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Nguyen Anh Ky

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Nedialka I. Stoilova

INTERNATIONAL ATOMIC ENERGY AGENCY

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FINITE–DIMENSIONAL REPRESENTATIONS OF THE QUANTUM SUPERALGEBRA $U_q[gl(2/2)]$ II: NONTYPICAL REPRESENTATIONS AT GENERIC q

Nguyen Anh Ky [1]
International Centre for Theoretical Physics, Trieste, Italy

and

Nedialka I. Stoilova
Institute for Nuclear Research and Nuclear Energy,
Blvd. Tsarigradsko chaussee 72, Sofia 1784, Bulgaria.

ABSTRACT

The construction approach proposed in the previous paper Ref.1 allows us there and in the present paper to construct at generic deformation parameter q all finite–dimensional representations of the quantum Lie superalgebra $U_q[gl(2/2)]$. The finite–dimensional $U_q[gl(2/2)]$–modules W^q constructed in Ref.1 are either irreducible or indecomposable. If a module W^q is indecomposable, i.e. when the condition (4.41) in Ref.1 does not hold, there exists an invariant maximal submodule of W^q, to say I_k^q, such that the factor–representation in the factor–module W^q/I_k^q is irreducible and called nontypical. Here, in this paper, indecomposable representations and nontypical finite–dimensional representations of the quantum Lie superalgebra $U_q[gl(2/2)]$ are considered and classified as their module structures are analyzed and the matrix elements of all nontypical representations are written down explicitly.

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[1]Fellow of the Ministry of Education and Training, Hanoi, Vietnam.
Permanent and mailing address: Centre for Polytechnic Educations, Vinh, Vietnam.

I. INTRODUCTION

As mentioned in Ref.1 (referred hereafter to as I), explicit representations (at generic q) are known only for some quantum superalgebras of particular types like $U_q[osp(1/2)]$, [2] $U_q[gl(1/n)]$, [3], etc., while for higher rank quantum superalgebras of nonparticular type, besides some q-oscillator representations (see, for example, Ref.4), other representations had not been explicitly constructed and completely investigated (at neither generic q nor q being roots of unity) with the exception of the module structure (and some general aspects) [5] and a class of representations [6] of the quantum superalgebra $U_q[gl(m/n)]$. In this paper we continue our investigations on finite–dimensional representations of the quantum Lie superalgebra $U_q[gl(2/2)]$ $U_q[gl(2/2)]$ started in I where a procedure for their explicit construction was proposed. Especially, all the typical representations of $U_q[gl(2/2)]$ were studied completely and constructed explicitly. This paper is devoted to indecomposable representations and nontypical representations of $U_q[gl(2/2)]$ which are classified in 5 classes. Nontypical representations of every class are investigated in detail as the matrix elements are presented in their explicit forms. In such a way, in I and the present paper, $U_q[gl(2/2)]$ becomes, to our best knowledge, the highest rank superalgebra of a nonparticular type $U_q[gl(m/n)]$, $m, n \geq 2$, with all finite–dimensional representations constructed explicitly at generic q .

Introduced by Drinfel'd [7] and Jimbo [8] the quantum deformation (q-deformation) of universal enveloping (super) algebras is one of four approaches to defining quantum (super) groups [9–11]. In particular, the quantum Lie superalgebra $U_q[gl(2/2)]$ as a quantum deformation of the universal enveloping algebra $U[gl(2/2)]$ is completely generated by the Cartan– Chevalley generators obeying a number of defining relations (see **I** and Sect.II), namely, the Cartan–Kac supercommutation relations, the Serre relations and the extra–Serre relations [12]. Representing a quantum extension of the induced representation method developed by Kac [13] in the case of Lie superalgebras (from now on, only superalgebras) the approach proposed in **I** allows us to construct at generic deformation parameter q all finite–dimensional representations of the quantum superalgebra $U_q[gl(2/2)]$ (and, certainly, is applicable to other higher rank quantum (super) groups). The induced $U_q[gl(2/2)]$-modules W^q constructed in **I** are either irreducible or indecomposable (see **I**, Proposition 2). Every (typical and nontypical) $U_q[gl(2/2)]$-module W^q is decomposed into a direct sum of 16 or less irreducible $U_q[gl(2)\oplus gl(2)]$-modules V_k^q, $0\le k\le 15$,

$$W^q=\bigoplus_{k=0}^{15}V_k^q. \tag{1.1}$$

Moreover, the module structure (1.1) of W^q reminds us of the module structure of the classical $gl(2/2)$-modules W constructed in Ref. 14 (see also Ref. 15 especially for nontypical modules W) and decomposed into a direct sum of 16, at most, finite-dimensional irreducible $gl(2)\oplus gl(2)$. For a basis of W^q we can choose, as explained in **I**, a union of the Gel'fand-Zetlin (GZ) bases of all the submodules V_i^q included in the decomposition (1.1) of W^q. In **I** all the analyses and the matrix elements were provided in the framework of such a basis which we suggest to be called the quasi-Gel'fand-Zetlin (QGZ) basis. The latter, as emphasized also in Ref.1, is convenient for a construction of finite-dimensional (typical and nontypical) $U_q[gl(2/2)]$-modules and spanned on all possible patterns of the form

$$\left[\begin{array}{cccc} m_{13} & m_{23} & m_{33} & m_{43}\\ m_{12} & m_{22} & m_{32} & m_{42}\\ m_{11} & 0 & m_{31} & 0\end{array}\right], \tag{1.2}$$

where m_{ij} are complex numbers such that $m_{13}-m_{23}$, $m_{33}-m_{43}$, $m_{12}-m_{11}$, $m_{11}-m_{22}$, $m_{32}-m_{31}$ and $m_{31}-m_{42}$ are non-negative integers:

$$m_{13},\ m_{23},\ m_{33},\ m_{43}\in \mathbf{C},$$

$$m_{13}-m_{23},\ m_{33}-m_{43},\ m_{12}-m_{11},\ m_{11}-m_{22},\ m_{32}-m_{31},\ m_{31}-m_{42}\in \mathbf{Z}_+. \tag{1.3}$$

The patterns (1.2) for a fixed second row

$$[m]_k:=[m_{12},m_{22},m_{32},m_{42}]_k \tag{1.4}$$

describe a GZ basis of a finite-dimensional irreducible module:

$$V_k^q\equiv V_i^q([m]_k) \tag{1.5}$$

of the quantum subalgebra $U_q[gl(2)\oplus gl(2)]$, while the first row

$$[m]\equiv[m_{13},m_{23},m_{33},m_{43}] \tag{1.6}$$

does not change through out the module W^q and characterizes the latter as the whole:

$$W^q\equiv W^q([m])=\bigoplus_{k=0}^{15}V_k^q([m]_k). \tag{1.7}$$

The basis vectors (1.2) and the subspaces V_k^q will be specified explicitly later in Sec.II. Thus, all the even Cartan-Chevalley generators of $U_q[gl(2/2)]$ can shift, at most, only the third row $[m_{11},0,m_{31},0]$, while the odd generators of $U_q[gl(2/2)]$ leave always invariant only the first row $[m]$. In other words, we can say that the basis description (1.2) relaxing the original Gel'fand-Zetlin basis description [16] corresponds to the branching rule

$$U_q[gl(2/2)]\supset U_q[gl(2)\oplus gl(2)]\supset U_q[gl(1)\oplus gl(1)]\,. \tag{1.8}$$

This is the reason of why the basis (1.2) is referred to as a quasi - Gel'fand - Zetlin basis. The basis description (1.2) was introduced for the case of the classical superalgebra $gl(2/2)$ in Ref.14 (see also Ref.15) and extended in **I** to the case of the quantum superalgebra $U_q[gl(2/2)]$. For some other extensions of the GZ basis description will be discussed in the Conclusion.

An induced module W^q and, therefore, the corresponding representation that W^q carries are irreducible and called typical if and only if the condition (I.4.41) (formula (4.41) in **I**)

$$(l_{13}+l_{33}+3)(l_{13}+l_{43}+3)(l_{23}+l_{33}+3)(l_{23}+l_{43}+3)\neq 0 \tag{1.9}$$

holds, where $l_{ij}=m_{ij}-i$, for $i=1,2$ and $l_{ij}=m_{ij}-i+2$, for $i=3,4$. The typical modules and representations of $U_q[gl(2/2)]$ were investigated in detail and all explicitly constructed at generic deformation parameter q in **I**. When the above-mentioned condition (1.9) (or I.4.41) is violated the modules W^q are no longer irreducible but indecomposable. It turns out that the indecomposable representations are divided into five classes $k=1,2,3,4,5$ (see (3.1)-(3.5)). Then in each indecomposable module W^q, belonging always to one of these five classes k, there exists a maximal invariant submodule, to say I_k^q, and the compliment to I_k^q subspace of W^q is not invariant under $U_q[gl(2/2)]$ transformations. The factor module $W_k^q = W^q/I_k^q$, however, carries an irreducible representation of $U_q[gl(2/2)]$ which is called nontypical (cf. Ref.15). Here, following a programme close to the classical one [14,15] we shall determine all nontypical $U_q[gl(2/2)]$-modules and choose within each of them an appropriate basis so that the decompositions of the nontypical representations of $U_q[gl(2/2)]$ into irreducible representations of $U_q[gl(2) \oplus gl(2)]$ are evident and their explicit matrix elements can be more easily written down. Going step by step we can imagine that the present paper is something like a quantum deformation of Ref.15, referred hereafter to as **I***. (Therefore, our plan here, in general, will go hand in hand with that of **I***.)

Since for an explicit construction of the finite-dimensional representations of $U_q[gl(2/2)]$ we have already made in **I** a relevant introduction to the problem, here, in Section II, in order to make the present paper more self-contained, we repeat only briefly some of the basic concepts and the main points in defining the quantum superalgebra $U_q[gl(2/2)]$ and its induced representations. According to the plan, we shall consider indecomposable representations of $U_q[gl(2/2)]$ in Section III where all the nontypical representations are classified and constructed explicitly at generic q. In Section IV we shall show that the class of the modules W^q determined in **I** and in the present paper, contains all finite-dimensional irreducible modules of $U_q[gl(2/2)]$. The conclusion is made in Section V where we state that the finite-dimensional irreducible representations of $U_q[gl(2/2)]$ are quantum deformations of the finite-dimensional irreducible representations of the classical $gl(2/2)$. In the Appendix we write down the matrix elements of the generators E_{23} and E_{32} for the nontypical representations of all the classes with the exception of the class I which is described in more detail in Sec. III.

Throughout the paper, for a convenient reading we shall keep as many as possible of the abbreviations and notations used in Refs.1, 14 and 15 among the following ones:

fidirmod(s) – finite-dimensional irreducible module(s),

GZ basis – Gel'fand-Zetlin basis,

QGZ basis – quasi- Gel'fand-Zetlin basis,

lin.env.{X} – linear envelope of X,

q – the deformation parameter,

$V_l^q \otimes V_r^q$ – tensor product between two linear spaces V_l^q and V_r^q or a tensor product between a $U_q[gl(2)_l]$-module V_l^q and a $U_q[gl(2)_r]$-module V_r^q,

$T^q \odot V_0^q$ – tensor product between two $U_q[gl(2) \oplus gl(2)]$-modules T^q and V_0^q,

$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$, where x is some number or operator,

$[x] \equiv [x]_{q^2}$,

$[E, F\}$ – supercommutator between E and F,

$[E, F\}_q \equiv EF - qFE$ – q-deformed supercommutator between E and F,

$[m] = [m_{13}, m_{23}, m_{33}, m_{43}]$ – the highest weight,

$l_{ij} = m_{ij} - i$ for $i = 1, 2$ and $l_{ij} = m_{ij} - i + 2$ for $i = 3, 4$,

I_k^q – the maximal invariant subspace in $W^q([m])$, corresponding to the class k, $k = 1, 2, 3, 4, 5$ (see (3.1)-(3.5)),

$W_k^q([m]) = W^q([m])/I_k^q$ – the class k nontypical module,

$(m;\ m_{kl} = \alpha)^{\pm ij}$ – a pattern obtained from (m) by replacing m_{kl} with α and by shifting $m_{ij} \to m_{ij} \pm 1$,

$[m;\ m_{kl} = \alpha]$ – a signature obtained from $[m]$ by replacing m_{kl} with α,

$\boxed{\text{-a,-b,c,d}}_{\ (s)}$ – a $U_q[gl(2/2)]$-fidirmod with a signature $[m_{13} - a, m_{23} - b, m_{33} + c, m_{43} + d]$ (see (3.7)),

$\overset{\text{inv}}{\boxed{\text{-a,-b,c,d}}}_{\ (s)}$ – a $\boxed{\text{-a,-b,c,d}}_{\ (s)}$ a $U_q[gl(2/2)]$-fidirmod with a signature

$[m_{13}-a, m_{23}-b, m_{33}+c, m_{43}+d]$ belonging to I_k^q (see (3.8)),

$\overset{\text{noninv}}{\boxed{-a,-b,c,d}}_{(s)}$ - a $\boxed{-a,-b,c,d}_{(s)}$ a $U_q[gl(2/2)]$- fidirmod with a signature

$[m_{13}-a, m_{23}-b, m_{33}+c, m_{43}+d]$ belonging to the compliments

to I_k^q subspaces in W^q (see (3.9)).

Note that the quantum deformation $[x] \equiv [x]_{q^2}$ of x must not be confused with the highest weight (signature) $[m]$ in the (quasi) GZ basis (m) or with the notation [,] for commutators.

II. $U_q[gl(2/2)]$ AND ITS INDUCED REPRESENTATIONS [1]

The quantum superalgebra $U_q[gl(2/2)]$ [1] as a quantum deformation of the universal enveloping algebra $U[gl(2/2)]$ of the superalgebra $gl(2/2)$ is completely generated by the Cartan–Chevalley generators E_{ii}, $i=1,2,3,4$, $E_{12}\equiv e_1$, $E_{23}\equiv e_2$, $E_{34}\equiv e_3$, $E_{21}\equiv f_1$, $E_{32}\equiv f_2$ and $E_{43}\equiv f_3$ satisfying [1]

a) the Cartan-Kac supercommutation relations ($1\le i, i+1, j, j+1 \le 4$):

$$\begin{aligned}
[E_{ii}, E_{jj}] &= 0,\\
[E_{ii}, E_{j,j+1}] &= (\delta_{ij}-\delta_{i,j+1})E_{j,j+1},\\
[E_{ii}, E_{j+1,j}] &= (\delta_{i,j+1}-\delta_{ij})E_{j+1,j},\\
\{E_{i,i+1}, E_{j+1,j}\} &= \delta_{ij}[h_i]_{q^2}, \quad h_i = (E_{ii} - \tfrac{d_{i+1}}{d_i}E_{i+1,i+1}),
\end{aligned} \tag{2.1}$$

with $d_1 = d_2 = -d_3 = -d_4 = 1$,

b) the Serre-relations:

$$\begin{aligned}
[E_{12}, E_{34}] &= [E_{21}, E_{43}] = 0,\\
E_{23}^2 &= E_{32}^2 = 0,\\
[E_{12}, E_{13}]_{q^2} &= [E_{24}, E_{34}]_{q^2} = 0,\\
[E_{21}, E_{31}]_{q^2} &= [E_{42}, E_{43}]_{q^2} = 0,
\end{aligned} \tag{2.2}$$

and

c) the extra-Serre relations [12]:

$$\begin{aligned}
\{E_{13}, E_{24}\} &= 0,\\
\{E_{31}, E_{42}\} &= 0,
\end{aligned} \tag{2.3}$$

where the operators

$$\begin{aligned}
E_{13} &:= [E_{12}, E_{23}]_{q^{-2}},\\
E_{24} &:= [E_{23}, E_{34}]_{q^{-2}},\\
E_{31} &:= -[E_{21}, E_{32}]_{q^{-2}},\\
E_{42} &:= -[E_{32}, E_{43}]_{q^{-2}}.
\end{aligned} \tag{2.4}$$

and the operators composed in the following way

$$\begin{aligned}
E_{14} &:= [E_{12}, [E_{23}, E_{34}]_{q^{-2}}]_{q^{-2}} \equiv [E_{12}, E_{24}]_{q^{-2}},\\
E_{41} &:= [E_{21}, [E_{32}, E_{43}]_{q^{-2}}]_{q^{-2}} \equiv -[E_{21}, E_{42}]_{q^{-2}}
\end{aligned} \tag{2.5}$$

were defined in **I** as new generators. The latter are odd and have vanishing squares. They, together with the Cartan-Chevalley generators, form a full system of q-analogues of the Weyl generators e_{ij}, $1\le i,j\le 4$,

$$(e_{ij})_{kl} = \delta_{ik}\delta_{jl}, \tag{2.6}$$

of the superalgebra $gl(2/2)$ whose universal enveloping algebra $U[gl(2/2)]$ is a classical limit of $U_q[gl(2/2)]$ when $q\to 1$. Other commutation relations between E_{ij}, used in different calculations through out **I** and the present paper, follow from the relations (2.1)-(2.3) and the definitions (2.4) and (2.5).

A. The induced modules

Since $U_q[gl(2/2)_0] \equiv U_q[gl(2)_l \oplus gl(2)_r]$ generated by E_{ij}, $1\le i,j\le 2$ or $3\le i,j\le 4$, is a stability subalgebra of $U_q[gl(2/2)]$ we can construct finite-dimensional representations of $U_q[gl(2/2)]$ induced from finite-dimensional representations of $U_q[gl(2/2)_0]$ which, as was shown by Rosso [17] and Lusztig [18], are simply quantum deformations (q-deformations) of finite–dimensional representations of the classical

$gl(2) \oplus gl(2)$. In **I**, the $U_q[gl(2/2)]$-module W^q induced from a finite-dimensional irreducible $U_q[gl(2/2)_0]$-module V_0^q such that

$$E_{23} V_0^q = 0, \tag{2.7}$$

by the construction, is the factor-space

$$W^q = [U_q \otimes V_0^q]/I^q, \tag{2.8}$$

where $U_q := U_q[gl(2/2)]$, while

$$I^q = \text{lin.env.}\{ub \otimes v - u \otimes bv \| \; u \in U_q, \; b \in U_q(B) \subset U_q, \; v \in V_0^q\} \tag{2.9}$$

and

$$U_q(B) = \text{lin.env.}\{E_{ij}, E_{23} \| i,j = 1,2 \; \text{ and } \; i,j = 3,4\}. \tag{2.10}$$

According to (2.8) any vector w from the induced module W^q has the form

$$w = u \otimes v, \quad u \in U_q, \quad v \in V_0^q. \tag{2.11}$$

Then W^q is a $U_q[gl(2/2)]$-module in the sense

$$gw \equiv g(u \otimes v) = gu \otimes v \in W^q \tag{2.12}$$

for $g, u \in U_q$, $w \in W^q$ and $v \in V_0^q$.

As explained in **I**, for a (GZ) basis of the $U_q[gl(2/2)_0]$-module V_0^q we can take the tensor product

$$(m) := \begin{bmatrix} m_{13} & m_{23} & m_{33} & m_{43} \\ m_{13} & m_{23} & m_{33} & m_{43} \\ m_{11} & 0 & m_{31} & 0 \end{bmatrix} := \begin{bmatrix} m_{13} & & m_{23} \\ & m_{11} & \end{bmatrix} \otimes \begin{bmatrix} m_{33} & & m_{43} \\ & m_{31} & \end{bmatrix} \tag{2.13}$$

between the GZ basis $(m)_l$ of $U_q[gl(2)_l]$

$$(m)_l := \begin{bmatrix} [m]_l \\ m_{11} \end{bmatrix} := \begin{bmatrix} m_{13} & & m_{23} \\ & m_{11} & \end{bmatrix} \tag{2.14}$$

and the GZ basis $(m)_r$ of $U_q[gl(2)_r]$

$$(m)_r := \begin{bmatrix} [m]_r \\ m_{31} \end{bmatrix} := \begin{bmatrix} m_{33} & & m_{43} \\ & m_{31} & \end{bmatrix}, \tag{2.15}$$

where m_{ij} are complex numbers satisfying (1.3). Then the decomposition (1.6) can be written explicitly as follows [1]

$$\begin{aligned} W^q([m]) \;=\; & V^q_{(00)}([m_{13}, m_{23}, m_{33}, m_{43}]) \\ & \bigoplus_{i=0}^{\min(1,2l)} \bigoplus_{j=0}^{\min(1,2l')} V^q_{(10)}([m_{13}-i, m_{23}+i-1, m_{33}-j+1, m_{43}+j]) \\ & \bigoplus_{i=0}^{\min(2,2l)} V^q_{(11)}([m_{13}-i, m_{23}+i-2, m_{33}+1, m_{43}+1]) \\ & \bigoplus_{j=0}^{\min(2,2l')} V^q_{(20)}([m_{13}-1, m_{23}-1, m_{33}-j+2, m_{43}+j]) \\ & \bigoplus_{i=0}^{\min(1,2l)} \bigoplus_{j=0}^{\min(1,2l')} V^q_{(21)}([m_{13}-i-1, m_{23}+i-2, m_{33}-j+2, m_{43}+j+1]) \\ & \bigoplus V^q_{(22)}([m_{13}-2, m_{23}-2, m_{33}+2, m_{43}+2]), \end{aligned} \tag{2.16}$$

where $V^q_{(00)} \equiv V^q_0$, $l = \frac{1}{2}(m_{13} - m_{23})$, $l' = \frac{1}{2}(m_{33} - m_{43})$. The basis within the module $W^q([m]) = W^q([m_{13}, m_{23}, m_{33}, m_{43}])$ is spanned on all the possible QGZ tableaux (see **I** and also Refs. 14 and 15)

$$(m)_{(hk)} \equiv \begin{bmatrix} m_{13} & m_{23} & m_{33} & m_{43} \\ m_{12} & m_{22} & m_{32} & m_{42} \\ m_{11} & 0 & m_{31} & 0 \end{bmatrix}_{(hk)}, \quad h,k \in \{0,1,2\} \tag{2.17}$$

such that

$$\begin{aligned} m_{12} &= m_{13} - r - \theta(h-2) - \theta(k-2) + 1, \\ m_{22} &= m_{23} + r - \theta(h-1) - \theta(k-1) - 1, \\ m_{32} &= m_{33} + h - s + 1, \\ m_{42} &= m_{43} + k + s - 1, \end{aligned} \tag{2.18}$$

where

$$r = 1, ..., 1 + \min(h-k, 2l'),$$

$$s = 1, ..., 1 + \min(\langle h\rangle + \langle k\rangle, 2l), \tag{2.19}$$

$$\theta(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0 \end{cases} \tag{2.20}$$

and

$$\langle i\rangle = \begin{cases} 1 & \text{for odd } i, \\ 0 & \text{for even } i. \end{cases} \tag{2.21}$$

Now we are ready to write down explicitly the matrix elements for all the (sufficiently, Cartan-Chevalley) generators of $U_q[gl(2/2)]$.

B. Typical representations

When the condition (1.9), i.e., (I.2.41), holds the finite-dimensional module W^q is irreducible and called typical. In that case, the transformations of the basis (2.17) of W^q under the actions of the generators of $U_q[gl(2/2)]$ are explicitly given in **I**. After using the formal rule (I.4.44)

$$\left|\frac{[x_1]...[x_i]}{[y_1]...[y_j]}\right| := \frac{[|x_1|]...[|x_i|]}{[|y_1|]...[|y_j|]} \tag{2.22}$$

assumed in **I** and removing the modulus, the expressions of the matrix elements of the even generators of $U_q[gl(2/2)]$ remain the same as in **I** (see (I.4.43)), while the matrix elements (I.4.45) and (I.4.46) of the generators E_{23} and E_{32} are reorganized (with a slight simplification) respectively as follows [1]:

$$\begin{aligned}
E_{23}(m)_{(00)} &= 0, \\
E_{23}(m)_{(10)} &= -[l_{3-r,3} + l_{s+2,3} + 3]\left(\frac{(-1)^{s+r}[l_{3-r,3} - l_{11}][l_{5-s,3} - l_{31} + 1]}{[l_{13} - l_{23}][l_{33} - l_{43}]}\right)^{1/2} (m)_{(00)}^{+i2-j2-31}, \\
E_{23}(m)_{(ab)} &= \sum_{i=max(1,b-r+2)}^{min(2,b-r+3)} \; \sum_{j=max(3,b+s+1)}^{min(4,b+s+2)} (-1)^{(b-1)i+b(j+1)} \\
&\times[l_{i3} + l_{j3} + \langle s\rangle - \langle r\rangle + 3] \\
&\times\left(\frac{(-1)^{i+j+1}[l_{i2} - l_{11} + 1][l_{7-j,2} - l_{31} + 1]}{[l_{12} - l_{22}][l_{32} - l_{42}]}\right)^{1/2} \\
&\times\left(\frac{[l_{i2} - l_{3-i,2} + \langle r\rangle - 1]}{[2 - \langle r\rangle][l_{i3} - l_{3-i,3} + (-1)^r]}\right)^{b/2} \\
&\times\left(\frac{[l_{j2} - l_{7-j,2} - \langle s\rangle + 1]}{[2 - \langle s\rangle][l_{j3} - l_{7-j,3} - (-1)^s]}\right)^{(1-b)/2} (m)_{(10)}^{+i2-j2-31}, \quad a + b = 2, \\
E_{23}(m)_{(21)} &= -\sum_{i=1}^{2}\sum_{j=3}^{4} \sum_{\langle s+j\rangle\leq k=0,1\leq\langle r+i\rangle} (-1)^{(1-k)i+kj} \\
&\times[l_{i3} + l_{j3} - (-1)^k \langle i + j + s + r\rangle + 3] \\
&\times\left(\frac{(-1)^{i+j+1}[l_{i2} - l_{11} + 1][l_{7-j,2} - l_{31} + 1]}{[l_{12} - l_{22}][l_{32} - l_{42}]}\right)^{1/2} \\
&\times\left(\frac{(-1)^{r+1}[l_{r2} - l_{i2} + 2k - 2]}{[2 - k][l_{13} - l_{23}]}\right)^{(i+r)/2} \\
&\times\left(\frac{(-1)^{s}[l_{5-s,2} - l_{j2} + 2k]}{[1 + k][l_{33} - l_{43}]}\right)^{(s+j+1)/2} (m)_{(1+k,1-k)}^{+i2-j2-31}, \\
E_{23}(m)_{(22)} &= \sum_{i=1}^{2}\sum_{j=3}^{4}(-1)^{i+j}[l_{i3} + l_{j3} + 3] \\
&\times\left(\frac{(-1)^{i+j+1}[l_{i3} - l_{11} - 1][l_{7-j,3} - l_{31} + 3]}{[l_{13} - l_{23}][l_{33} - l_{43}]}\right)^{1/2} (m)_{(21)}^{+i2-j2-31},
\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}
E_{32}(m)_{(00)} &= -\sum_{i=1}^{2}\sum_{j=3}^{4}\left(\frac{(-1)^{i+j+1}[l_{i3} - l_{11}][l_{7-j,3} - l_{31}]}{[l_{13} - l_{23}][l_{33} - l_{43}]}\right)^{1/2} (m)_{(10)}^{-i2+j2+31}, \\
E_{32}(m)_{(10)} &= \sum_{i=1}^{2}\sum_{j=3}^{4} \sum_{\langle r+i\rangle\leq k=0,1\leq\langle s+j\rangle} (-1)^{(1-k)i+k(j+1)} \\
&\times\left(\frac{(-1)^{i+j+1}[l_{i2} - l_{11}][l_{7-j,2} - l_{31}]}{[l_{12} - l_{22}][l_{32} - l_{42}]}\right)^{1/2} \\
&\times\left(\frac{(-1)^{r}[l_{3-i,3} - l_{i3} + 2k - 1]}{[1 + k][l_{13} - l_{23}]}\right)^{(i+r+1)/2} \\
&\times\left(\frac{(-1)^{s+1}[l_{7-j,3} - l_{j3} + 2k - 1]}{[2 - k][l_{33} - l_{43}]}\right)^{(s+j)/2} (m)_{(2-k,k)}^{-i2+j2+31},
\end{aligned}$$

$$
\begin{aligned}
E_{32}(m)_{(ab)} &= -\sum_{i=max(1,r-b)}^{min(2,r-b+1)} \sum_{j=max(3,5-b-s)}^{min(4,6-b-s)} (-1)^{bi+(1-b)j} \\
&\times \left(\frac{(-1)^{i+j+1}[l_{i2}-l_{11}][l_{7-j,2}-l_{31}]}{[l_{12}-l_{22}][l_{32}-l_{42}]} \right)^{1/2} \\
&\times \left(\frac{[l_{i2}-l_{3-i,2}-(r)+1]}{[2-(r)][l_{i3}-l_{3-i,3}-(-1)^r]} \right)^{b/2} \\
&\times \left(\frac{[l_{j2}-l_{7-j,2}+(s)-1]}{[2-(s)][l_{j3}-l_{7-j,3}+(-1)^s]} \right)^{(1-b)/2} (m)_{(21)}^{-i2+j2+31}, \quad a+b=2, \\
E_{32}(m)_{(21)} &= -(-1)^{r+s} \left(\frac{(-1)^{r+s}[l_{r3}-l_{11}-1][l_{s+2,3}-l_{31}+2]}{[l_{13}-l_{23}][l_{33}-l_{43}]} \right)^{1/2} \\
&\times (m)_{(22)}^{-r2+j2+31}, \quad j=5-s, \\
E_{32}(m)_{(22)} &= 0, \qquad (2.24))
\end{aligned}
$$

where the rescaling

$$
\begin{aligned}
E_{23} &\longrightarrow q^2 E_{23}, \\
E_{32} &\longrightarrow q^{-2} E_{32},
\end{aligned}
\qquad (2.25))
$$

and the rescaling (I.4.47) in Ref.1 have already been taken into account. Unless stated otherwise, hereafter these rescalings will be kept and understood throughout the paper.

If the condition (1.9) (i.e., (I.4.41)) is not fulfilled, namely, if for certain values of $i = 1, 2$ and $j = 3, 4$

$$
l_{i3} + l_{j3} + 3 = 0; \quad i = 1, 2 \text{ and } j = 3, 4 \qquad (2.26)
$$

the representations of $U_q[gl(2/2)]$ in W^q are no longer irreducible but indecomposable (cf. Refs.14,15). As we can see later, there are five possibilities of occurring the equalities (2.26) leading to five classes of nontypical representations of $U_q[gl(2/2)]$ which are irreducible representations extracted from indecomposable representations in modules W^q. In the next section, where our plan in many points coincides with the one of I*, we shall classify and consider indecomposable representations and all the nontypical representations of $U_q[gl(2/2)]$.

III. NONTYPICAL REPRESENTATIONS

When a module W^q is indecomposable, it contains a maximal invariant subspace I_k^q and simultaneously the compliment to I_k^q subspace in W^q is not invariant under the actions of $U_q[gl(2/2)]$. However, as mentioned in the introduction, the factor module $W_k^q = W^q/I_q^k$ carries an irreducible representation of $U_q[gl(2/2)]$ which is called nontypical. It turns out that all the assertions proved in I* can be extended to take place in the present case of the quantum deformation at generic q. Because the proofs, some of which represent direct computations, are cumbersome, in this section we shall not prove all these assertions but only some of them.

Since $m_{13} - m_{23} \in \mathbf{Z}_+$ and $m_{33} - m_{43} \in \mathbf{Z}_+$ or, equivalently, $l_{13} - l_{23} \in \mathbf{N}$ and $l_{33} - l_{43} \in \mathbf{N}$ the indecomposable representations and, therefore, the nontypical representations of $U_q[gl(2/2)]$ are classified, as in the classical case [15], in five following classes (see (2.26))

$$
\begin{aligned}
&\text{class 1} \quad l_{13} + l_{43} + 3 = 0 \quad \Leftrightarrow \quad m_{13} + m_{43} = 0, \\
&\qquad\qquad l_{23} + l_{33} + 3 \neq 0 \quad \Leftrightarrow \quad m_{23} + m_{33} \neq 0; \qquad (3.1) \\
&\text{class 2} \quad l_{13} + l_{43} + 3 \neq 0 \quad \Leftrightarrow \quad m_{13} + m_{43} \neq 0, \\
&\qquad\qquad l_{23} + l_{33} + 3 = 0 \quad \Leftrightarrow \quad m_{23} + m_{33} = 0; \qquad (3.2) \\
&\text{class 3} \quad l_{23} + l_{43} + 3 = 0 \quad \Leftrightarrow \quad m_{23} + m_{43} - 1 = 0; \qquad (3.3) \\
&\text{class 4} \quad l_{13} + l_{33} + 3 = 0 \quad \Leftrightarrow \quad m_{13} + m_{33} + 1 = 0; \qquad (3.4) \\
&\text{class 5} \quad l_{13} + l_{43} + 3 = 0 \quad \Leftrightarrow \quad m_{13} + m_{43} = 0, \\
&\qquad\qquad l_{23} + l_{33} + 3 = 0 \quad \Leftrightarrow \quad m_{23} + m_{33} = 0. \qquad (3.5)
\end{aligned}
$$

Let us consider an induced module $W^q([m])$, $[m] = [m_{13}, m_{23}, m_{33}, m_{43}]$, of $U_q[gl(2/2)]$. According to (2.16) the module $W^q([m])$ is decomposed into a sum of all possible 16 or less $U_q[gl(2) \oplus gl(2)]$ - fidimods $V_{(hk)}([m]_{(hk)})$, where $h, k \in \{0, 1, 2\}$, while the signature $[m]_{(hk)} \equiv [m_{12}, m_{22}, m_{32}, m_{42}]_{(hk)}$ determined by the same formula (and also by (2.18)-(2.21)) has the form

$$[m_{12}, m_{22}, m_{32}, m_{42}]_{(hk)} = [m_{13}-a, m_{23}-b, m_{33}+c, m_{43}+d]_{(hk)}, \quad a, b c, d \in \mathbf{Z}_+. \quad (3.6)$$

The subscript (hk) can be omitted when there is no a confusion caused by some degenerations. As in $\mathbf{I}^*$, we assume the notation (with a slight modification)

$$V^q_{(s)}([m_{13}-a, m_{23}-b, m_{33}+c, m_{43}+d]) := \boxed{-a,-b,c,d}_{\ (s)} \quad (3.7)$$

for an arbitrary $V^q_{(s)}([m_{13}-a, m_{23}-b, m_{33}+c, m_{43}+d])$ which is denoted by

$$V^q_{(s)}([m_{13}-a, m_{23}-b, m_{33}+c, m_{43}+d]) := \overset{\text{inv}}{\boxed{-a,-b,c,d}}_{\ (s)} \quad (3.8)$$

if

$$V^q_{(s)}([m_{13}-a, m_{23}-b, m_{33}+c, m_{43}+d]) \subseteq I^q_k$$

or by

$$V^q_{(s)}([m_{13}-a, m_{23}-b, m_{33}+c, m_{43}+d]) := \overset{\text{noninv}}{\boxed{-a,-b,c,d}}_{\ (s)} \quad (3.9)$$

if otherwise,

$$V^q_{(s)}([m_{13}-a, m_{23}-b, m_{33}+c, m_{43}+d]) \cap I^q_k = V,$$

where by (s) we denote (hk), upper, lower or any other indices specifying the subspaces $V^q_{(s)}$. As in the classical case [15], for some $V^q_{(s)}$ neither (3.8) nor (3.9) holds.

As mentioned above, although the matrix elements and other expressions are deformed all the propositions proved in $\mathbf{I}^*$ have in the present case quantum analogues. Bellow, we shall prove some of the most important Propositions.

Proposition 1: For any $k = 1,2,3,4,5$ the submodule $V^q_{(22)}([m_{13}-2, m_{23}-2, m_{33}+2, m_{43}+2])$ always belongs to the maximal $U_q[gl(2/2)]$-invariant subspace I^q_k of the class k indecomposable induced module W^q

$$V^q_{(22)}([m_{13}-2, m_{23}-2, m_{33}+2, m_{43}+2]) = \overset{\text{inv}}{\boxed{-2,-2,2,2}}_{\ (22)} \subset I^q_k. \quad (3.10)$$

Proof. This proposition is a quantum analogue of Proposition 4 in $\mathbf{I}^*$. Following the latter we suppose

$$0 \neq x \in I^q_k. \quad (3.11)$$

Then from $\mathbf{I}$ we have

$$x = \sum_{\theta_i=0,1} \sum_{(m)\in V^q_{(0)}} \alpha(\theta_1,\theta_2,\theta_3,\theta_4;(m))\,(E_{31})^{\theta_1}(E_{32})^{\theta_2}(E_{41})^{\theta_3}(E_{42})^{\theta_4} \otimes (m). \quad (3.12)$$

Let k be a number (which is never bigger than 4) such that if $\sum_{i=1}^4 \theta_i < k$ all the coefficients $\alpha(\theta_1,\theta_2,\theta_3,\theta_4;(m)) = 0$ and for certain (m^0) and θ_i^0, $\sum_{i=1}^4 \theta_i^0 = k$, $\alpha(\theta_1^0,\theta_2^0,\theta_3^0,\theta_4^0;(m^0)) \neq 0$. Thus the first sum in (3.12) is truncated and spreads over only all θ_i, $i = 1,2,3,4$, for which $\sum_{i=1}^4 \theta_i \geq k$. From (3.11) we have also

$$0 \neq (E_{31})^{1-\theta_1^0}(E_{32})^{1-\theta_2^0}(E_{41})^{1-\theta_3^0}(E_{42})^{1-\theta_4^0} x := y \in I^q_k. \quad (3.13)$$

Using (2-1)-(2.5) (or (I.3.1)-(I.3.5)), (I.4.33) and (I.4.34) (i.e., (2.16)) we derive

$$(E_{31})^{1-\theta_1^0}(E_{32})^{1-\theta_2^0}(E_{41})^{1-\theta_3^0}(E_{42})^{1-\theta_4^0}(E_{31})^{\theta_1^0}(E_{32})^{\theta_2^0}(E_{41})^{\theta_3^0}(E_{42})^{\theta_4^0}$$

$$= (-1)^{(1-\theta_4^0)(\theta_1^0+\theta_2^0+\theta_3^0)+(1-\theta_3^0)(\theta_1^0+\theta_2^0)+(1-\theta_2^0)\theta_1^0} E_{31}E_{32}E_{41}E_{42}$$

and according to (I.4.33) and (2.16) (or (I.4.34))

$$y \in E_{31}E_{32}E_{41}E_{42} \otimes V^q_0([m]) \equiv V^q_{22}([m_{13}-2, m_{23}-2, m_{33}-2, m_{43}-2]). \quad (3.14)$$

In other words,

$$V^q_{(22)}([m_{13}-2, m_{23}-2, m_{33}+2, m_{43}+2]) \subset I^q_k,$$

since $V^q_{(22)}$ is a $U_q[gl(2)\oplus gl(2)]$-fidirmod and I^q_k is the maximal $U_q[gl(2/2)]$ invariant subspace of the indecomposable module W^q. Therefore (3.10) holds.

Note that for a proof of the latter proposition we can choose, for example,

$$x = \sum_{\eta_i=0,1} \sum_{(m)\in V^q_{(0)}} \alpha(\eta_1,\eta_2,\eta_3,\eta_4;(m))\,(E_{41})^{\eta_1}(E_{31})^{\eta_2}(E_{42})^{\eta_3}(E_{32})^{\eta_4} \otimes (m)$$

instead (3.12).

The following proposition is a quantum analogue of Proposition 5 in $\mathbf{I}^*$:

Proposition 2: For any $k = 1,2,3,4,5$

$$V^q_{(00)}([m_{13},m_{23},m_{33},m_{43}]) = \overset{\text{noninv}}{\boxed{0,0,0,0}}_{(00)} \cap I^q_k = \vee. \tag{3.15}$$

Proof. If

$$V^q_{(00)}([m_{13},m_{23},m_{33},m_{43}]) \cap I^q_k \neq \vee$$

then

$$V^q_{(00)}([m_{13},m_{23},m_{33},m_{43}]) \subset I^q_k.$$

Therefore, according to (1.4.19') and the fact that the proper subspace I^q_k of W^q is invariant under the actions of $U_q[gl(2/2)]$ we would have the contradiction

$$W^q = T^q \odot V^q_{(00)} \subset I^q_k$$

which means I^q_k $(= W^q)$ is not a proper subspace of W^q as it has to be.

We are also in a position to prove the following assertion:

Proposition 3: The transformation of the factor space $W^q_k([m]) \equiv W^q([m])/I^q_k$ under the action of $U_q[gl(2/2)]$ can be obtained by replacing in the corresponding transformation of $W^q([m])$ all the basis vectors belonging to the maximal invariant subspace I^q_k by zero.

The proof of this proposition is analogous with the one of Proposition 6 in $\mathbf{I}^*$.

Let us now consider the nontypical representations of $U_q[gl(2/2)]$ of all the classes corresponding to the conditions (3.1)-(3.5). Since the decomposition structures of the (quantum) indecomposable and nontypical modules are the same as those of the classical ones, below, in order to make the present paper less cumbersome we shall expose as examples only some of their decomposition schemes but not all which can be found in $\mathbf{I}^*$. In the meantime the matrix elements for the nontypical representations of all the classes are given explicitly in the next subsections or in the

Appendix. A more detailed description will be made only for the class 1 nontypical representations.

A. The class 1 nontypical representations

This class corresponds to the case (3.1)

$$\begin{aligned} \text{class 1} \quad & l_{13}+l_{43}+3=0 \quad \Leftrightarrow \quad m_{13}+m_{43}=0, \\ & l_{23}+l_{33}+3\neq 0 \quad \Leftrightarrow \quad m_{23}+m_{33}\neq 0. \end{aligned} \tag{3.1}$$

Then we have to replace everywhere m_{43} with $-m_{13}$

$$W^q([m;\ m_{43}=-m_{13}]) := W^q([m_{13},m_{23},m_{33},-m_{13}]) \tag{3.16}$$

and keep valid the conditions

$$\begin{aligned} m_{13} > m_{23} \quad & \Leftrightarrow \quad l_{13}-l_{23}-1>0, \\ m_{33} > -m_{13} \quad & \Leftrightarrow \quad l_{13}+l_{33}+2>0 \end{aligned} \tag{3.17}$$

throughout this subsection. Once these conditions are violated for some vectors we have to put the latter equal zero.

As mentioned above, there are some subspaces $V^q_{(s)}$, namely, the submodules $V^q_{(11)}([m_{13}-1,m_{22}-1,m_{32}+1,m_{42}+1])$ and $V^q_{(20)}([m_{13}-1,m_{22}-1,m_{32}+1,m_{42}+1])$, for which neither (3.8) nor (3.9) holds. However, we can find their linear combinations satisfying either (3.8) or (3.9). Indeed, being $U_q[gl(2)\oplus gl(2)]$ - fidirmods with a signature

$$[m\pm 1;\ m_{43}=-m_{13}] := [m_{13}-1,m_{23}-1,m_{33}+1,-m_{13}+1] \tag{3.18}$$

the following linear spaces

$$\begin{aligned} \overset{\text{inv}\ q}{V}_{(1)} &:= \overset{\text{inv}\ q}{V}_{(1)}([m\pm 1;\ m_{43}=-m_{13}]) \\ &= \text{lin.env.}\{\overset{\text{inv}}{(m_{11},m_{31})_{(1)}} \parallel m_{13}-m_{11}-1, m_{11}-m_{23}+1, \\ &\qquad m_{33}-m_{31}+1, m_{31}+m_{13}-1 \in \mathbf{Z}_+\} \end{aligned} \tag{3.19}$$

and

$$V^q_{(1)} \;:=\; V^q_{(1)}([m\pm 1;\; m_{43}=-m_{13}])$$
$$= \text{lin.env.}\{(m_{11},m_{31})_{(1)} \| \; m_{13}-m_{11}-1,\, m_{11}-m_{23}+1,$$
$$m_{33}-m_{31}+1,\, m_{31}+m_{13}-1 \in \mathbf{Z}_+\}, \tag{3.20}$$

belong always to the invariant subspace I^q_k and the compliment to it subspace of W^q, respectively, where

$$\overset{\text{inv}}{(m_{11},m_{31})}_{(1)} \;=\; \frac{[2]}{2}\left\{\left(\frac{[l_{13}-l_{23}+1]}{[l_{13}-l_{23}-1]}\right)^{1/2}(m;\; m_{43}=-m_{13})^{-13-23+33+43}_{(11)}\right.$$
$$\left. +\left(\frac{[l_{13}+l_{33}+4]}{[l_{13}+l_{33}+2]}\right)^{1/2}(m;\; m_{43}=-m_{13})^{-13-23+33+43}_{(20)}\right\} \tag{3.21}$$

and

$$(m_{11},m_{31})_{(1)} \;:=\; (m;\; m_{43}=-m_{13})^{-13-23+33+43}_{(1)}$$
$$= \frac{[2]}{2}\left\{\left(\frac{[l_{13}-l_{23}+1]}{[l_{13}-l_{23}-1]}\right)^{1/2}(m;\; m_{43}=-m_{13})^{-13-23+33+43}_{(11)}\right.$$
$$\left. -\left(\frac{[l_{13}+l_{33}+4]}{[l_{13}+l_{33}+2]}\right)^{1/2}(m;\; m_{43}=-m_{13})^{-13-23+33+43}_{(20)}\right\}. \tag{3.22}$$

In (3.21) and (3.22) and hereafter by $(m;\; m_{kl}=\alpha)^{\pm ij}_{(s)}$ we denote a QGZ basis vector obtained from (m) by replacing the elements m_{kl} and m_{ij} of the latter by α and $m_{ij}\pm 1$, respectively, while $[m;\; m_{kl}=\alpha]^{\pm ij}$ is obtained from the signature $[m]$ by the same way. The index (s) indicates the subspace to which the considered vector belongs.

Proposition 4: Let I^q_1 be a maximal invariant subspace in a class 1 indecomposable induced $U_q[gl(2/2)]$-module $W^q([m;\; m_{43}=-m_{13}])$, then

$$\overset{\text{inv}}{V}{}^q_{(1)}([m\pm 1;\; m_{43}=-m_{13}]) = \overset{\text{inv}}{\boxed{-1,-1,1,1}}_{\,(1)}, \tag{3.23}$$

i.e.,

$$\overset{\text{inv}}{V}{}^q_{(1)}([m\pm 1;\; m_{43}=-m_{13}]) \subset I^q_1$$

and

$$V^q_{(1)}([m\pm 1;\; m_{43}=-m_{13}]) = \overset{\text{noninv}}{\boxed{-1,-1,1,1}}_{\,(1)}, \tag{3.24}$$

i.e.,

$$V^q_{(1)}([m\pm 1;\; m_{43}=-m_{13}]) \cap I^q_1 = V.$$

The module $W^q([m;\; m_{43}=-m_{13}])$ is decomposed exactly in the same way as the classical module $W([m;\; m_{43}=-m_{13}])$ (see $\mathbf{I}^*$)

$$\begin{aligned} W^q([m;\; m_{43}=-m_{13}]) \;=\; & \overset{\text{noninv}}{\boxed{0,0,0,0}}_{(00)} \oplus \overset{\text{noninv}}{\boxed{-1,0,1,0}}_{(10)} \oplus \overset{\text{noninv}}{\boxed{0,-1,1,0}}_{(10)} \\ & \oplus \overset{\text{inv}}{\boxed{-1,0,0,1}}_{(10)} \oplus \overset{\text{noninv}}{\boxed{0,-1,0,1}}_{(10)} \oplus \overset{\text{noninv}}{\boxed{-1,-1,2,0}}_{(20)} \\ & \oplus \overset{\text{inv}}{\boxed{-1,-1,0,2}}_{(20)} \oplus \overset{\text{inv}}{\boxed{-2,0,1,1}}_{(11)} \oplus \overset{\text{noninv}}{\boxed{0,-2,1,1}}_{(11)} \\ & \oplus \overset{\text{inv}}{\boxed{-1,-1,1,1}}_{(1)} \oplus \overset{\text{noninv}}{\boxed{-1,-1,1,1}}_{(1)} \oplus \overset{\text{inv}}{\boxed{-2,-1,2,1}}_{(21)} \\ & \oplus \overset{\text{noninv}}{\boxed{-1,-2,2,1}}_{(21)} \oplus \overset{\text{inv}}{\boxed{-2,-1,1,2}}_{(21)} \oplus \overset{\text{inv}}{\boxed{-1,-2,1,2}}_{(21)} \\ & \oplus \overset{\text{inv}}{\boxed{-2,-2,2,2}}_{(22)}\,. \end{aligned} \tag{3.25}$$

The maximal invariant subspace I^q_k is a sum of all the terms $\overset{\text{inv}}{\boxed{-a,-b,c,d}}$ in (3.25) and represents an irreducible $U_q[gl(2)\oplus gl(2)]$ module with a signature $[m_{13}-1, m_{22}, m_{32}, -m_{13}+1]$, while the compliment to I^q_k subspace W^q_1 is a sum of all the terms remaining there, i.e., of all the terms $\overset{\text{noninv}}{\boxed{-a,-b,c,d}}$.

Following, in the general line, $\mathbf{I}^*$, the proof of the latest proposition (and the proofs of all the similar propositions for the next classes which often represent direct computations) is of a rather technical nature and, therefore, can be omitted.

As in the classical case [15], the transformations of the nontypical modules W^q_1 under the actions of $U_q[gl(2/2)]$ (more precisely, of its Cartan-Chevalley generators) are obtained from (2.23) and (2.24) (or (I.4.45) and (I.4.46)) by

(3.A.1) inserting everywhere there $m_{43}=-m_{13}$ (see (3.1)), (3.A.2) expressing the vectors $(m)^{-13-23+33+43}_{\underset{\rm inv}{(11)}}$ and $(m)^{-13-23+33+43}_{(20)}$ in terms of the vectors $(m_{11}, m_{31})_{(1)}$ and $(m_{11}, m_{31})_{(1)}$ (see (3.21) and (3.22)),

(3.A.3) taking into account Proposition 3, i.e. replacing all the basis vectors from the maximal invariant subspace by zero.

The actions of the even generators in the nontypical module $W_1^q([m;\ m_{43} = -m_{13}])$ which is a direct sum of irreducible $U_q[gl(2)\oplus gl(2)]$ submodules are standard and given in (I.4.43). As far as the matrix elements of the odd generators E_{32} and E_{23} are concerned, in spite of the deformations they have the same structure with the classical ones and are the following:

Transformations under the action of E_{32}:

$$E_{32}(m;\ m_{43} = -m_{13})_{(00)} =$$

$$-\left(\frac{[l_{13}-l_{11}][l_{13}+l_{31}+3]}{[l_{13}-l_{23}][l_{13}+l_{33}+3]}\right)^{1/2}(m;\ m_{43} = -m_{13})^{-13+33+31}_{(10)}$$

$$-\left(\frac{[l_{11}-l_{23}][l_{13}+l_{31}+3]}{[l_{13}-l_{23}][l_{13}+l_{33}+3]}\right)^{1/2}(m;\ m_{43} = -m_{13})^{-23+33+31}_{(10)}$$

$$-\left(\frac{[l_{11}-l_{23}][l_{33}-l_{31}]}{[l_{13}-l_{23}][l_{13}+l_{33}+3]}\right)^{1/2}(m;\ m_{43} = -m_{13})^{-23+43+31}_{(10)},$$

$$E_{32}(m;\ m_{43} = -m_{13})^{-13+33}_{(10)} =$$

$$\left(\frac{[l_{11}-l_{23}][l_{13}+l_{31}+3]}{[l_{13}-l_{23}][l_{13}+l_{33}+4]}\right)^{1/2}(m;\ m_{43} = -m_{13})^{-13-23+33+33+31}_{(20)}$$

$$-\frac{1}{2}\left(\frac{[2][l_{11}-l_{23}][l_{33}-l_{31}+1][l_{13}+l_{33}+3]}{[l_{13}-l_{23}][l_{13}+l_{33}+4][l_{13}+l_{33}+4]}\right)^{1/2}$$

$$\times(m;\ m_{43} = -m_{13})^{-13-23+33+43+31}_{(1)},$$

$$E_{32}(m;\ m_{43} = -m_{13})^{-23+33}_{(10)} =$$

$$-\left(\frac{[l_{13}-l_{11}][l_{13}+l_{31}+3]}{[l_{13}-l_{23}][l_{13}+l_{33}+4]}\right)^{1/2}(m;\ m_{43} = -m_{13})^{-13-23+33+33+31}_{(20)}$$

$$-\left(\frac{[l_{11}-l_{23}+1][l_{33}-l_{31}+1]}{[l_{13}-l_{23}+1][l_{13}+l_{33}+3]}\right)^{1/2}(m;\ m_{43} = -m_{13})^{-23-23+33+43+31}_{(11)}$$

$$+\frac{[l_{23}+l_{33}+3]}{[l_{13}+l_{33}+4][l_{13}-l_{23}+1]}\left(\frac{[l_{13}-l_{11}][l_{33}-l_{31}+1]}{[2][l_{13}-l_{23}][l_{13}+l_{33}+3]}\right)^{1/2}$$

$$\times(m;\ m_{43} = -m_{13})^{-13-23+33+43+31}_{(1)},$$

$$E_{32}(m;\ m_{43} = -m_{13})^{-23+43}_{(10)} =$$

$$\left(\frac{[l_{11}-l_{23}+1][l_{13}+l_{31}+2]}{[l_{13}-l_{23}+1][l_{13}+l_{33}+3]}\right)^{1/2}(m;\ m_{43} = -m_{13})^{-23-23+33+43+31}_{(11)}$$

$$+\frac{1}{[l_{13}-l_{23}+1]}\left(\frac{[l_{13}-l_{11}][l_{13}-l_{23}][l_{13}+l_{31}+2]}{[2][l_{13}+l_{33}+3]}\right)^{1/2}$$

$$\times(m;\ m_{43} = -m_{13})^{-13-23+33+43+31}_{(1)},$$

$$E_{32}(m;\ m_{43} = -m_{13})^{-13-23+33+33}_{(20)} =$$

$$-\left(\frac{[l_{11}-l_{23}+1][l_{33}-l_{31}+2]}{[l_{13}-l_{23}][l_{13}+l_{33}+4]}\right)^{1/2}(m;\ m_{43} = -m_{13})^{-13-23-23+33+33+43+31}_{(21)},$$

$$E_{32}(m;\ m_{43} = -m_{13})^{-13-23+33+43}_{(1)} =$$

$$-\left(\frac{[2][l_{11}-l_{23}+1][l_{13}+l_{31}+2]}{[l_{13}-l_{23}][l_{13}+l_{33}+3]}\right)^{1/2}(m;\ m_{43} = -m_{13})^{-13-23-23+33+33+43+31}_{(21)},$$

$$E_{32}(m;\ m_{43} = -m_{13})^{-23-23+33+43}_{(11)} =$$

$$\left(\frac{[l_{13}-l_{11}][l_{13}+l_{31}+2]}{[l_{13}-l_{23}+1][l_{13}+l_{33}+3]}\right)^{1/2}(m;\ m_{43} = -m_{13})^{-13-23-23+33+33+43+31}_{(21)},$$

$$E_{32}(m;\ m_{43} = -m_{13})^{-13-23-23+33+33+43}_{(21)} = 0. \tag{3.26}$$

Transformations under the action of E_{23}:

$$E_{23}(m;\ m_{43} = -m_{13})_{(21)}^{-13-23-23+33+33+43} =$$

$$\left(\frac{[l_{13}-l_{11}][l_{13}+l_{31}+1][l_{13}+l_{33}+3]}{[l_{13}-l_{23}+1]}\right)^{1/2}(m;\ m_{43} = -m_{13})_{(11)}^{-23-23++33+43-31}$$

$$+\left(\frac{[l_{11}-l_{23}+1][l_{33}-l_{31}+3][l_{13}-l_{23}]}{[l_{13}+l_{33}+4]}\right)^{1/2}(m;\ m_{43} = -m_{13})_{(20)}^{-13-23+33+33-31}$$

$$-[l_{23}+l_{33}+3]\frac{([2][l_{11}-l_{23}+1][l_{13}+l_{31}+1][l_{13}-l_{23}][l_{13}+l_{33}+3])^{1/2}}{[2][l_{13}-l_{23}+1][l_{13}+l_{33}+4]}$$

$$\times(m;\ m_{43} = -m_{13})_{(1)}^{-13-23+33+43-31},$$

$$E_{23}(m;\ m_{43} = -m_{13})_{(11)}^{-23-23+33+43} =$$

$$[l_{13}-l_{23}]\left(\frac{[l_{11}-l_{23}+1][l_{33}-l_{31}+2]}{[l_{13}-l_{23}+1][l_{13}+l_{33}+3]}\right)^{1/2}(m;\ m_{43} = -m_{13})_{(10)}^{-23++33-31}$$

$$+[l_{23}+l_{33}+3]\left(\frac{[l_{11}-l_{23}+1][l_{13}+l_{31}+1]}{[l_{13}-l_{23}+1][l_{13}+l_{23}+3]}\right)^{1/2}(m;\ m_{43} = -m_{13})_{(10)}^{-23+43-31},$$

$$E_{23}(m;\ m_{43} = -m_{13})_{(1)}^{-13-23+33+43} =$$

$$-\left(\frac{[2][l_{13}-l_{11}][l_{33}-l_{31}+2]}{[l_{13}-l_{23}][l_{13}+l_{33}+3]}\right)^{1/2}(m;\ m_{43} = -m_{13})_{(10)}^{-23+33-31}$$

$$+2[l_{13}-l_{23}+1]\left(\frac{[l_{11}-l_{23}][l_{33}-l_{31}+2]}{[2][l_{13}+l_{33}+3][l_{13}-l_{23}]}\right)^{1/2}(m;\ m_{43} = -m_{13})_{(10)}^{-13+33-31}$$

$$+2[l_{13}+l_{33}+4]\left(\frac{[l_{13}-l_{11}][l_{13}+l_{31}+1]}{[2][l_{13}-l_{23}][l_{13}+l_{33}+3]}\right)^{1/2}(m;\ m_{43} = -m_{13})_{(10)}^{-23+43-31},$$

$$E_{23}(m;\ m_{43} = -m_{13})_{(20)}^{-13-23+33+33} =$$

$$-[l_{13}+l_{33}+3]\left(\frac{[l_{13}-l_{11}][l_{13}+l_{31}+2]}{[l_{13}-l_{23}][l_{13}+l_{33}+4]}\right)^{1/2}(m;\ m_{43} = -m_{13})_{(10)}^{-23+33-31}$$

$$+[l_{23}+l_{33}+3]\left(\frac{[l_{11}-l_{23}][l_{13}+l_{31}+2]}{[l_{13}-l_{23}][l_{13}+l_{33}+4]}\right)^{1/2}(m;\ m_{43} = -m_{13})_{(10)}^{-13+33-31},$$

$$E_{23}(m;\ m_{43} = -m_{13})_{(10)}^{-23+43} =$$

$$\left(\frac{[l_{13}-l_{23}][l_{11}-l_{23}][l_{33}-l_{31}+1]}{[l_{13}+l_{33}+3]}\right)^{1/2}(m;\ m_{43} = -m_{13})_{(00)}^{-31},$$

$$E_{23}(m;\ m_{43} = -m_{13})_{(10)}^{-23+33} =$$

$$-[l_{23}+l_{33}+3]\left(\frac{[l_{11}-l_{23}][l_{13}+l_{31}+2]}{[l_{13}-l_{23}][l_{13}+l_{33}+3]}\right)^{1/2}(m;\ m_{43} = -m_{13})_{(00)}^{-31},$$

$$E_{23}(m;\ m_{43} = -m_{13})_{(10)}^{-23+33} =$$

$$-\left(\frac{[l_{13}-l_{11}][l_{13}+l_{33}+3][l_{13}+l_{31}+2]}{[l_{13}-l_{23}]}\right)^{1/2}(m;\ m_{43} = -m_{13})_{(00)}^{-31},$$

$$E_{23}(m;\ m_{43} = -m_{13})_{(00)} = 0. \tag{3.27}$$

It is not difficult to show that at the classical limit when $q = 1$ the matrix elements (3.26) and (3.27) take respectively the values of the matrix elements of the $gl(2/2)$-generators e_{32} and e_{23} given in **I*** (from (I*.82) to (I*.97)).

As far as Propositions 8 and 9 in **I*** (as well as all the next Propositions there) are concerned they can also be extended to the present case of the q-deformations, namely, if we put $m_{13} = m_{23}$ and $m_{33} > -m_{13}$ or $m_{13} > m_{23}$ and $m_{33} = -m_{13}$ the class 1 indecomposable module $W^q([m;\ m_{43} = -m_{13}])$ is decomposed into irreducible $U_q[gl(2) \oplus gl(2)]$ modules in the same way as its classical analogue $W([m;\ m_{43} = -m_{13}])$ is decomposed into irreducible $gl(2) \oplus gl(2)$ modules in (I*.99) or (I*.103), respectively. Therefore, these decompositions and the decompositions schemes for all the remaining indecomposable and nontypical modules are not necessary to be exposed again here unlike the matrix elements which, although have similar structures with their classical analogues, are not the same. In order to obtain the transformations of a nontypical module of any class we simply apply the procedure (3.A.1)–(3.A.3) to the corresponding case. The direct computations show that they can be obtained immediately from their classical analogues in $\mathbf{I}^*$ if we replace the brackets "( )" by the notation for the quantum deformation "[ ]" and then remove the modulus after using the rule (2.22), i.e., the rule (I.4.44). As examples, we gave explicitly only the matrix elements for the class 1 nontypical representation (see (3.26) and (3.27)). From now on, the matrix elements for the nontypical representations of all the next classes will be placed in the Appendix , since they are too cumbersome.

B. The class 2 nontypical representations

This class corresponds to the case (3.2), namely,

$$\begin{aligned} \text{class 2} \quad & l_{13} + l_{43} + 3 \neq 0 \quad \Leftrightarrow \quad m_{13} + m_{43} \neq 0, \\ & l_{23} + l_{33} + 3 = 0 \quad \Leftrightarrow \quad m_{23} + m_{33} = 0; \end{aligned} \tag{3.2}$$

The signature of the induced modules is $[m;\ m_{33} = -m_{23}] := [m_{13}, m_{23}, -m_{23}, m_{43}]$:

$$W([m;\ m_{33} = -m_{23}]) = W([m_{13}, m_{23}, -m_{23}, m_{43}]) \tag{3.28}$$

Proposition 5: The vectors

$$\begin{aligned} \overset{\text{inv}}{(m_{11}, m_{31})}_{(2)} = \frac{[2]}{2} \Bigg\{ & \left(\frac{[l_{13} - l_{23} - 1]}{[l_{13} - l_{23} + 1]} \right)^{1/2} (m;\ m_{33} = -m_{23})^{-13-23+33+43}_{(11)} \\ & + \left(\frac{[l_{23} + l_{43} + 4]}{[l_{23} + l_{43} + 2]} \right)^{1/2} (m;\ m_{33} = -m_{23})^{-13-23+33+43}_{(20)} \Bigg\} \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} (m_{11}, m_{31})_{(2)} & := (m;\ m_{33} = -m_{23})^{-13-23+33+43}_{(2)} \\ & = \frac{[2]}{2} \Bigg\{ \left(\frac{[l_{13} - l_{23} - 1]}{[l_{13} - l_{23} + 1]} \right)^{1/2} (m;\ m_{33} = -m_{23})^{-13-23+33+43}_{(11)} \\ & \quad - \left(\frac{[l_{23} + l_{43} + 4]}{[l_{23} + l_{43} + 2]} \right)^{1/2} (m;\ m_{33} = -m_{23})^{-13-23+33+43}_{(20)} \Bigg\} \end{aligned} \tag{3.30}$$

form respectively linear spaces

$$\begin{aligned} \overset{\text{inv}}{V}{}^{q}_{(2)} & := \overset{\text{inv}}{V}{}^{q}_{(2)}([m \pm 1;\ m_{33} = -m_{23}]) \\ & = \text{lin.env.}\{\overset{\text{inv}}{(m_{11}, m_{31})}_{(2)} \parallel m_{13} - m_{11} - 1, m_{11} - m_{23} + 1, \\ & \quad -m_{23} - m_{31} + 1, m_{31} - m_{43} - 1 \in \mathbf{Z}_+\} \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} V^q_{(2)} & := V^q_{(2)}([m \pm 1;\ m_{33} = -m_{23}]) \\ & = \text{lin.env.}\{(m_{11}, m_{31})_{(2)} \parallel m_{13} - m_{11} - 1, m_{11} - m_{23} + 1, \\ & \quad -m_{23} - m_{31} + 1, m_{31} - m_{43} - 1 \in \mathbf{Z}_+\}, \end{aligned} \tag{3.32}$$

which being irreducible $U_q[gl(2) \oplus gl(2)]$ modules satisfy

$$\overset{\text{inv}}{V}{}^{q}_{(2)}([m \pm 1;\ m_{33} = -m_{23}]) = \overset{\text{inv}}{\boxed{-1,-1,1,1}}_{(2)}, \tag{3.33}$$

i.e.,

$$\overset{\text{inv}}{V}{}^{q}_{(2)}([m \pm 1;\ m_{33} = -m_{23}]) \subset I^q_2$$

and

$$V^q_{(2)}([m \pm 1;\ m_{33} = -m_{23}]) = \overset{\text{noninv}}{\boxed{-1,-1,1,1}}_{(2)}, \tag{3.34}$$

i.e.,

$$V^q_{(2)}([m \pm 1;\ m_{33} = -m_{23}]) \cap I^q_2 = \vee.$$

C. The class 3 nontypical representations

This class corresponds to the case (3.3)

$$\text{class 3} \quad l_{23} + l_{43} + 3 = 0 \quad \Leftrightarrow \quad m_{23} + m_{43} - 1 = 0; \tag{3.3}$$

and the signature $[m;\ m_{43} = -m_{23} + 1] := [m_{13}, m_{23}, m_{33}, -m_{23} + 1]$:

$$W([m;\ m_{43} = -m_{23} + 1]) = W([m_{13}, m_{23}, m_{33}, -m_{23} + 1]) \tag{3.35}$$

Proposition 6: The vectors

$$\overset{\text{inv}}{(m_{11}, m_{31})}_{(3)} = \frac{[2]}{2}\left\{\left(\frac{[l_{13} - l_{23} - 1]}{[l_{13} - l_{23} + 1]}\right)^{1/2} (m;\ m_{43} = -m_{23} + 1)_{(11)}^{-13-23+33+43}\right.$$

$$\left. - \left(\frac{[l_{23} + l_{33} + 4]}{[l_{23} + l_{33} + 2]}\right)^{1/2} (m;\ m_{43} = -m_{23} + 1)_{(20)}^{-13-23+33+43}\right\} \qquad (3.36)$$

and

$$(m_{11}, m_{31})_{(3)} := (m;\ m_{43} = -m_{23} + 1)_{(3)}^{-13-23+33+43}$$

$$= \frac{[2]}{2}\left\{\left(\frac{[l_{13} - l_{23} - 1]}{[l_{13} - l_{23} + 1]}\right)^{1/2} (m;\ m_{43} = -m_{23} + 1)_{(11)}^{-13-23+33+43}\right.$$

$$\left. + \left(\frac{[l_{23} + l_{33} + 4]}{[l_{23} + l_{33} + 2]}\right)^{1/2} (m;\ m_{43} = -m_{23} + 1)_{(20)}^{-13-23+33+43}\right\} \qquad (3.37)$$

form respectively linear spaces

$$\overset{\text{inv}}{V}{}^{q}_{(3)} := \overset{\text{inv}}{V}{}^{q}_{(3)}([m \pm 1;\ m_{43} = -m_{23} + 1])$$

$$= \text{lin.env.}\{\overset{\text{inv}}{(m_{11}, m_{31})}_{(3)} \parallel m_{13} - m_{11} - 1, m_{11} - m_{23} + 1,$$

$$m_{33} - m_{31} + 1, m_{31} + m_{23} - 2 \in \mathbf{Z}_+\} \qquad (3.38)$$

and

$$V^{q}_{(3)} := V^{q}_{(3)}([m \pm 1;\ m_{43} = -m_{23} + 1])$$

$$= \text{lin.env.}\{(m_{11}, m_{31})_{(3)} \parallel m_{13} - m_{11} - 1, m_{11} - m_{23} + 1,$$

$$m_{33} - m_{31} + 1, m_{31} + m_{23} - 2 \in \mathbf{Z}_+\}, \qquad (3.39)$$

which being irreducible $U_q[gl(2) \oplus gl(2)]$ modules satisfy

$$\overset{\text{inv}}{V}{}^{q}_{(3)}([m \pm 1;\ m_{43} = -m_{23} + 1]) = \overset{\text{inv}}{\boxed{-1,-1,1,1}}_{\ (3)}, \qquad (3.40)$$

i.e.,

$$\overset{\text{inv}}{V}{}^{q}_{(3)}([m \pm 1;\ m_{43} = -m_{23} + 1]) \subset I_3^q$$

and

$$V^{q}_{(3)}([m \pm 1;\ m_{43} = -m_{23} + 1]) = \overset{\text{noninv}}{\boxed{-1,-1,1,1}}_{\ (3)}, \qquad (3.41)$$

i.e.,

$$V^{q}_{(3)}([m \pm 1;\ m_{43} = -m_{23} + 1]) \cap I_3^q = \vee.$$

D. The class 4 nontypical representations

This class corresponds to the case (3.4)

$$\text{class 4} \quad l_{13} + l_{33} + 3 = 0 \quad \Leftrightarrow \quad m_{13} + m_{33} + 1 = 0; \qquad (3.4)$$

and the signature $[m;\ m_{33} = -m_{13} - 1] := [m_{13}, m_{23}, -m_{13} - 1, m_{43}]$:

$$W([m;\ m_{13} = -m_{13} - 1]) = W([m_{13}, m_{23}, -m_{13} - 1, m_{43}]) \qquad (3.42)$$

Proposition 7: The vectors

$$\overset{\text{inv}}{(m_{11}, m_{31})}_{(4)} = \frac{[2]}{2}\left\{-\left(\frac{[l_{13} - l_{23} + 1]}{[l_{13} - l_{23} - 1]}\right)^{1/2} (m;\ m_{33} = -m_{13} - 1)_{(11)}^{-13-23+33+43}\right.$$

$$\left. + \left(\frac{[l_{13} + l_{43} + 4]}{[l_{13} + l_{43} + 2]}\right)^{1/2} (m;\ m_{33} = -m_{13} - 1)_{(20)}^{-13-23+33+43}\right\} \qquad (3.43)$$

and

$$(m_{11}, m_{31})_{(4)} := (m;\ m_{33} = -m_{13} - 1)_{(4)}^{-13-23+33+43}$$

$$= \frac{[2]}{2}\left\{-\left(\frac{[l_{13} - l_{23} + 1]}{[l_{13} - l_{23} - 1]}\right)^{1/2} (m;\ m_{33} = -m_{13} - 1)_{(11)}^{-13-23+33+43}\right.$$

$$\left. - \left(\frac{[l_{13} + l_{43} + 4]}{[l_{13} + l_{43} + 2]}\right)^{1/2} (m;\ m_{33} = -m_{13} - 1)_{(20)}^{-13-23+33+43}\right\} \qquad (3.44)$$

form respectively linear spaces

$$\overset{\text{inv}}{V}{}^{q}_{(4)} := \overset{\text{inv}}{V}{}^{q}_{(4)}([m \pm 1;\ m_{33} = -m_{13} - 1])$$
$$= \text{lin.env.}\{\overset{\text{inv}}{(m_{11}, m_{31})}_{(4)} \parallel m_{13} - m_{11} - 1, m_{11} - m_{23} + 1,$$
$$-m_{13} - m_{31}, m_{31} - m_{43} - 1 \in \mathbf{Z}_+\} \qquad (3.45)$$

and

$$V^q_{(4)} := V^q_{(4)}([m \pm 1;\ m_{33} = -m_{13} - 1])$$
$$= \text{lin.env.}\{(m_{11}, m_{31})_{(4)} \| m_{13} - m_{11} - 1, m_{11} - m_{23} + 1,$$
$$-m_{13} - m_{31}, m_{31} - m_{43} - 1 \in \mathbf{Z}_+\}, \qquad (3.46)$$

which being irreducible $U_q[gl(2) \oplus gl(2)]$ modules satisfy

$$\overset{\text{inv}}{V}{}^{q}_{(4)}([m \pm 1;\ m_{33} = -m_{13} - 1]) = \overset{\text{inv}}{\boxed{-1,-1,1,1}}_{(4)}, \qquad (3.47)$$

i.e.,

$$\overset{\text{inv}}{V}{}^{q}_{(4)}([m \pm 1;\ m_{33} = -m_{13} - 1]) \subset I^q_4$$

and

$$V^q_{(4)}([m \pm 1;\ m_{33} = -m_{13} - 1]) = \overset{\text{noninv}}{\boxed{-1,-1,1,1}}_{(4)}, \qquad (3.48)$$

i.e.,

$$V^q_{(4)}([m \pm 1;\ m_{33} = -m_{13} - 1]) \cap I^q_4 = \vee.$$

E. The class 5 nontypical representations

This class corresponds to the case (3.5)

$$\text{class 5} \quad l_{13} + l_{43} + 3 = 0 \quad \Leftrightarrow \quad m_{13} + m_{43} = 0,$$
$$l_{23} + l_{33} + 3 = 0 \quad \Leftrightarrow \quad m_{23} + m_{33} = 0. \qquad (3.5)$$

and the signature $[m;\ m_{33} = -m_{23}, m_{43} = -m_{13}] := [m_{13}, m_{23}, -m_{23}, -m_{13}]$:

$$W([m;\ m_{33} = -m_{23}, m_{43} = -m_{13}]) = W([m_{13}, m_{23}, -m_{23}, -m_{13}]) \qquad (3.49)$$

Proposition 8: The vectors

$$\overset{\text{inv}}{(m_{11}, m_{31})}_{(5)} = \frac{[2]}{2}\left(\frac{[l_{13} - l_{23} - 1]}{[l_{13} - l_{23} + 1]}\right)^{1/2}$$
$$\times \left\{(m;\ m_{33} = -m_{23}, m_{43} = -m_{13})^{-13-23+33+43}_{(11)}\right.$$
$$\left. + (m;\ m_{33} = -m_{23}, m_{43} = -m_{13})^{-13-23+33+43}_{(20)}\right\}$$
$$(3.50)$$

and

$$(m_{11}, m_{31})_{(5)} := (m;\ m_{33} = -m_{23}, m_{43} = -m_{13})^{-13-23+33+43}_{(5)}$$
$$= \frac{[2]}{2}\left(\frac{[l_{13} - l_{23} - 1]}{[l_{13} - l_{23} + 1]}\right)^{1/2}$$
$$\times \left\{(m;\ m_{33} = -m_{23}, m_{43} = -m_{13})^{-13-23+33+43}_{(11)}\right.$$
$$\left. - (m;\ m_{33} = -m_{23}, m_{43} = -m_{13})^{-13-23+33+43}_{(20)}\right\}$$
$$(3.51)$$

form respectively linear spaces

$$\overset{\text{inv}}{V}{}^{q}_{(5)} := \overset{\text{inv}}{V}{}^{q}_{(5)}([m \pm 1;\ m_{33} = -m_{23}, m_{43} = -m_{13}])$$
$$= \text{lin.env.}\{\overset{\text{inv}}{(m_{11}, m_{31})}_{(5)} \parallel m_{13} - m_{11} - 1, m_{11} - m_{23} + 1,$$
$$-m_{23} - m_{31} + 1, m_{31} + m_{13} - 1 \in \mathbf{Z}_+\} \qquad (3.52)$$

and

$$V^q_{(5)} := V^q_{(5)}([m \pm 1;\ m_{33} = -m_{23}, m_{43} = -m_{13}])$$
$$= \text{lin.env.}\{(m_{11}, m_{31})_{(5)} \| m_{13} - m_{11} - 1, m_{11} - m_{23} + 1,$$
$$-m_{23} - m_{31} + 1, m_{31} + m_{13} - 1 \in \mathbf{Z}_+\}, \qquad (3.53)$$

which being irreducible $U_q[gl(2) \oplus gl(2)]$ modules satisfy

$$\overset{\text{inv}}{V}{}^{q}_{(5)}([m \pm 1;\ m_{33} = -m_{23}, m_{43} = -m_{13}]) = \overset{\text{inv}}{\boxed{-1,-1,1,1}}_{(5)}, \qquad (3.54)$$

i.e.,

$$\overset{\text{inv } q}{V}_{(5)}([m \pm 1;\ m_{33} = -m_{23}, m_{43} = -m_{13}]) \subset I_5^q$$

and

$$V_{(5)}^q([m \pm 1;\ m_{33} = -m_{23}, m_{43} = -m_{13}]) = \overset{\text{noninv}}{\boxed{-1,-1,1,1}}_{(5)}, \qquad (3.55)$$

i.e.,

$$V_{(5)}^q([m \pm 1;\ m_{33} = -m_{23}, m_{43} = -m_{13}]) \cap I_5^q = \vee.$$

The decomposition of $W^q([m;\ m_{33} = -m_{23}, m_{43} = -m_{13}])$ is the same as given in ($\mathbf{I}^*$.227). Unlike the maximal invariant subspaces I_k^q , $k = 1,2,3,4$, of the previous classes the maximal invariant subspace I_5^q is indecomposable and decomposed into three invariant subspaces $I_5^{q,0}$, $I_5^{q,1}$, $I_5^{q,2}$, where $I_5^{q,0}$ is an irreducible and nontypical module with a signature

$$[m \pm 1;\ m_{33} = -m_{23}, m_{43} = -m_{13}] \qquad (3.56)$$

while every of $I_5^{q,1}$ and $I_5^{q,2}$ is indecomposable and contains $I_5^{q,0}$ as a maximal invariant subspace. The factor spaces $I_5^{q,1}/I_5^{q,0}$ and $I_5^{q,2}/I_5^{q,0}$ in turn carry nontypical representations of $U_q[gl(2/2)]$ with signatures

$$[m;\ m_{33} = -m_{23}, m_{43} = -m_{13}]^{-23+33} \qquad (3.57)$$

and

$$[m;\ m_{33} = -m_{23}, m_{43} = -m_{13}]^{-13+43}, \qquad (3.58)$$

respectively.

IV. FINITE–DIMENSIONAL REPRESENTATIONS OF $\mathbf{U_q[gl(2/2)]}$

As in $\mathbf{I}^*$ we denote by $\Im$ the class of finite–dimensional irreducible $U_q[gl(2/2)]$ modules $W^q([m_{13}, m_{23}, m_{33}, m_{43}])$ determined in $\mathbf{I}$ and in this paper. The modules $W^q \in \Im$ are characterized by the signatures (1.6) which represent ordered sets of all possible complex numbers m_{13}, m_{23}, m_{33} and m_{43} satisfying the conditions (1.3)

$$m_{13} - m_{23},\ m_{33} - m_{43} \in \mathbf{Z}_+. \qquad (4.1)$$

The typical modules are those for which (1.9) holds, while the indecomposable modules carrying nontypical representations of $U_q[gl(2/2)]$, are those for which one of the conditions (3.1)–(3.5) is fulfilled. The transformations of all the nontypical modules under the actions of $U_q[gl(2/2)]$ are completely defined through the actions (I.4.43) of the even generators and the actions of the odd Chevalley generators E_{23} and E_{32} written down in (3.26) and (3.27) for the class 1 and in the Appendix for the other classes. The transformations of all the typical modules have already given in $\mathbf{I}$ by the formulae (I.4.43)-(I.4.46).

Proposition 9: If W^q is a $U_q[gl(2/2)]$ fidirmod, then $W^q \in \Im$.

We shall sketch the proof. Let us denote by $\{x_{\Lambda_k}\}$ the set of all the eigenvectors x_{Λ_k}, $k \in \mathbf{Z}_+$, of the Cartan generators E_{ii}, $i = 1,2,3,4$, such that they are annihilated by the generators E_{12} and E_{34}. Acting on every $x_{\Lambda_k} \in \{x_{\Lambda_k}\}$ by all possible elements of $U_q^0 := U_q[gl(2) \oplus gl(2)]$ (cf. (I.4.15)) we obtain a set $\{x^k\}$

$$\{x^k\} = U_q^0 x_{\Lambda_k} \qquad (4.2)$$

which is a finite–dimensional irreducible $U_q[gl(2) \oplus gl(2)]$ module and spanned on all basis vectors x^k determined up to multiplicative constants in the forms

$$x^k = c(x^k)(E_{21})^{n_1}(E_{43})^{n_2} x_{\Lambda_k}\ ;\quad n_1,\ n_2 \in \mathbf{Z}_+. \qquad (4.3)$$

Since q is generic (i.e., there are not cyclic representations) and W^q is a finite–dimensional irreducible $U_q[gl(2/2)]$ module it can be shown that the union $\{x\}$ of all the sets ($U_q[gl(2/2)]$ fidirmods) $\{x^k\}$ cover W^q whole and every of the odd generators (e.g., E_{23}) intertwining the sets $\{x^k\}$ has to vanish in some $\{x^0\}$. Hence, there exists always in W^q an eigenvector x_Λ^0 of the Cartan generators E_{ii}, $i = 1,2,3,4$,

$$E_{ii} x_\Lambda^0 = m_{i3} x_\Lambda^0 \qquad (4.4)$$

such that it is annihilated by the generators E_{12} and E_{34}

$$E_{12} x_\Lambda^0 = E_{34} x_\Lambda^0 = 0 \qquad (4.5)$$

and simultaneously by the generator E_{23}

$$E_{23}x_{\Lambda}^0 = 0 \tag{4.6}$$

Therefore, x_{Λ}^0 is a highest weight vector of the highest weight $[m] = [m_{13}, m_{23}, m_{33}, m_{43}]$ with respect to both $U_q[gl(2) \oplus gl(2)]$ and $U_q[gl(2/2)]$. In other words, the $U_q[gl(2) \oplus gl(2)]$ fidirmod

$$V^q_{(00)} := \{x^0\} = U^0_q x^0_{\Lambda} \tag{4.7}$$

and the $U_q[gl(2/2)]$ fidirmod W^q have one and the same signature $[m]$. The module $V^q_{(00)}$, however, is a tensor product

$$V^q_{(00)} = V^q_{(00),l} \otimes V^q_{(00),r} \tag{4.8}$$

between a $U_q[gl(2)_l]$ fidirmod $V^q_{(00),l}$ and a $U_q[gl(2)_r]$ fidirmod $V^q_{(00),r}$ which in turn are labeled by the signatures $[m]_l = [m_{13}, m_{23}]$ and $[m]_r = [m_{33}, m_{43}]$, respectively. As is well known that finite-dimensional representations of $U_q[gl(2)]$ are q-deformations of the finite-dimensional representations of the classical $gl(2)$ and the GZ bases are invariant under the q-deformations. Therefore, the classical conditions

$$m_{13} - m_{23} \in \mathbf{Z}_+, \quad m_{33} - m_{43} \in \mathbf{Z}_+$$

imposed on the complex numbers m_{13}, m_{23}, m_{33}, m_{43} are still valid in the present case and coincide with (4.1), i.e., $W^q \in \Im$.

We can conclude that a $U_q[gl(2/2)]$ fidirmod W^q is also a $U_q[sl(2/2)]$ fidirmod and vice versa. The quantum superalgebra $U_q[sl(2/2)]$ is generated completely by the Cartan generators

$$h_1 = E_{11} - E_{22}, \ h_2 = E_{22} + E_{33}, \ h_3 = E_{33} - E_{44} \tag{4.9}$$

defined in (2.1) (or (I.3.1)) and by the other $U_q[gl(2/2)]$–Chevalley generators E_{ij}, $i \neq j = 1, 2, 3, 4$, satisfying the defining relations (2.1)- (2.4) (or (I.3.1)-(I.3.4). A signature (highest weight) of a $U_q[sl(2/2)]$ fidirmod is defined again as an ordered set of eigenvalues of the Cartan generators h_i, $i = 1, 2, 3$, on a highest weight vector. Since $U_q[gl(2/2)]$ and $U_q[sl(2/2)]$ have ones and the same Chevalley (nondiagonal)

generators a $U_q[gl(2/2)]$–highest weight vector is also a $U_q[sl(2/2)]$–highest weight vector and vice versa. The $U_q[gl(2/2)]$ fidirmod $W^q[m_{13}, m_{23}, m_{33}, m_{43}]$ considered as a $U_q[sl(2/2)]$ fidirmod is labeled by the numbers

$$\alpha_1 = m_{13} - m_{23}, \quad \alpha_2 = m_{23} + m_{33}, \quad \alpha_3 = m_{33} - m_{43}, \tag{4.10}$$

where α_i, $i = 1, 2, 3$, are eigenvalues of h_i. Therefore, any $U_q[sl(2/2)]$–signature (i.e., the signature of a module W^q considered as a $U_q[sl(2/2)]$ fidirmod) is uniquely determined from some $U_q[gl(2/2)]$–signature. If m_{13}, m_{23}, m_{33}, m_{43} labeling $U_q[gl(2/2)]$ fidirmods take all values consistent with (4.1) then the triple $(\alpha_1, \alpha_2, \alpha_3)$ runs over all labels (signatures) for the $U_q[sl(2/2)]$ fidirmods. Vice versa, a $U_q[sl(2/2)]$ fidirmod can be extended to (inequivalent copies of) $U_q[gl(2/2)]$ fidirmods with signatures determined from the $U_q[sl(2/2)]$–signature and by setting $E_{44}x^0_{\Lambda} = \alpha x^0_{\Lambda}$, ($\alpha \equiv m_{43}$), on the highest weight vector x^0_{Λ}. We have shown that the following Proposition holds

Proposition 10: The class $\Im$ contains all finite–dimensional irreducible $U_q[sl(2/2)]$ modules.

The next step one can make is to consider finite–dimensional irreducible representations of the quantum superalgebra $U_q[A(1/1)]$.

V. CONCLUSION

We have completed our programme on an explicit construction at generic deformation parameter q of all (typical and nontypical) finite–dimensional representations of the quantum Lie superalgebra $U_q[gl(2/2)]$. The construction method proposed in **I** allowed us to construct the induced modules of (certainly, not only) $U_q[gl(2/2)]$ with basis systems convenient for finding all possible finite- dimensional irreducible modules and representations of this quantum superalgebra. The previous paper I is devoted to the general construction procedure with an accent on the typical representations, while in this paper the indecomposable representations were considered and classified. Here, the nontypical representations of all classes as irreducible representations extracted from the indecomposable ones were constructed explicitly. As we can see, all the results obtained in **I** and in the present paper coincide at $q = 1$ with the classical ones in Refs.14 and 15. It turns out that although the quantum deformation gives rise to some specific difficulties and makes the present construction more cumbersome the resemblance between the quantum structures and the non-deformation ones [14,15] is remarkable. As far as the case of q being roots of unity is concerned it is a subject of a later publication. In the latter case, however, the structures of $U_q[gl(2/2)]$-modules are drastically different in comparison with the structures of $gl(2/2)$-modules [14,15].

Putting the results of Refs.1, 14, 15 and the present paper altogether we arrive at the following conclusion

Proposition 11: The finite-dimensional representations of the quantum superalgebra $U_q[gl(2/2)]$ are quantum deformations (q-deformations) of the finite-dimensional representations of the superalgebra $gl(2/2)$.

Certainly, our construction procedure proposed in **I** and the present paper for $U_q[gl(2/2)]$ is applicable to higher rank $U_q[gl(m/n)]$ and other quantum (super) groups, for which Proposition 11 may remain valid. Then our approach may have some advantage as it is worthy to mention that the theory of representations and especially of the nontypical ones is far from being complete even for the nondeformed superalgebras. In particular, the dimensions of the nontypical representations are unknown unless the ones for $sl(1/n)$ computed recently in Ref.19. Based on the generalizations of the concept of the GZ basis (see, for example, Refs.20 and 21) the matrix elements of all nontypical representations were computed only for $sl(1/n)$ and $gl(1/n)$ (see Refs. 22). Later, the essentially typical representations of $gl(m/n)$ were also constructed [23]. So far, however, the GZ basis concept was not not defined and presumably cannot be defined for nontypical $gl(m/n)$-modules with $m, n \geq 2$. This was the reason to try to describe the nontypical modules in terms of the basis of

the even subalgebras. This approach was developed so far for $gl(2/2)$ in Ref.14 and for $U_q[gl(2/2)]$ in **I** and it turned appropriate for explicit descriptions of all nontypical modules of $gl(2/2)$ (see Ref.15) and $U_q[gl(2/2)]$, respectively. The approach in **I**, unlike some earlier approaches [14,20], avoids, however, the use of the Clebsch–Gordan coefficients which are not always known for higher rank (quantum and classical) algebras. Other extensions were made in Ref.3 for all finite-dimensional representations of $U_q[gl(1/n)]$ and in Ref.6 for a class of finite-dimensional representations of $U_q[gl(m/n)]$. To our best knowledge, we give for the first time in **I** and the present papers, respectively, the explicit expressions for all typical representations and all nontypical representations of a quantum superalgebra $U_q[gl(m/n)]$ with $m, n \geq 2$.

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APPENDIX: Nontypical repersentation matrix elements of E_{32} and E_{23}

A.1. Class 1

All the matrix elements of E_{32} and E_{23} for this class nontypical representations have already given in (3.26) and (3.27), respectively.

A.2. Class 2

The matrix elements of E_{32}:

$$E_{32}(m;\ m_{33}=-m_{23})_{(00)} =$$

$$-\left(\frac{[l_{13}-l_{11}][l_{31}-l_{43}]}{[l_{13}-l_{23}][l_{23}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{23})^{-13+33+31}_{(10)}$$

$$-\left(\frac{[l_{13}-l_{11}][l_{23}+l_{31}+3]}{[l_{13}-l_{23}][l_{23}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{23})^{-13+31+43}_{(10)}$$

$$-\left(\frac{[l_{11}-l_{23}][l_{23}+l_{31}+3]}{[l_{13}-l_{23}][l_{23}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{23})^{-23+43+31}_{(10)},$$

$$E_{32}(m;\ m_{33}=-m_{23})^{-13+33}_{(10)} =$$

$$-\left(\frac{[l_{13}-l_{11}-1][l_{23}+l_{31}+2]}{[l_{13}-l_{23}-1][l_{23}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{23})^{-13-13+33+43+31}_{(11)}$$

$$-\frac{1}{(l_{13}-l_{23}-1)}\left(\frac{[l_{13}-l_{23}][l_{11}-l_{23}][l_{23}+l_{31}+2]}{[2][l_{23}+l_{43}+3]}\right)^{1/2}$$

$$\times(m;\ m_{33}=-m_{23})^{-13-23+33+43+31}_{(2)},$$

$$E_{32}(m;\ m_{33}=-m_{23})^{-13+43}_{(10)} =$$

$$\left(\frac{[l_{11}-l_{23}][l_{23}+l_{31}+3]}{[l_{13}-l_{23}][l_{23}+l_{43}+4]}\right)^{1/2}(m;\ m_{33}=-m_{23})^{-13-23+43+43+31}_{(20)}$$

$$+\left(\frac{[l_{13}-l_{11}-1][l_{31}-l_{43}-1]}{[l_{13}-l_{23}-1][l_{23}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{23})^{-13-13+33+43+31}_{(11)}$$

$$+\frac{[l_{13}+l_{43}+3]}{[l_{23}+l_{43}+4][l_{13}-l_{23}-1]}\left(\frac{[l_{11}-l_{23}][l_{31}-l_{43}-1]}{[2][l_{13}-l_{23}][l_{23}+l_{43}+3]}\right)^{1/2}$$

$$\times(m;\ m_{33}=-m_{23})^{-13-23+33+43+31}_{(2)},$$

$$E_{32}(m;\ m_{33}=-m_{23})^{-23+43}_{(10)} =$$

$$-\left(\frac{[l_{13}-l_{11}][l_{23}+l_{31}+3]}{[l_{13}-l_{23}][l_{23}+l_{43}+4]}\right)^{1/2}(m;\ m_{33}=-m_{23})^{-13-23+43+43+31}_{(20)}$$

$$+\frac{1}{[l_{23}+l_{43}+4]}\left(\frac{[l_{13}-l_{11}][l_{31}-l_{43}-1][l_{23}+l_{43}+3]}{[2][l_{13}-l_{23}]}\right)^{1/2}$$

$$\times(m;\ m_{33}=-m_{23})^{-13-23+33+43+31}_{(2)},$$

$$E_{32}(m;\ m_{33}=-m_{23})^{-13-23+43+43}_{(20)} =$$

$$\left(\frac{[l_{13}-l_{11}-1][l_{31}-l_{43}-2]}{[l_{13}-l_{23}][l_{23}+l_{43}+4]}\right)^{1/2}(m;\ m_{33}=-m_{23})^{-13-13-23+33+43+43+31}_{(21)},$$

$$E_{32}(m;\ m_{33}=-m_{23})^{-13-23+33+43}_{(2)} =$$

$$\left(\frac{[2][l_{13}-l_{11}-1][l_{23}+l_{31}+2]}{[l_{13}-l_{23}][l_{23}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{23})^{-13-13-23+33+43+43+31}_{(21)},$$

$$E_{32}(m;\ m_{33}=-m_{23})^{-13-13+33+43}_{(11)} =$$

$$-\left(\frac{[l_{11}-l_{23}][l_{23}+l_{31}+2]}{[l_{13}-l_{23}-1][l_{23}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{23})^{-13-13-23+33+43+43+31}_{(21)},$$

$$E_{32}(m;\ m_{43}=-m_{13})^{-13-13-23+33+43+43}_{(21)} = 0. \tag{A.1}$$

The matrix elements of of E_{23}:

$$E_{23}(m;\ m_{33}=-m_{23})^{-13-13-23+33+43+43}_{(21)} =$$

$$-\left(\frac{[l_{11}-l_{23}][l_{23}+l_{31}+1][l_{23}+l_{43}+3]}{[l_{13}-l_{23}-1]}\right)^{1/2}(m;\ m_{33}=-m_{23})^{-13-13+33+43-31}_{(11)}$$

$$+\frac{([l_{13}-l_{11}-1][l_{31}-l_{43}-3][l_{13}-l_{23}])^{1/2}}{[l_{23}+l_{43}+4]}(m;\ m_{33}=-m_{23})^{-13-23+43+43-31}_{(20)}$$

$$+[l_{13}+l_{43}+3]\frac{([2][l_{13}-l_{23}][l_{13}-l_{11}-1][l_{31}+l_{23}+1][l_{23}+l_{43}+3])^{1/2}}{[2][l_{13}-l_{23}-1][l_{23}+l_{43}+4]}$$

$$\times(m;\ m_{33}=-m_{23})^{-13-23+33+43-31}_{(2)},$$

$$E_{23}(m;\ m_{33}=-m_{23})^{-13-23+33+43}_{(2)} =$$

$$-[l_{23}+l_{43}+4]\left(\frac{[2][l_{11}-l_{23}][l_{23}+l_{31}+1]}{[l_{13}-l_{23}][l_{23}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{23})_{(10)}^{-13+33-31}$$

$$+[l_{13}-l_{23}-1]\left(\frac{[2][l_{13}-l_{11}][l_{31}-l_{43}-2]}{[l_{13}-l_{23}][l_{23}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{23})_{(10)}^{-23+43-31}$$

$$+\left(\frac{[2][l_{11}-l_{23}][l_{31}-l_{43}-2]}{[l_{13}-l_{23}][l_{23}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{23})_{(10)}^{-13+43-31},$$

$$E_{23}(m;\ m_{33}=-m_{23})_{(11)}^{-13-13+33+43}=$$

$$-[l_{13}+l_{43}+3]\left(\frac{[l_{13}-l_{11}-1][l_{23}+l_{31}+1]}{[l_{13}-l_{23}-1][l_{23}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{23})_{(10)}^{-13+33-31}$$

$$+[l_{13}-l_{23}]\left(\frac{[l_{13}-l_{11}-1][l_{31}-l_{43}-2]}{[l_{13}-l_{23}-1[l_{23}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{23})_{(10)}^{-13+43-31},$$

$$E_{23}(m;\ m_{33}=-m_{23})_{(20)}^{-13-23+43+43}=$$

$$-[l_{13}+l_{43}+3]\left(\frac{[l_{13}-l_{11}][l_{31}+l_{23}+2]}{[l_{13}-l_{23}][l_{23}+l_{43}+4]}\right)^{1/2}(m;\ m_{33}=-m_{23})_{(10)}^{-23+43-31}$$

$$+[l_{23}+l_{43}+3]\left(\frac{[l_{11}-l_{23}][l_{23}+l_{31}+2]}{[l_{13}-l_{23}][l_{23}+l_{43}+4]}\right)^{1/2}(m;\ m_{33}=-m_{23})_{(10)}^{-13+43-31},$$

$$E_{23}(m;\ m_{33}=-m_{23})_{(10)}^{-23+43}=$$

$$-\left(\frac{[l_{11}-l_{23}][l_{23}+l_{31}+2][l_{23}+l_{43}+3]}{[l_{13}-l_{23}]}\right)^{1/2}(m;\ m_{33}=-m_{23})_{(00)}^{-31},$$

$$E_{23}(m;\ m_{33}=-m_{23})_{(10)}^{-13+43}=$$

$$-[l_{13}+l_{43}+3]\left(\frac{[l_{13}-l_{11}][l_{23}+l_{31}+2]}{[l_{13}-l_{23}][l_{23}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{23})_{(00)}^{-31},$$

$$E_{23}(m;\ m_{33}=-m_{23})_{(10)}^{-13+33}=$$

$$-\left(\frac{[l_{13}-l_{11}][l_{13}-l_{23}][l_{31}-l_{43}-1]}{[l_{23}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{23})_{(00)}^{-31},$$

$$E_{23}(m;\ m_{33}=-m_{23})_{(00)}=0. \qquad (A.2)$$

A.3. Class 3

The matrix elements of E_{32}:

$$E_{32}(m;\ m_{43}=-m_{23}+1)_{(00)}=$$

$$-\left(\frac{[l_{13}-l_{11}][l_{23}+l_{31}+3]}{[l_{13}-l_{23}][l_{23}+l_{33}+3]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(10)}^{-13+33+31}$$

$$-\left(\frac{[l_{13}-l_{11}][l_{33}-l_{31}]}{[l_{13}-l_{23}][l_{23}+l_{33}+3]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(10)}^{-13+43+31}$$

$$-\left(\frac{[l_{11}-l_{23}][l_{23}+l_{31}+3]}{[l_{13}-l_{23}][l_{23}+l_{33}+3]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(10)}^{-23++33+31},$$

$$E_{32}(m;\ m_{43}=-m_{23}+1)_{(10)}^{-13+33}=$$

$$\left(\frac{[l_{11}-l_{23}][l_{23}+l_{31}+3]}{[l_{13}-l_{23}][l_{23}+l_{33}+4]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(20)}^{-13-23+33+33+31}$$

$$-\left(\frac{[l_{13}-l_{11}-1][l_{33}-l_{31}+1]}{[l_{13}-l_{23}-1][l_{23}+l_{33}+3]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(11)}^{-13-13+33+43+31}$$

$$-\frac{[l_{13}+l_{33}+3]}{[l_{13}-l_{23}-1][l_{23}+l_{33}+4]}\left(\frac{[l_{11}-l_{23}][l_{33}-l_{31}+1]}{[2][l_{13}-l_{23}][l_{23}+l_{33}+3]}\right)^{1/2}$$

$$\times(m;\ m_{43}=-m_{23}+1)_{(3)}^{-13-23+33+43+31},$$

$$E_{32}(m;\ m_{43}=-m_{23}+1)_{(10)}^{-13+43}=$$

$$\left(\frac{[l_{13}-l_{11}-1][l_{23}+l_{31}+2]}{[l_{13}-l_{23}-1][l_{23}+l_{33}+3]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(11)}^{-13-13+33+43+31}$$

$$+\frac{1}{[l_{13}-l_{23}-1]}\left(\frac{[l_{13}-l_{23}][l_{11}-l_{23}][l_{23}+l_{31}+2]}{[2][l_{23}+l_{33}+3]}\right)^{1/2}$$

$$\times(m;\ m_{43}=-m_{23}+1)_{(3)}^{-13-23+33+43+31},$$

$$E_{32}(m;\ m_{43}=-m_{23}+1)_{(10)}^{-23+33}=$$

$$-\left(\frac{[l_{13}-l_{11}][l_{23}+l_{31}+3]}{[l_{13}-l_{23}][l_{23}+l_{33}+4]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(20)}^{-13-23+33+33+31}$$

$$+\frac{1}{[l_{23}+l_{33}+4]}\left(\frac{[l_{13}-l_{11}][l_{33}-l_{31}+1][l_{23}+l_{33}+3]}{[2][l_{13}-l_{23}]}\right)^{1/2}$$

$$\times(m;\ m_{43}=-m_{23}+1)_{(3)}^{-13-23+33+43+31},$$

$$E_{32}(m;\ m_{43}=-m_{23}+1)_{(20)}-13-23+33+33=$$

$$-\left(\frac{[l_{13}-l_{11}-1][l_{33}-l_{31}+2]}{[l_{13}-l_{23}][l_{23}+l_{33}+4]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(21)}^{-13-13-23+33+33+43+31},$$

$$E_{32}(m;\ m_{43}=-m_{23}+1)_{(3)}^{-13-23+33+43}=$$

$$\left(\frac{[2][l_{13}-l_{11}-1][l_{23}+l_{31}+2]}{[l_{13}-l_{23}][l_{23}+l_{33}+3]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(21)}^{-13-13-23+33+33+43+31},$$

$$E_{32}(m;\ m_{43}=-m_{23}+1)_{(11)}^{-13-13+33+43}=$$

$$-\left(\frac{[l_{11}-l_{23}][l_{23}+l_{31}+2]}{[l_{13}-l_{23}-1][l_{23}+l_{33}+3]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(21)}^{-13-13-23+33+33+43+31},$$

$$E_{32}(m;\ m_{43}=-m_{13})_{(21)}^{-13-13-23+33+33+43}=0. \qquad (A.3)$$

The matrix elements of of E_{23}:

$$E_{23}(m;\ m_{43}=-m_{23}+1)_{(21)}^{-13-13-23+33+33+43}=$$

$$-\left(\frac{[l_{11}-l_{23}][l_{23}+l_{31}+1][l_{23}+l_{33}+3]}{[l_{13}-l_{23}-1]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(11)}^{-13-13+33+43-31}$$

$$-\left(\frac{[l_{13}-l_{11}-1][l_{33}-l_{31}+3][l_{13}-l_{23}]}{[l_{23}+l_{33}+4]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(20)}^{-13-23+33+33-31}$$

$$+[l_{13}+l_{33}+3]\frac{([2][l_{13}-l_{11}-1][l_{13}-l_{23}][l_{23}+l_{31}+1][l_{23}+l_{33}+3])^{1/2}}{[2][l_{13}-l_{23}-1][l_{23}+l_{33}+4]}$$

$$\times(m;\ m_{43}=-m_{23}+1)_{(3)}^{-13-23+33+43-31},$$

$$E_{23}(m;\ m_{43}=-m_{23}+1)_{(3)}^{-13-23+33+43}=$$

$$-[l_{13}-l_{23}-1]\left(\frac{[2][l_{13}-l_{11}][l_{33}-l_{31}+2]}{[l_{13}-l_{23}][l_{23}+l_{33}+3]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(10)}^{-23+33-31}$$

$$-\left(\frac{[2][l_{11}-l_{23}][l_{33}-l_{31}+2]}{[l_{13}-l_{23}][l_{23}+l_{33}+3]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(10)}^{-13+33-31}$$

$$+[l_{23}+l_{33}+4]\left(\frac{[2][l_{11}-l_{23}][l_{23}+l_{31}+1]}{[l_{13}-l_{23}][l_{23}+l_{33}+3]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(10)}^{-13+43-31},$$

$$E_{23}(m;\ m_{43}=-m_{23}+1)_{(11)}^{-13-13+33+43}=$$

$$-[l_{13}-l_{23}]\left(\frac{[l_{13}-l_{11}-1][l_{33}-l_{31}+2]}{[l_{13}-l_{23}-1][l_{23}+l_{33}+3]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(10)}^{-13+33-31}$$

$$+[l_{13}+l_{33}+3]\left(\frac{[l_{13}-l_{11}-1][l_{23}+l_{31}+1]}{[l_{13}-l_{23}-1[l_{23}+l_{33}+3]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(10)}^{-13+43-31},$$

$$E_{23}(m;\ m_{43}=-m_{23}+1)_{(20)}^{-13-23+33+33}=$$

$$-[l_{13}+l_{33}+3]\left(\frac{[l_{13}-l_{11}][l_{23}+l_{31}+2]}{[l_{13}-l_{23}][l_{23}+l_{33}+4]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(10)}^{-23+33-31}$$

$$+[l_{23}+l_{33}+3]\left(\frac{[l_{11}-l_{23}][l_{23}+l_{31}+2]}{[l_{13}-l_{23}][l_{23}+l_{33}+4]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(10)}^{-13+33-31},$$

$$E_{23}(m;\ m_{43}=-m_{23}+1)_{(10)}^{-23+33}=$$

$$-\left(\frac{[l_{11}-l_{23}][l_{23}+l_{31}+2][l_{23}+l_{33}+3]}{[l_{13}-l_{23}]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(00)}^{-31},$$

$$E_{23}(m;\ m_{43}=-m_{23}+1)_{(10)}^{-13+43}=$$

$$-\left(\frac{[l_{13}-l_{11}][l_{13}-l_{23}][l_{33}-l_{31}+1]}{[l_{23}+l_{33}+3]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)_{(00)}^{-31},$$

$$E_{23}(m;\ m_{43}=-m_{23}+1)_{(10)}^{-13+33}=$$

$$-[l_{13}+l_{33}+3]\left(\frac{[l_{13}-l_{11}][l_{23}+l_{31}+2]}{[l_{13}-l_{23}][l_{23}+l_{33}+3]}\right)^{1/2}(m;\ m_{43}=-m_{23}+1)^{-31}_{(00)},$$

$$E_{23}(m;\ m_{43}=-m_{23}+1)_{(00)}=0. \qquad (A.4)$$

A.4. Class 4

The matrix elements of E_{32}:

$$E_{32}(m;\ m_{33}=-m_{13}-1)_{(00)}=$$

$$-\left(\frac{[l_{13}-l_{11}][l_{13}+l_{31}+3]}{[l_{13}-l_{23}][l_{13}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-13+43+31}_{(10)}$$

$$-\left(\frac{[l_{11}-l_{23}][l_{31}-l_{43}]}{[l_{13}-l_{23}][l_{13}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-13+33+31}_{(10)}$$

$$-\left(\frac{[l_{11}-l_{23}][l_{13}+l_{31}+3]}{[l_{13}-l_{23}][l_{13}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-23+43+31}_{(10)},$$

$$E_{32}(m;\ m_{33}=-m_{13}-1)^{-13+43}_{(10)}=$$

$$\left(\frac{[l_{11}-l_{23}][l_{13}+l_{31}+3]}{[l_{13}-l_{23}][l_{13}+l_{43}+4]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-13-23+43+43+31}_{(20)}$$

$$-\frac{1}{(l_{13}+l_{43}+4)}\left(\frac{[l_{11}-l_{23}][l_{31}-l_{43}-1][l_{13}+l_{43}+3]}{[2][l_{13}-l_{23}]}\right)^{1/2}$$

$$\times(m;\ m_{33}=-m_{13}-1)^{-13-23+33+43+31}_{(4)},$$

$$E_{32}(m;\ m_{33}=-m_{13}-1)^{-23+33}_{(10)}=$$

$$-\left(\frac{[l_{11}-l_{23}+1][l_{13}+l_{31}+2]}{[l_{13}-l_{23}+1][l_{13}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-23-23+33+43+31}_{(11)}$$

$$+\frac{1}{[l_{13}-l_{23}+1]}\left(\frac{[l_{13}-l_{11}][l_{13}-l_{23}][l_{13}+l_{31}+2]}{[2][l_{13}+l_{43}+3]}\right)^{1/2}$$

$$\times(m;\ m_{33}=-m_{13}-1)^{-13-23+33+43+31}_{(4)},$$

$$E_{32}(m;\ m_{33}=-m_{13}-1)^{-23+43}_{(10)}=$$

$$-\left(\frac{[l_{13}-l_{11}][l_{13}+l_{31}+3]}{[l_{13}-l_{23}][l_{13}+l_{43}+4]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-13-23+43+43+31}_{(20)}$$

$$+\left(\frac{[l_{11}-l_{23}-1][l_{31}-l_{43}-1]}{[l_{13}-l_{23}+1][l_{13}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-23-23+33+43+31}_{(11)}$$

$$+\frac{[l_{23}+l_{43}+3]}{[l_{13}+l_{43}+4][l_{13}-l_{23}+1]}\left(\frac{[l_{13}-l_{11}][l_{31}-l_{43}-1]}{[2][l_{13}-l_{23}][l_{13}+l_{43}+3]}\right)^{1/2}$$

$$\times(m;\ m_{33}=-m_{13}-1)^{-13-23+33+43+31}_{(4)},$$

$$E_{32}(m;\ m_{33}=-m_{13}-1)^{-13-23+43+43}_{(20)}=$$

$$\left(\frac{[l_{11}-l_{23}+1][l_{31}-l_{43}-2]}{[l_{13}-l_{23}][l_{13}+l_{43}+4]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-13-23-23+33+43+43+31}_{(21)},$$

$$E_{32}(m;\ m_{33}=-m_{13}-1)^{-13-23+33+43}_{(4)}=$$

$$\left(\frac{[2][l_{11}-l_{23}+1][l_{13}+l_{31}+2]}{[l_{13}-l_{23}][l_{13}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-13-23-23+33+43+43+31}_{(21)},$$

$$E_{32}(m;\ m_{33}=-m_{13}-1)^{-23-23+33+43}_{(11)}=$$

$$\left(\frac{[l_{13}-l_{11}][l_{13}+l_{31}+2]}{[l_{13}-l_{23}+1][l_{13}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-13-23-23+33+43+43+31}_{(21)},$$

$$E_{32}(m;\ m_{43}=-m_{13})^{-13-23-23+33+43+43}_{(21)}=0. \qquad (A.5)$$

The matrix elements of of E_{23}:

$$E_{23}(m;\ m_{33}=-m_{13}-1)^{-13-23-23+33+43+43}_{(21)}=$$

$$\left(\frac{[l_{13}-l_{11}][l_{13}+l_{31}+1][l_{13}+l_{43}+3]}{[l_{13}-l_{23}-1]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-23-23+33+43-31}_{(11)}$$

$$-\left(\frac{[l_{11}-l_{23}+1][l_{31}-l_{43}-3][l_{13}-l_{23}]}{[l_{13}+l_{43}+4]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-13-23+43+43-31}_{(20)}$$

$$+\frac{[l_{23}+l_{43}+3]\,([2][l_{11}-l_{23}+1][l_{13}-l_{23}][l_{13}+l_{31}+1][l_{13}+l_{43}+3])^{1/2}}{[2][l_{13}-l_{23}+1][l_{13}+l_{43}+4]}$$

$$\times(m;\ m_{33}=-m_{13}-1)^{-13-23+33+43-31}_{(4)},$$

$$E_{23}(m;\ m_{33}=-m_{13}-1)^{-13-23+33+43}_{(4)}=$$

$$[l_{13}+l_{43}+4]\left(\frac{[2][l_{13}-l_{11}][l_{13}+l_{31}+1]}{[l_{13}-l_{23}][l_{13}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-23+33-31}_{(10)}$$

$$-\left(\frac{[2][l_{13}-l_{11}][l_{31}-l_{43}-2]}{[l_{13}-l_{23}][l_{13}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-23+43-31}_{(10)}$$

$$+[l_{13}-l_{23}+1]\left(\frac{[2][l_{11}-l_{23}][l_{31}-l_{43}-2]}{[l_{13}-l_{23}][l_{13}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-13+43-31}_{(10)},$$

$$E_{23}(m;\ m_{33}=-m_{13}-1)^{-23-23+33+43}_{(11)}=$$

$$-[l_{23}+l_{43}+3]\left(\frac{[l_{11}-l_{23}+1][l_{13}+l_{31}+1]}{[l_{13}-l_{23}+1][l_{13}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-23+33-31}_{(10)}$$

$$+[l_{13}-l_{23}]\left(\frac{[l_{11}-l_{23}+1][l_{31}-l_{43}-2]}{[l_{13}-l_{23}+1[l_{13}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-23+43-31}_{(10)},$$

$$E_{23}(m;\ m_{33}=-m_{13}-1)^{-13-23+43+43}_{(20)}=$$

$$-[l_{13}+l_{43}+3]\left(\frac{[l_{13}-l_{11}][l_{13}+l_{31}+2]}{[l_{13}-l_{23}][l_{13}+l_{43}+4]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-23+43-31}_{(10)}$$

$$+[l_{23}+l_{43}+3]\left(\frac{[l_{11}-l_{23}][l_{13}+l_{31}+2]}{[l_{13}-l_{23}][l_{13}+l_{43}+4]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-13+43-31}_{(10)},$$

$$E_{23}(m;\ m_{33}=-m_{13}-1)^{-23+43}_{(10)}=$$

$$-[l_{23}+l_{43}+3]\left(\frac{[l_{11}-l_{23}][l_{13}+l_{31}+2]}{[l_{13}-l_{23}][l_{13}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-31}_{(00)},$$

$$E_{23}(m;\ m_{33}=-m_{13}-1)^{-23+33}_{(10)}=$$

$$\left(\frac{[l_{11}-l_{23}][l_{13}-l_{23}][l_{31}-l_{43}-1]}{[l_{13}+l_{43}+3]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-31}_{(00)},$$

$$E_{23}(m;\ m_{33}=-m_{13}-1)^{-13+43}_{(10)}=$$

$$-\left(\frac{[l_{13}-l_{11}][l_{13}+l_{31}+2][l_{13}+l_{43}+3]}{[l_{13}-l_{23}]}\right)^{1/2}(m;\ m_{33}=-m_{13}-1)^{-31}_{(00)},$$

$$E_{23}(m;\ m_{33}=-m_{13}-1)_{(00)}=.0 \tag{A.6}$$

A.5. Class 5

The matrix elements of E_{32}:

$$E_{32}(m;\ m_{33}=-m_{23},\ m_{43}=-m_{13})_{(00)}=$$

$$-\frac{([l_{13}-l_{11}][l_{13}+l_{31}+3])^{1/2}}{[l_{13}-l_{23}]}(m;\ m_{33}=-m_{23},\ m_{43}=-m_{13})^{-13+33+31}_{(10)}$$

$$-\frac{([l_{11}-l_{23}][l_{23}+l_{31}+3])^{1/2}}{[l_{13}-l_{23}]}(m;\ m_{33}=-m_{23},\ m_{43}=-m_{13})^{-23+43+31}_{(10)},$$

$$E_{32}(m;\ m_{33}=-m_{23},\ m_{43}=-m_{13})^{-13+33}_{(10)}=$$

$$-\frac{([2][l_{11}-l_{23}][l_{23}+l_{31}+2])^{1/2}}{[2][l_{13}-l_{23}-1]}(m;\ m_{33}=-m_{23},\ m_{43}=-m_{13})^{-13-23+33+43+31}_{(5)},$$

$$E_{32}(m;\ m_{33}=-m_{23},\ m_{43}=-m_{13})^{-23+43}_{(10)}=$$

$$\frac{([2][l_{13}-l_{11}][l_{13}+l_{31}+2])^{1/2}}{[2][l_{13}-l_{23}-1]}(m;\ m_{33}=-m_{23},\ m_{43}=-m_{13})^{-13-23+33+43+31}_{(5)},$$

$$E_{32}(m;\ m_{33}=-m_{23},\ m_{43}=-m_{13})^{-13-23+33+43}_{(5)}=0. \tag{A.7}$$

The matrix elements of of E_{23}:

$$E_{23}(m;\ m_{33}=-m_{23},\ m_{43}=-m_{13})^{-13-23+33+43}_{(5)}=$$

$$\frac{[l_{13}-l_{23}-1]}{[l_{13}-l_{23}]}\left([2][l_{13}-l_{11}][l_{13}+l_{31}+1]\right)^{1/2}$$

$$\times(m;\ m_{33}=-m_{23},\ m_{43}=-m_{13})^{-23+43-31}_{(10)}$$

$$+\frac{[l_{13}-l_{23}-1]}{[l_{13}-l_{23}]}\left([2][l_{11}-l_{23}][l_{23}+l_{31}+1]\right)^{1/2}$$

$$\times(m;\ m_{33}=-m_{23},\ m_{43}=-m_{13})^{-13+33-31}_{(10)},$$

$$E_{23}(m;\ m_{33}=-m_{23},\ m_{43}=-m_{13})^{-23+43}_{(10)}=$$

$$\left([l_{11}-l_{23}][l_{23}+l_{31}+2]\right)^{1/2}(m;\ m_{33}=-m_{23},\ m_{43}=-m_{13})^{-31}_{(00)},$$

$$E_{23}(m;\ m_{33}=-m_{23},\ m_{43}=-m_{13})^{-13+33}_{(10)}=$$

$$-\left([l_{13}-l_{11}][l_{13}+l_{31}+2]\right)^{1/2}(m;\ m_{33}=-m_{23},\ m_{43}=-m_{13})^{-31}_{(00)},$$

$$E_{23}(m;\ m_{33}=-m_{23},\ m_{43}=-m_{13})_{(00)}=0. \tag{A.8}$$

REFERENCES

1. Nguyen Anh Ky, J. Math. Phys. **35**, 2583 (1994).
2. P. P. Kulish and N. Yu. Reshetikhin, *Lett. Math. Phys.* **18**, 143 (1989) ; E. Celeghini, Tch. Palev and M. Tarlini, Mod. Phys. Lett. **B 5**, 187 (1991).
3. Tch. D. Palev and V. N. Tolstoy, Comm. Math. Phys. **141**, 549 (1991).
4. R. Floreanini, V. Spiridonov and L. Vinet, Comm. Math. Phys. **137**, 149 (1991); E. D'Hoker, R. Floreanini and L. Vinet, J. Math. Phys., **32**, 1427 (1991); Tch. Palev J.Math.Phys. **34**, 4872 (1993)..
5. R. B. Zhang, J. Math. Phys., **34**, 1236 (1993).
6. Tch. Palev, N. Stoilova and J. Van der Jeugt, Preprint IC/93/157, to appear in Comm. Math. Phys.
7. V. D. Drinfel'd, "Quantum groups", in *Proceedings of the International Congress of Mathematicians*, 1986, Berkeley, vol. **1**, 798-820 (The American Mathematical Society, 1987).
8. M. Jimbo, Lett. Math. Phys. **10**, 63 (1985); *ibil* **11**, 247 (1986).
9. L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtajan, Algebra and Analys, **1**, 178 (1987).
10. Yu. I. Manin, *Quantum groups and non-commutative geometry*, Centre des Recherchers Mathématiques, Montréal (1988); *Topics in non-commutative geometry*, Princeton University Press, Princeton, New Jersey (1991).
11. S. I. Woronowicz, Comm. Math. Phys., **111**, 613 (1987); **122**, 125 (1989).
12. R. Floreanini, D. Leites and L. Vinet, Lett. Math. Phys. **23**, 127 (1991); M. Scheunert, Lett. Math. Phys. **24**, 173 (1992); S. M. Khoroshkin and V. N. Tolstoy, Comm. Math. Phys. **141**, 599 (1991).
13. V. Kac, Comm. Math. Phys., **53**, 31 (1977); Adv. Math. **26**, 8 (1977); *Lecture Notes in Mathematics* **676**, 597 (Springer - Verlag, Berlin 1978).
14. A. H. Kamupingene, Nguyen Anh Ky and Tch. D. Palev, J. Math. Phys. **30**, 553 (1989).
15. Tch. Palev and N. Stoilova, J. Math. Phys., **31**, 953 (1990).

16. I. M. Gel'fand and M. L. Zetlin, *Dokl. Akad. Nauk USSR*, **71**, 825 (1950), (in Russian); for a detailed description of the Gel'fand-Zetlin basis see also G. E. Baird and L. C. Biedenharn, *J. Math. Phys.*, **4**, 1449 (1963); A. O. Barut and R. Raczka, *Theory of Group Representations and Applications*, Polish Scientific Publishers, Warszawa, 1980.

17. M. Rosso, Comm. Math. Phys. **117**, 581 (1987).

18. G. Lusztig, Adv. in Math **70**, 237 (1988).

19. H. Schlosser, Seminar Sophus Lie **3**,15 (1993).

20. Tch. Palev, J. Math. Phys. **27**, 1994 (1986).

21. Tch. Palev, J. Math. Phys. **28**, 2280 (1987); Funct. Anal. Appl. **21**, 245 (1987).

22. Tch. Palev, J. Math. Phys. **29**, 2589 (1988); J. Math. Phys. **30**, 1433 (1989).

23. Tch. Palev, Funct. Anal. Appl. **23**, 141 (1989).