Corollary 3.9 shows that :

Theorem (Kalman R., Ho Y., Narendra K., [4])

L is controllable $\Leftrightarrow$ $dim.Span\{b_1, \dots, b_p, Ab_1, \dots, Ab_p, \dots, A^{n-1}b_1, \dots, A^{n-1}b_\kappa\} = n$. □

Remark : We know that for $G = \mathbb{R}^n$ the element of the normalizer has the form :

$$b + A \in \mathbb{R}^n \otimes M_n(\mathbb{R}) \ .$$

In particular,

$$b + A \text{ has a singularity} \Leftrightarrow b \in Im(A) \ .$$

2. Linear control system on a matrix Lie group.

Markus introduces in [5] the notion of linear control system Σ on a q^2-dimensional matrix Lie subgroup G of $GL_n(\mathbb{R})$, and which dynamic is defined by a family of differential equations on G, of the form:

$$\dot{P} = X^*(P) + \sum_{j=1}^{\kappa} u_j Y^j P \ .$$

Here, the drift vector field X^* defined by $X^*(P) = AP - PA$ is induced by an element A in the Lie algebra $M_n(\mathbb{R})$ of all $n \times n$ real matrices and for each $j = 1, 2, \dots, \kappa$, Y^j is defined by left multiplication, (right multiplication in our definition) are also elements in $M_n(\mathbb{R})$. In this case, for each $t \in \mathbb{R}$, the flow

$$X_t^*(P) = e^{tA} P e^{-tA}, \ \forall t \in \mathbb{R}$$

is an automorphism of G. Again, $X = (0, X^*) \in \eta$. For this class of system the author obtains the following results:

Theorem (Markus L. [5])
The Σ-reachable set of the neutral element e of G has the closure in G:

$$\bigcup_{0\leq t} G_t \subset \overline{S_\Sigma(e)},$$

where

$$G_t = e^{tA} G_0 e^{-tA}$$

is a Lie group with Lie algebra

$$L(G_t) = \mathrm{Span}_{\mathcal{L}.A.}\{e^{-tA} Y^j e^{tA} | j = 1, 2, \ldots, \kappa\}$$

□

This result comes from Theorem 3.7.

Theorem (Markus L. [5])
If Σ is controllable on G then Σ satisfies the Lie algebra rank condition. □

This result comes from Theorem 3.6.

4 Examples

In this section we compute some examples. We use the Lie algebra rank condition, Theorem 3.6 and the rank condition Theorem 3.8 to study controllability.

1. Let G be the Heisenberg group of dimension 3,

$$G = \left\{ \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \Big| x_1, x_2, x_3 \in \mathbb{R} \right\} ,$$

with Lie algebra

$$L(G) = \mathrm{Span}_{\mathcal{L}.A.} \left\{ Y^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Y^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

In this case $[Y^1, Y^2] = Y^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

We consider the linear control $\Sigma = (G, \mathcal{D})$ with

$$\mathcal{D} = \{X + uY^2 | u \in \mathbb{R}\} ,$$

where the infinitesimal automorphism X is defined by

$$X(x) = x_2 Y^3, \quad \text{for } x = \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \in G .$$

A simple computation shows that

$$\langle X | \mathcal{H} \rangle = \mathrm{Span}_{\mathcal{L}.A.}\{Y^2, Y^3\} = \mathrm{Span}\{Y^2, Y^3\} .$$

Then Σ is not transitive. Indeed

$$G_\Sigma(e) = \left\{ \begin{pmatrix} 1 & 0 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \Big| x_2, x_3 \in \mathbb{R} \right\} .$$

Theorem 3.8 shows that the linear control system $\Sigma_1 = (G_\Sigma(e), \mathcal{D})$ induced by Σ on the orbit of the neutral element is controllable.

2. Let $G = SL_2(\mathbb{R})$ be the Lie subgroup of $GL_2(\mathbb{R})$ whose elements are matrices of determinant 1. The Lie algebra $sl_2(\mathbb{R})$ of G is given by

$$L(G) = \mathrm{Span}_{\mathcal{L}.A.} \left\{ Y^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Y^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

where $[Y^1, Y^2] = Y^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

We consider the matrix $A = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{R})$ and the 1-parameter group of automorphisms (X_t) of G induced by A, i.e.

$$X_t(x) = e^{tA}xe^{-tA}, \ t \in \mathbb{R}, \ x \in G .$$

To the linear control system $\Sigma = (G, \mathcal{D})$ with

$$\mathcal{D} = \{X + uY^2 | u \in \mathbb{R}\}$$

we have:

$$\mathrm{ad}(X)(Y^2) = Y^3 \quad \text{and} \quad \mathrm{ad}^2(X)(Y^2) = Y^1 .$$

Since

$$\langle X | \mathcal{H} \rangle = sl_2(\mathbb{R})$$

then by Theorem 3.8 Σ is controllable, i.e.

$$S_\Sigma(e) = SL_2(\mathbb{R}) .$$

3. Let us consider on $\mathbb{R}^2$ the non-linear control system $\mathcal{N}$ defined by the family of differential equations

$$\dot{x_1} = u$$
$$\dot{x_2} = x_1^2$$

parametrized by the unrestricted piecewise constant controls $u \in \mathcal{U}$. Of course, the positive orbit of $\mathcal{N}$ from the origin is not $\mathbb{R}^2$. So $\mathcal{N}$ can not be controllable.

On the other hand, we can describe these equations by

$$(x_1, \dot{x_2}) = (p + uq)(x_1, x_2), \quad u \in \mathcal{U}$$

where p and q are vector fields on $\mathbb{R}^2$ defined by

$$p = x_1^2 \frac{\partial}{\partial x_2} \quad and \quad q = u \frac{\partial}{\partial x_1} .$$

We have

$$[p, q] = -2ux_1 \frac{\partial}{\partial x_2}$$
$$[q, [p, q]] = -2u^2 \frac{\partial}{\partial x_2} .$$

In particular,

$$Span_{\mathcal{L}.A}\{q, [p, q]\}$$

is a nilpotent Lie algebra of dimension 3. We can realize this algebra as a subalgebra g of $M_3(\mathbb{R})$ via the representation defined by

$$\rho(p) = X^* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(q) = Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} .$$

Thus, the group G associated to g is the connected and simply connected Heisenberg Lie group. We obtain a linear control system $\Sigma = (G, \mathcal{D})$ which has dynamic is given by

$$\dot{x} = ad(X^*)(x) + uY(x), \quad x \in G, \quad u \in \mathcal{U} \ .$$

Indeed $ad(X^*)$ is an infinitesimal automorphism of G.

It follows that Σ satisfies the Lie algebra rank condition, in fact,

$$\rho[p, q] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ .$$

But, Σ can not be controllable. Otherwise, we have

$$S_\Sigma(e) = G \ .$$

Thus, by the non-linear action of G on $\mathbb{R}^2$ given by the representation, we obtain

$$S_\Sigma(0, 0) = S_N(0, 0) = \mathbb{R}^2$$

which is a contradiction.

Just observe that Σ does not satisfy the rank condition. Indeed,

$$[p, [p, q]] = 0 \ .$$

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