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ON THE EXISTENCE OF SOLUTIONS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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1 Introduction

Boundary value problems for functional differential equations for first and second order have been studied by many authors, see for instance [3, 4, 7, 8, 12] for first order and [2, 6, 10, 11], among others, for second order.

In [3, 4, 7, 8, 12] and also in [2, 6] the following B-property is required

"The appearance function maps bounded sets of its domain into bounded sets in its range" (B)

Although this condition is essential for first order boundary value problems, see for instance [2, 10] it can be omitted for second order boundary value problems (see [10, 11]). In this paper, the authors use the Granas Topological Transversality Method, based on the motion of an essential mapping [5, 13, 9].

The aim of this paper is to extend, the previous method of Granas, in the case of n-th order functional differential equations.

The paper is organized as follows. Section 1 contains some preliminaries results for th linear problems. In Sec. 3 we state a general existence result. Sec. 4 contains some applications of the result of Sec. 3.

2 Preliminaries

Let C_r, $r > 0$, be the space of all continuous functions $x : [-r, 0] \to \mathbb{R}$, endowed with the sup norm

$$\|x\|_r = \sup_{\theta\in[-r,0]} |x(\theta)|$$

Clearly C_r is a Banach space. For any continuous function x defined on $[-r, T]$; $T > 0$, and any $t \in [0, T]$, the function x_t is a element of C_r defined by

$$x_t(\theta) = x(t + \theta), \quad -r \le \theta \le 0$$

We shall consider the n-th order boundary value problem BVP:

$$x^{(n)}(t) = f(t,\ x_t,\ x(t)) \tag{1}$$

$$\begin{aligned} &x(t) = \varphi(t), \quad -r \le t \le 0 \\ &x^{(i)}(0) = 0 \quad i = 0, 1, \ldots, k-1 \\ &x^{(i)}(T) = 0, \quad i = k, k+1, \ldots, n-1,\ n \ge 2 \end{aligned} \tag{2}$$

where $k \in \{1, 2, \ldots, n-1\}$ is given, and $f : [0, T] \times C_r \times \mathbb{R} \to \mathbb{R}$ is assumed to be a continuous function.

We consider the Banach space

$$\mathbb{B} = C([-r, T], \mathbb{R}) \cap C^n([0, T], \mathbb{R})$$

with norm

$$\|x\| \quad \max\{\|x\|_r, \|x\|_n\}$$

where

$$\|x\|_n = \max_{0\le t\le T} |x^{(i)}(t)|, \quad 0 \le i \le n$$

By a solution of BVP (1)-(2), we mean a function $x \in \mathbb{B}$ such that x satisfies Eq.(1) with boundary conditions (2).

For each $t \in [0,T]$ we consider the equation

$$x^{(n)}(t) = 0 \tag{3}$$

with conditions

$$\begin{array}{ll} x(t) = \varphi(t), & -r \leq t \leq 0 \\ x^{(i)}(0) = 0 & 0 \leq i \leq k-1 \\ x^{(i)}(T) = 0, & k \leq i \leq n-1 \end{array} \tag{2}$$

Agarwal [1], and also Eloe et al. [14, 15] show that the Green function of BVP (1)-(3) exists and is given by

$$G(t,s) = \frac{1}{(n-1)!} \begin{cases} \sum_{i=0}^{k-1} \binom{n-1}{i} t^i (-s)^{n-i-1}; & 0 \leq s \leq t \leq T \\ -\sum_{i=0}^{k-1} \binom{n-1}{i} t^i (-s)^{n-i-1}, & 0 \leq t \leq s \leq T \\ 0 & t \notin [0,T] \end{cases}$$

and the following inequalities hold

$$(-1)^{n-k} G^{(i)}(t,s) \geq 0, \quad 0 \leq i \leq k-1$$

$$(-1)^{n-i} G^{(i)}(t,s) \geq 0, \quad k \leq i \leq n-1 \tag{4}$$

where $G^{(i)}(t,s)$ denotes the ith derivative $\frac{\partial^i}{\partial t^i} G(t,s)$. Hence, the BVP (1)-(2) can be transformed into the integral equation

$$x(t) = \begin{cases} \int_0^T G(t,s) f(s, x_s, x(s)) ds + \varphi(0), & 0 \leq t \leq T \\ \varphi(t) & -r \leq t \leq 0 \end{cases}$$

Now, the problem consists of finding a fixed point for the mapping $M : \mathbb{B} \to \mathbb{B}$ defined by

$$M(x)(t) = \begin{cases} \int_0^T G(t,s) f(s, x_s, x(s)) ds + \varphi(0), & 0 \leq t \leq T \\ \varphi(t) & -r \leq t \leq 0 \end{cases}$$

3 Existence Result

We are now in a position to state the main result of this paper:

Theorem 3.1. Let $f : [0,T] \times C_r \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that:
(a) $f(t, x_t, x(t)) \geq 0$ for any $(t, x_t, x) \in [0,T] \times C_r \times \mathbb{R}$
(b) There exists a constant $M > 0$ such that:

$$\|x\| \leq M$$

for every solution of the BVP

$$x^{(n)}(t) = \lambda f(t, x_t, x(t)), t \in [0,T], \lambda \in [0,1] \tag{1$_\lambda$}$$

$$\begin{array}{ll} x(t) = \varphi(t) & -r \leq t \leq 0 \\ x^{(i)}(0) = 0 & 0 \leq i \leq k-1 \\ x^{(i)}(T) = 0 & k \leq i \leq n-1 \end{array} \tag{2}$$

Then, the BVP (1)-(2) has at least one solution.

For the proof of our Theorem 3.1, we shall need the following lemma which is referred to as the Luay-Schauder alternative [13].

Lemma 3.1. Let C be a convex subset of a normed linear space E and assume $0 \in C$. Let $F : C \to C$ be a completely continuous operator and let

$$\mathcal{S}(F) = \{x \in C : x = \lambda F(x) \text{ for some } \lambda \in (0,1)\}$$

Then either $\mathcal{S}(F)$ is unbounded or F has a fixed point.

Proof of Theorem 3.1.
Suppose first that $\varphi(0) = 0$. Using the sign properties of the Green's function, we can define a cone $\mathbb{P}_1$ in the Banach space

$$\mathbb{B}_1 = \{x \in C^{n-1}[0,T] : x^{(i)}(0) = 0, i = 0, 1, \ldots k-1\}$$

by

$$\mathbb{P}_1 = \{x \in \mathbb{B}_1 : (-1)^{n-k} x^{(k)}(t) \geq 0\}$$

One can remark that if $x \in \mathbb{P}_1$, then $(-1)^{n-k} x^{(i)}(t) \geq 0$ for every $i \in [0,k]$ and $t \in [0,T]$. Hence $0 \in \mathbb{P}_1$. Now let $F : \mathbb{P}_1 \to \mathbb{B}_1$ be an operator defined by

$$F(x)(t) = \int_0^T G(t,s) f(s, x_s, x(s)); \; 0 \leq t \leq T$$

where

$$x_s(\theta) = \begin{cases} x(s+\theta) & s+\theta \geq 0 \\ \varphi(s+\theta) & s+\theta < 0 \end{cases}$$

Clearly, $F(\mathbb{P}_1) \subseteq \mathbb{P}_1$. We show now that F is completely continuous. The continuity of F follows from that of f and that of the integral operator. Let $D \subset \mathbb{P}_1$ be a bounded subset. Then there exists a constant $d \geq 0$ such that $\|x\|_{n-1} \leq d$ for each $x \in D$; this means that $\max_{0 \leq t \leq T} |x^{(i)}(t)| \leq d$ for $0 \leq i \leq n-1$. Clearly D is an equicontinuous subset of $C^{n-1}([0,T], \mathbb{R})$.

Let $\bar{D}$ be a subset of C_r defined by

$$\bar{D} = \{x_t : x \in D\}$$

We argue as in [10] that there exists a compact subset K of C_r such that

$$\bar{D} \subseteq K$$

$\bar{D}$ is uniformly bounded since, for each $x_t \in \bar{D}$ and $t \in [0,T]$.

$$\sup_{t \in [-r,0]} |x_t(\theta)| = \sup_{t \in [-r,0]} |x(t+\theta)| \leq \max\{d, \|\varphi\|_r\}$$

We shall prove that $\bar{D}$ is uniformly equicontinuous. Let $\theta_1, \theta_2 \in [-r, 0]$, then

$$|x_t(\theta_1) - x_t(\theta_2)| = \begin{cases} |x(t+\theta_1) - x(t+\theta_2)|, & t+\theta_1 \geq 0, \quad t+\theta_2 \geq 0 \\ |\varphi(t+\theta_1) - x(t+\theta_2)|, & t+\theta_1 < 0, \quad t+\theta_2 \geq 0 \\ |\varphi(t+\theta_1) - \varphi(t+\varphi_2)|, & t+\theta_1 < 0, \quad t+\theta_2 < 0 \\ |x(t+\theta_1) - \varphi(t+\theta_2)|, & t+\theta_1 \geq 0 \quad t+\theta_2 < 0 \end{cases}$$

Let $\varepsilon > 0$ be arbitrarily chosen, then if $t+\theta_1 \geq 0$, $t+\theta_2 \geq 0$ there exists a $\delta_0(\varepsilon) > 0$ such that

$$|x(t+\theta_1) - x(t+\theta_2)| < \varepsilon \text{ provided } |\theta_2 - \theta_1| < \delta_0(\varepsilon),$$

by equicontinuity of D.

If $(t+\theta_1) < 0$ and $(t+\theta_2) \geq 0$, then

$$|\varphi(t+\theta_1) - x(t+\theta_2)| \leq |\varphi(t+\theta_1) - \varphi(0)| + |x(0) - x(t+\theta_2)|$$

Since φ is uniformly continuous in $[-r, 0]$ and D is equicontinuous, there exists $\delta_1(\varepsilon) > 0$ and $\delta_2(\varepsilon)$ such that

$$|\varphi(\tau_1) - \varphi(\tau_2)| < \frac{\varepsilon}{2} \text{ provided } |\tau_1 - \tau_2| < \delta_1(\varepsilon)$$

$$|x(\tau_1) - x(\tau_2)| < \frac{\varepsilon}{2} \text{ provided } |\tau_1 - \tau_2| < \delta_1(\varepsilon)$$

Thus if $|\tau_1 - \tau_2| < \min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\} = \delta(\varepsilon)$, then $|t+\theta_1| < \delta(\varepsilon)$ and $|t+\theta_2| < \delta(\varepsilon)$ since $t+\theta_1 < 0 \leq t+\theta_2$. Hence

$$|\varphi(t+\theta_1) - x(t+\theta_2)| \leq |\varphi(t+\theta_1) - \varphi(0) + |(0) - x(t+\theta-2)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

The case where $t+\theta_1 \geq 0 > t+\theta_2$ is analogous, and the case where $t+\theta_1 \geq 0 > t+\theta_2$ is analogous, and the case where $t+\theta_1 < 0$, $t+\theta_2 < 0$ is obvious since φ is uniformly continuous in $[-r, 0]$. Hence for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$|x_t(\theta_1) - x_t(\theta_2)| < \varepsilon \text{ provided } |\theta_1 - \theta_2| < \delta(\varepsilon)\ .$$

so that $\bar{D}$ is uniformly equicontinuous. Hence there exists a compact set K of C_r such that $\bar{D} \subseteq K$.

Let $\{x_m\}$ be a bounded set in $\mathbb{P}_1$. Then the sequence $\{x_{m_t}\}$ is bounded in C_r, and there exists a compact subset K of C_r such that $x_{mt} \in K$ for all m and $t \in [0, T]$. If $d_1 > 0$ is a constant such that $\|x_m\| \leq d_1$ for every m, then the set

$$X = [0, T] \times K \times [-d_1, d_1] \text{ is a compact subset of}$$

$[0, T] \times C_r \times \mathbb{R}$.

Let

$$F(x_m)(t) = \int_0^T G(t, s) f(s, x_{ms}, x_m(s)) ds$$

then

$$(F(x_m))^{(n)}(t) = f(t, x_{m_t}, x_m(t))$$

Thus for every (t, x_{m_t}, x_m) in X, we have

$$|F(x_m)(t)| = \max_{0 \leq t \leq T} \int_0^T |G(t, s)| ds . F_m$$

$$\leq \frac{T^n}{n!} \cdot F_m = M_1$$

by using interpolation formula (see [1], p. 76), and

$$|F(x_m))^{(n)}(t)| \leq F_m\ .$$

where F_m is the maximum of $f(t, u, v)$ on the compact set X. Hence $F(x_m) \in C^n[0, T]$. By Green's function properties $F(x_m)$ satisfies the two-point condition

$$(F(x_m))^{(i)}(0) = 0$$

$$(F(x_m))^{(i)}(T) = 0$$

Thus, using interpolation formula ([1], theorem 8.5, p. 83) we have

$$|(F(x_m))^{(i)}(t)| \leq \frac{T^{n-i}}{(n-i)!} \max_{0 \leq t \leq T} |(F(x_m))^{(n)}(t)| \tag{5}$$

$$\leq \frac{T^{n-i}}{(n-i)!} F_m$$

Hence

$$\|F(x_m)\| \leq M$$

where

$$M = \max_{0 \leq i \leq n-1} \frac{T^{n-i}}{(n-i)!} \cdot F_m$$

The above inequality (5) implies easily that the set $\{F(x_m))^{(i)}(t) : x_m(t) \in X\}$, $0 \leq i \leq n-1$ is uniformly equicontinuous. Hence $\overline{F(X)}$ is compact by Arzela-Ascoli Theorem, and F is completely continuous. Since

$$\mathcal{S}(F) = \{x \in \mathbb{P}_1 : x = \lambda F(x), 0 < \lambda < 1\}$$

is bounded by assumption, lemma 3.1 implies that F has a fixed point x in $\mathbb{P}_1$. And any function

$$z(t) = \begin{cases} x(t) & t \in [0, T] \\ \varphi(t) & t \in [-r, 0] \end{cases}$$

is a solution of the BVP (1)-(2)

Now, if $\varphi(0) \neq 0$, we can define

$$y = x - \varphi(0) \tag{6}$$

The transformation (6) reduces the BVP (1)-(2) to

$$y^{(n)}(t) = f(t, y_t + \varphi(0), y(t) + \varphi(0)) = \hat{f}(t, y_t, y(t))$$

$$y(t) = x(t) - \varphi(0) = \hat{\varphi}(t)$$

$$y^{(i)}(0) = 0, \quad 0 \leq i \leq k-1$$

$$y^{(i)}(T) = 0, \quad k \leq i \leq n-1$$

Here we have $\hat{\varphi}(0) = 0$. Thus the proof is complete.

4 Applications

The applicability of Theorem 3.1, depends upon the existence of an a-priori $\|\cdot\|$-norm bound for the solutions of the family of BVPs $(1)_\lambda$-(2) which is independent of λ. In the next results we shall provide sufficient conditions under which such bound exists.

Lemma 4.1. Let $f : [0,T] \times C_r \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that
(a) There is $M_0 > 0$ such that

$$\max_{-r\le t\le T} |x(t)| \le M_0$$

for every solution x of the BVP $(1)_\lambda$-(2), $\lambda \in [0,1]$
(b) The subset $D \subset C_r$ defined by

$$D = \{x_t : \|x_t\|_r \le M_0,\ t \in [0,T]\}$$

is compact.
Then there exists $M > 0$ independent of λ such that

$$\|x\| \le M$$

for every solution x of the BVP $(1)_\lambda$-(2)

Proof. The set $x = [0,T] \times D \times [-M_0, M_0]$ is a compact subset of $[0,T] \times C_r \times \mathbb{R}$. Let x be a solution of $(1)_\lambda$-(2) in X, then $x \in C^n[0,T]$ and satisfies (2). Since any such function can be written as

$$x(t) = \int_0^T G(t,s)x^{(n)}(s)ds + \varphi(0)$$

Thus

$$|x^{(i)}(t)| \le \max_{0\le t\le T}\int_0^T |G^{(i)}(t,s)|ds. \max_{0\le t\le T}|x^{(n)}(t)|$$
$$\le \frac{T^{n-i}}{(n-i)!}\cdot Q$$

where $Q = \max\{|f(t,u,v)| : (t,u,v) \in X\}$. Hence

$$\|x\|_{n-1} \le M_1$$

Let $M = \max\{M_0, M_1\}$ then $\|x\| \le M$, independent of λ for every solution x of the BVP $(1)_\lambda$-(2). Using theorem 3.1, we have the following theorem

Theorem 4.1. Let $f : [0,T] \times C_r \times \mathbb{R} \to \mathbb{R}$ be a continuous and nonnegative function. If assumptions (a), (b) of Lemma 4.1 are satisfied. Then the BVP (1)-(2) has at least one solution.

Theorem 4.2. Let $N_0 > 0$ and $E(N_0) = \{(t,u) : t \in [0,T],\ \|u\| \le N_0\}$ be a compact set. Let $f : E(N_0) \times \mathbb{R} \to \mathbb{R}$ be a continuous and nonnegative function. if there exists a positive real-valued function $H \in C([0,\infty),(0,\infty))$ such that

$$f(t,u,v) \le H(|v|)$$

for each $(t,u,v) \in E(N_0) \times \mathbb{R}$ with

$$\int_0^\infty \frac{s}{H(s)} = +\infty \tag{7}$$

Then, the BVP (1)-(2) has at least one solution.

Proof. By Theorem 3.1, the proof is complete if we can find an a-priori $\|\cdot\|$-norm bound for all solutions x of the family of differential equations

$$x^{(n)}(t) = \lambda f(t, x_t, x(t)) \tag{1}$$

satisfying the boundary conditions (2)
Let x be a solution of $(1)_\lambda$-(2), then

$$x(t) = \int_0^T G(t,s)x^{(n)}ds + \varphi(0)$$

Without loss of generality we can suppose $\varphi(0) = 0$, thus

$$\begin{aligned} |x(t)| &\le \max_{0\le t\le T}\int_0^T |G(t,s)|ds \cdot \max_{0\le t\le T}|x^{(n)}(t)| \\ &= \frac{T^n}{n!}\max_{0\le t\le T}|\lambda f(t,x_t,x(t)| \\ &\le \frac{T^n}{n!}H(|x|) \end{aligned}$$

Hence

$$\frac{|x(t)|}{H(|x|)} \le \frac{T^n}{n!}$$

By (7), this implies that there exists $N_1 > 0$ independent of λ such that

$$\max_{0\le t\le T}|x(t)| \le N_1$$

Let

$$X = \{(t,u,v) : (t,u) \in E(N_0),\ \max_{0\le t\le T}|v(t)| \le N_1\}$$

then clearly X is a compact subset of $E(N_0) \times \mathbb{R}$. Since f is continuous on $E(N_0) \times \mathbb{R}$, there exists $N_2 > 0$ which is the maximum of $|f(t,u,v)|$, in X. Since

$$\begin{aligned} |x^{(i)}(t)| &\le \max_{0\le t\le T}\int_0^T |G^{(i)}(t,s)|ds\ \max|x^{(n)}(t)| \\ &= \frac{T^{(n-i)}}{(n-1)!}\max_{0\le t\le T}|\lambda f(t,x_t,x)| \\ &\le \frac{T^{(n-i)}}{(n-i)!}\cdot N_2 \end{aligned}$$

Hence there exists $N_3 > 0$ such that

$$\max_{0\le t\le T}|x^{(i)}(t)| \le N_3$$

Put $N = \max\{N_0, N_3\}$, then we have

$$\|x\| \le N$$

for every solution x of the BVP $(1)_\lambda$-(2) and we are done.

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