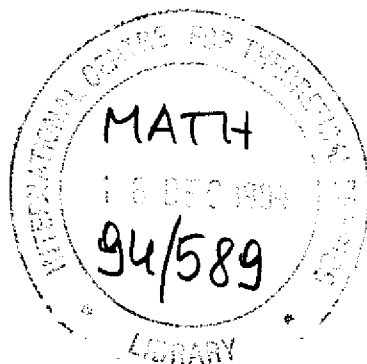


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**INTERNATIONAL CENTRE FOR  
THEORETICAL PHYSICS**

**CANONICAL GENERATORS  
OF THE COHOMOLOGY OF MODULI  
OF PARABOLIC BUNDLES ON CURVES**

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OF MODULI OF PARABOLIC BUNDLES ON CURVES**

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**ABSTRACT**

We determine generators of the rational cohomology algebras of moduli spaces of parabolic vector bundles on a curve, under some ‘primality’ conditions on the parabolic datum. These generators are canonical in a precise sense. Our results are new even for usual vector bundles (i.e., vector bundles without parabolic structure) whose rank is greater than 2 and is coprime to the degree; in this case, they are generalizations of a theorem of Newstead on the moduli of vector bundles of rank 2 and odd degree.

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## 1 Introduction

The aim of this paper is to determine generators of the rational cohomology algebras of moduli spaces of parabolic vector bundles on a curve, under some ‘primality’ conditions (see Assumptions 1.1 and 1.2) on the parabolic datum. These generators are canonical in a sense which will be made precise below. Our results are new even for usual vector bundles (i.e., vector bundles without parabolic structure) whose rank is greater than 2 and is coprime to the degree; in this case, they are generalizations of a theorem of Newstead [8], where the case of vector bundles of rank 2 and odd degree is studied.

Let  $X$  be a compact Riemann surface, fix integers  $n$  and  $d$  with  $n$  positive, and let  $\Delta$  be a parabolic datum of rank  $n$  on  $X$  (see Section 3 below). Denote by  $\mathcal{U}_X(n, d, \Delta)$  the moduli space of parabolic vector bundles of rank  $n$  and degree  $d$ , which are parabolic semistable with respect to  $\Delta$ . Fix a holomorphic line bundle  $L$  of degree  $d$  on  $X$ , and let  $S\mathcal{U}_X(n, L, \Delta)$  be the subvariety of  $\mathcal{U}_X(n, d, \Delta)$  consisting of vector bundles with determinant isomorphic to  $L$ . We make the following two hypotheses on the parameters  $n$ ,  $d$  and  $\Delta$ .

**Assumption 1.1** Every parabolic vector bundle of rank  $n$  and degree  $d$  on  $X$  which is parabolic semistable with respect to the parabolic datum  $\Delta$  is in fact parabolic stable.

**Assumption 1.2** There exists a universal parabolic bundle (or briefly, a universal bundle) on  $\mathcal{U}_X(n, d, \Delta) \times X$ .

Recall that a universal bundle on  $\mathcal{U}_X(n, d, \Delta) \times X$  is a vector bundle  $U$  on  $\mathcal{U}_X(n, d, \Delta) \times X$  together with a flag of subbundles  $j_X^* U = U^{s,1} \supset$

$U^{x,2} \supset \dots \supset U^{x,k_x} \supset U^{x,k_x+1} = 0$  in  $j_x^*U$  for each  $x \in J$  ( $J$  being the set of parabolic points), where  $j_x : \mathcal{U}_X(n, d, \Delta) \rightarrow \mathcal{U}_X(n, d, \Delta) \times X$  is the map  $E \mapsto (E, x)$ , such that for each  $E \in \mathcal{U}_X(n, d, \Delta)$ , the restriction of  $U$  to  $\{E\} \times X$  is parabolically isomorphic to  $E$ . We use the same symbols to denote the restrictions of  $U$  and  $U^{x,i}$  etc. to  $S\mathcal{U}_X(n, L, \Delta)$ .

**Notation 1.3**

- If  $S$  and  $T$  are topological spaces, and if  $\alpha \in H^*(S \times T, \mathbf{Q})$ , then  $\sigma(\alpha) : H_*(T, \mathbf{Q}) \rightarrow H^*(S, \mathbf{Q})$  is the map  $\sigma(\alpha)z = \alpha/z$ , where  $/$  denotes the slant product.
- If  $V$  is a vector bundle, then  $\mathbf{P}(V)$  denotes its projectivization.
- All cohomologies in the paper have  $\mathbf{Q}$ -coefficients.

Having settled on the notation, we now have the following two theorems which are the main results of this paper.

**Theorem 1.4** *Suppose that Assumptions 1.1 and 1.2 hold, and let  $U$  be any universal bundle on  $\mathcal{U}_X(n, d, \Delta) \times X$ . Then, the rational cohomology algebra of  $\mathcal{U}_X(n, d, \Delta)$  is generated by the Chern classes  $c_j(\text{Hom}(U^{x,i}, U^{x,i-1}))$  ( $x \in J$ ) and the images of*

$$\sigma(c_1(U)) : H_1(X) \rightarrow H^1(\mathcal{U}_X(n, d, \Delta)) \quad \text{and}$$

$$\sigma(a_i(\mathbf{P}(U))) : H_r(X) \rightarrow H^{2i-r}(\mathcal{U}_X(n, d, \Delta)) \quad (2 \leq i \leq n, 0 \leq r \leq 2)$$

where  $a_i(\cdot)$  denote the characteristic classes of projective bundles introduced in Definition 2.4 below.

**Theorem 1.5** *Suppose that Assumptions 1.1 and 1.2 hold, and let  $U$  be any universal bundle on  $S\mathcal{U}_X(n, L, \Delta) \times X$ . Then the rational cohomology algebra of  $S\mathcal{U}_X(n, L, \Delta)$  is generated by the Chern classes  $c_j(\text{Hom}(U^{x,i}, U^{x,i-1}))$  ( $x \in J$ ) and the images of*

$$\sigma(a_i(\mathbf{P}(U))) : H_r(X) \rightarrow H^{2i-r}(S\mathcal{U}_X(n, L, \Delta)) \quad (2 \leq i \leq n, 0 \leq r \leq 2).$$

Note that the generators given in Theorems 1.4 and 1.5 are *canonical*, i.e., independent of the choice of a universal bundle (which is easily seen to be non-unique). Indeed, if  $U'$  is another universal bundle, then there exists a line bundle  $\xi$  on  $\mathcal{U}_X(n, d, \Delta)$  such that  $U' \cong U \otimes p^*\xi$ , where  $p : \mathcal{U}_X(n, d, \Delta) \times X \rightarrow \mathcal{U}_X(n, d, \Delta)$  is the canonical projection. Now, for every  $z \in H_1(X)$ , we have  $\sigma(c_1(U'))z = \sigma(c_1(U))z$ , since  $\sigma(c_1(p^*\xi))z = 0$ ; on the other hand, it is obvious that  $\mathbf{P}(U') \cong \mathbf{P}(U)$  and  $\mathcal{H}om((U')^{x,i}, (U')^{x,i-1}) \cong \mathcal{H}om(U^{x,i}, U^{x,i-1})$ .

We now relate the above theorems to certain results of Atiyah and Bott [1]. Let  $\mathcal{U}_X(n, d)$  be the moduli space of stable vector bundles of rank  $n$  and degree  $d$ , where  $n$  and  $d$  are coprime. In this case, Atiyah and Bott [1] proved that the Kunneth components (with respect to any basis of  $H^*(X, \mathbf{Q})$ ) of the Chern classes of any universal bundle on  $\mathcal{U}_X(n, d) \times X$  generate the rational cohomology algebra of  $\mathcal{U}_X(n, d)$ . Theorem 1.4 above differs from this result in the following respects. Firstly, we work throughout in the setup of parabolic bundles, whereas Atiyah and Bott were working with usual vector bundles. Secondly, as we have observed above, the generators we obtain are canonical, i.e., they are independent of the choice of a universal bundle, whereas the Kunneth components of the Chern classes of a universal bundle  $U$  do depend on the choice of  $U$ . Finally, by specializing to the case where the parabolic set is empty, and applying Lemma 2.6 below, we obtain the above result of Atiyah and Bott from Theorem 1.4; whereas it does not seem possible to deduce Theorem 1.4 from the result of Atiyah and Bott: the difficulty is due to the fact that the slant product does not behave well with the cup product.

We should remark that Beauville [2] has given another proof of the above result of Atiyah and Bott. In the parabolic setup, and under Assumptions 1.1 and 1.2, the method of Beauville can be used to deduce that the Kunneth components of the Chern classes of any universal bundle  $U$  and the Chern classes  $c_j(\text{Hom}(U^{x,i}, U^{x,i-1}))$  generate

the cohomology of  $\mathcal{U}_X(n, d, \Delta)$ , a statement which, as we have seen above, is weaker than Theorem 1.4.

The following is a consequence of the above theorems.

**Corollary 1.6** *Let  $n=2$ , suppose Assumptions 1.1 and 1.2 are true, and let  $U$  be any universal bundle on  $SU_X(2, L, \Delta) \times X$ . Then, the rational cohomology algebra of  $SU_X(2, L, \Delta)$  is generated by the Chern classes  $c_j(\text{Hom}(S^x, Q^x))$  ( $x \in J$ ), and the image of*

$$\sigma(c_2(\mathcal{E}ndU)) : H_r(X) \longrightarrow H^{4-r}(SU_X(n, L, \Delta)) \quad (0 \leq r \leq 2),$$

where  $j_x^*U = U^{x,1} \supset U^{x,2} = S^x$  is the flag in  $j_x^*U$  ( $x \in J$ ), and  $Q^x = j_x^*U/S^x$ .

The above corollary is a generalization to parabolic bundles of a theorem of Newstead [8].

In our approach, Assumption 1.1 is natural and indispensable; it is a technical necessity which guarantees that the action of a certain gauge group on a certain space of holomorphic structures is free. Granted this, Assumption 1.2 is not too stringent a restriction, as the following observation shows.

**Proposition 1.7** *Suppose the parameters  $n, d$  and  $\Delta$  satisfy Assumption 1.1. Then, they satisfy Assumption 1.2 if any one of the following three conditions is satisfied:*

- *The rank  $n$  and the degree  $d$  are coprime.*
- *There exists a parabolic point  $x \in J$  such that  $\sum_{i=1}^{k_x} n_{x,i}$  is coprime to  $n$  for some  $j$  ( $1 \leq j \leq k_x$ ), where  $n_{x,1}, \dots, n_{x,k_x}$  denote the parabolic multiplicities at  $x$ .*
- *There exists a parabolic point  $x \in J$  such that  $\sum_{i=1}^{k_x} n_{x,i}$  and  $n+d$  are coprime for some  $j$ .*

Here is a brief outline of the contents of the paper. In Section 2, we define the characteristic classes of projective bundles which occur in the statements of the theorems. The next section contains a description of generators of the rational cohomology of the classifying spaces of certain gauge groups. The final section contains proofs of the above results.

## 2 Projective Bundles

This preliminary section deals with some universal aspects of projective bundles, the aim being to define explicit characteristic classes for these bundles.

If  $G$  is a topological group, then  $EG \rightarrow BG$  will denote a universal principal  $G$ -bundle. Cohomology groups will have rational coefficients throughout, unless otherwise indicated. Fix a positive integer  $n$ .

The natural epimorphism  $\pi : U(n) \rightarrow PU(n)$  induces a fibration  $B\pi : BU(n) \rightarrow BPU(n)$  with fibre  $BU(1)$ . Let  $x_1, \dots, x_n$  be the Chern roots of  $EU(n)$ , so  $H^*(BU(n))$  is the algebra  $S[x_1, \dots, x_n]$  of symmetric polynomials in the  $x_i$  with rational coefficients (or, equivalently,  $H^*(BU(n))$  is the polynomial algebra  $\mathbf{Q}[c_1, \dots, c_n]$ , where  $c_i = c_i(EU(n))$ ). The Leray-Hirsch theorem implies that the map  $(B\pi)^* : H^*(BPU(n)) \rightarrow H^*(BU(n))$  is injective, and (see [3], Section 15.2) its image equals the subalgebra  $I[x_1, \dots, x_n]$  of  $S[x_1, \dots, x_n]$  consisting of symmetric polynomials invariant under the affine change of variables  $x_i \mapsto x_i + d$ , where  $d$  is an indeterminate.

**Remark 2.1** The above fact means that if  $E \rightarrow M$  is a vector bundle, the characteristic classes of its projectivization  $\mathbf{P}(E)$  are precisely the characteristic classes of  $E$  which are invariant under tensoring by a line bundle.

**Lemma 2.2** *The above algebra  $I[x_1, \dots, x_n]$  is a polynomial algebra*

$\mathbf{Q}[z_2, \dots, z_n]$ , where

$$z_k = \sum_{1 \leq j_1 < \dots < j_k \leq n} y_{j_1} \dots y_{j_k}, \quad 2 \leq k \leq n,$$

are the elementary symmetric functions in  $y_1, \dots, y_n$ , where

$$y_i = nx_i - \sum_{j=1}^n x_j, \quad 1 \leq i \leq n.$$

The proof of the lemma is quite easy, and we omit it. Note that the first elementary symmetric polynomial in the  $y_i$ , namely their sum, is zero. The following assertion follows immediately from Lemma 2.2.

**Corollary 2.3** For  $k = 2, \dots, n$ , define  $a_k \in H^{2k}(BPU(n))$  by  $(B\pi)^*a_k = z_k$ . Then  $H^*(BPU(n))$  is the polynomial algebra  $\mathbf{Q}[a_2, \dots, a_n]$ .

**Definition 2.4** Let  $P$  be a principal  $PU(n)$ -bundle on a CW-complex  $M$ . Then, for  $k = 2, \dots, n$ , the  $k$ -th characteristic class  $a_k(P)$  of  $P$  is, by definition, the element  $f^*a_k \in H^{2k}(M)$ , where  $f : M \rightarrow BPU(n)$  is some classifying map for  $P$  and  $a_k$  is as in Corollary 1.3. (As usual,  $a_k(P)$  is independent of the choice of  $f$ .)

### Examples 2.5

1. If  $E \rightarrow M$  is a vector bundle of rank 2, with Chern roots  $x_1, x_2$ , then

$$a_2(\mathbf{P}(E)) = -(x_1 - x_2)^2 = c_2(\mathcal{E}nd E),$$

where  $\mathbf{P}(E)$  is the projectivization and  $\mathcal{E}nd E$  is the endomorphism bundle of  $E$ .

2. Let  $E \rightarrow M$  be a vector bundle of rank 3 such that  $c_i(E) = 0$  ( $i = 1, 2$ ),  $c_3 \neq 0$ , and let  $x_1, x_2, x_3$  be the Chern roots of  $E$ . Then

$$a_3(\mathbf{P}(E)) = 27x_1x_2x_3 = 27c_3(E) \neq 0,$$

while  $c_1(\mathcal{E}nd E) = c_3(\mathcal{E}nd E) = 0$ ,  $\mathcal{E}nd E$  being the complexification of a real vector bundle.

The above examples illustrate the fact that any characteristic class of  $\mathcal{E}nd E$  can be written as a polynomial in  $a_i(\mathbf{P}(E))$ , but the converse is not true, for  $\mathbf{P}(E)$  has more characteristic classes than  $\mathcal{E}nd E$ .

**Lemma 2.6** If  $n \geq k \geq 2$ , there exist a polynomial  $P_{n,k}(T_1, \dots, T_{k-1})$  with rational coefficients, and a non-zero constant  $\lambda_{n,k} \in \mathbf{Q}$  such that for every vector bundle  $E$  of rank  $n$  on a CW-complex  $M$ , we have

$$a_k(\mathbf{P}(E)) = P_{n,k}(c_1(E), a_2(\mathbf{P}(E)), \dots, a_{k-1}(\mathbf{P}(E))) + \lambda_{n,k}c_k(E). \quad (1)$$

*Proof.* It suffices to find  $P_{n,k}$  and non-zero  $\lambda_{n,k}$  such that (1) holds for  $EU(n)$ . Let  $u_i = a_i(\mathbf{P}(EU(n)))$  and  $c_i = c_i(EU(n))$ ; then  $u_i = (BU)^*a_i$ . We prove the result only for  $k = 2$ , the general case following easily by induction on  $k$ . Since  $c_1^2$  and  $c_2$  generate  $H^4(BU(n))$ , we can find  $\alpha, \lambda \in \mathbf{Q}$  such that  $u_2 = \alpha c_1^2 + \lambda c_2$ . If  $\lambda = 0$ , then  $u_2 - \alpha c_1^2 = 0$ ; since  $\{1, c_1, c_1^2, \dots\}$  is an  $H^*(BPU(n))$ -basis of  $H^*(BU(n))$ , this implies that  $u_2 = 0$ , contradicting the injectivity of  $(B\pi)^*$ .  $\square$

**Lemma 2.7** Let  $1 \leq r \leq 2$ , and fix a  $C^\infty$  vector bundle  $E$  of rank  $n$  on the  $r$ -sphere  $S^r$ . Then, for each  $k = 2, \dots, n$ , there exists a  $C^\infty$  vector bundle  $V$  on  $S^{2k-r} \times S^r$  such that:

1. For each  $t \in S^{2k-r}$ , we have  $V_t \simeq E$ , where  $V_t = i_t^*V$ ,  $i_t : S^r \rightarrow S^{2k-r} \times S^r$  being the map  $x \mapsto (t, x)$ .
2. For each  $x \in S^r$ ,  $j_x^*V$  is trivial, where  $j_x : S^{2k-r} \rightarrow S^{2k-r} \times S^r$  is the map  $t \mapsto (t, x)$ .
3.  $a_k(\mathbf{P}(V)) \neq 0$ .

*Proof.* The existence of  $V$  satisfying (1) and (2) with  $c_k(V) \neq 0$  follows by standard arguments of K-theory. If  $a_k(V) = 0$ , then Lemma 2.6 implies that  $c_k(V)$  is a multiple of  $c_1(V)^k$ , which is zero since  $c_1(V)$  is the pull-back of  $c_1(E)$  by the second projection, a contradiction.  $\square$

### 3 Cohomology of Some Classifying Spaces

In this section we describe, for later use, generators of the rational cohomology algebras of certain mapping spaces. The computations here are motivated by Section 2 of [1], [5], and Section 5.1 of [4]. All CW-complexes here are assumed to be finite, and cohomologies are over  $\mathbf{Q}$ .

Let  $M$  be a pointed CW-complex, and let  $E$  be a complex vector bundle of rank  $n$  over  $M$ . Fix a base point  $b_0 \in BU(n)$ . Let  $\mathcal{G}(M, E)$  denote the complex gauge group of  $E$ , with the compact-open topology. Then as shown in [1] and [5], the space  $\text{Map}_E(M, BU(n))$  of all maps  $f : M \rightarrow BU(n)$  such that  $f^*EU(n) \simeq E$  is a classifying space for  $\mathcal{G}(M, E)$ , so let us denote  $\text{Map}_E(M, BU(n))$  by  $B\mathcal{G}(M, E)$ . The subspace  $\text{Map}_E^*(M, BU(n))$  consisting of all pointed maps is closed in  $B\mathcal{G}(M, E)$ . We denote  $\text{Map}_E^*(M, BU(n))$  by  $\mathcal{B}(M, E)$ . When there is no scope for confusion, we shall write just  $\mathcal{G}$  for  $\mathcal{G}(M, E)$ ,  $\mathcal{B}$  for  $\mathcal{B}(M, E)$ , etc.

Let  $\varepsilon : B\mathcal{G} \times M \rightarrow BU(n)$  denote the evaluation map,  $(f, x) \mapsto f(x)$ . Then the bundle  $\varepsilon^*EU(n)$  is called the *universal bundle* on  $B\mathcal{G} \times M$ , and is denoted  $\mathcal{E}(M, E)$ . We denote the restriction of  $\mathcal{E}(M, E)$  to  $\mathcal{B} \times M$  also by  $\mathcal{E}(M, E)$ .

If  $\phi : M' \rightarrow M$  is a pointed map of CW-complexes, and if  $E$  is a vector bundle over  $M$ , then, denoting  $E' = \phi^*E$ , there is a natural map  $\phi^\# : B\mathcal{G}(M, E) \rightarrow B\mathcal{G}(M', E')$ ,  $f \mapsto f \circ \phi$ , carrying  $\mathcal{B}(M, E)$  into  $\mathcal{B}(M', E')$ . These constructs have the following functorial property.

**Proposition 3.1** *Suppose  $\phi : M \rightarrow M'$  is a pointed map of CW-complexes, let  $E$  be a complex vector bundle over  $M$ , and let  $E' = \phi^*E$ . Write  $\mathcal{G}$  for  $\mathcal{G}(M, E)$ ,  $\mathcal{G}'$  for  $\mathcal{G}(M', E')$ , etc. If  $d(\cdot)$  denotes  $c_k(\cdot)$  or  $\alpha_k(\mathbf{P}(\cdot))$  for some  $k$ , then the diagram*

$$\begin{array}{ccc} H_*(M') & \xrightarrow{\phi_*} & H_*(M) \\ \sigma(d(\mathcal{E}')) \downarrow & & \downarrow \sigma(d(\mathcal{E})) \\ H^*(B\mathcal{G}') & \xrightarrow{H^*(\phi^\#)} & H^*(B\mathcal{G}) \end{array}$$

commutes, where  $\sigma(\alpha)z = \alpha/z$  is the slant product with  $\alpha$ .

*Proof.* The commutativity of the diagram

$$\begin{array}{ccc} B\mathcal{G} \times M' & \xrightarrow{\phi^\# \times 1} & B\mathcal{G}' \times M' \\ 1 \times \phi \downarrow & & \downarrow \varepsilon' \\ B\mathcal{G} \times M & \xrightarrow{\varepsilon} & BU(n) \end{array}$$

taken with the functorial properties of the slant product, leads directly to the desired conclusion.  $\square$

The space  $\mathcal{B}(M, E)$  has the following semi-universal property.

**Proposition 3.2** *Suppose  $E$  is a complex vector bundle on a pointed CW-complex, and  $V$  a vector bundle on  $T \times M$  such that:*

- *For each  $t \in T$ , we have  $V_t \cong E$ , where  $V_t = i_t^*V$ ,  $i_t : M \rightarrow T \times M$  being the map  $x \mapsto (t, x)$ .*
- *If  $x_0$  is the base point of  $M$ , and if  $j_{x_0} : T \rightarrow T \times M$  is the map  $t \mapsto (t, x_0)$ , then  $j_{x_0}^*V$  is trivial.*

*Then, there exists a map  $\psi : T \rightarrow \mathcal{B}(M, E)$  such that  $(\psi \times 1)^*\mathcal{E}(M, E) \cong V$ .*

*Proof.* General properties of classifying spaces give a map  $\theta : (T \times M, T \times \{x_0\}) \rightarrow (BU(n), b_0)$  such that  $\theta^*EU(n) \cong V$ , where  $b_0$  is the base point in  $BU(n)$ . If we define  $\psi : T \rightarrow \mathcal{B}$  by  $\psi(t)x = \theta(t, x)$ , then  $\varepsilon \circ (\psi \times 1) = \theta$ , proving that  $(\psi \times 1)^*\mathcal{E} \cong V$ .  $\square$

Let us study the space  $\mathcal{B}(M, E)$  a little more when  $M = S^r$ ,  $r = 1, 2$ . Take  $M = S^1$  first.

**Proposition 3.3** *Let  $E$  be a (necessarily trivial) complex vector bundle of rank  $n$  on  $S^1$ . Then the cohomology algebra of  $\mathcal{B}(S^1, E)$  is generated by  $c_1(\mathcal{E})/[S^1]$  and  $a_i(\mathbf{P}(\mathcal{E}))/[S^1]$ ,  $i = 2, \dots, n$ , where  $[S^1]$  denotes the fundamental class of  $S^1$ .*

*Proof.* Note first that  $\mathcal{B}(S^1, E)$  equals  $\Omega BU(n)$ , which is homotopically equivalent to  $U(n)$ . Thus  $H^*(\mathcal{B})$  is an exterior algebra on  $n$  generators  $\theta_i \in H^{2i-1}(\mathcal{B})$ ,  $i = 1, \dots, n$ . Introduce the ad-hoc notation  $\omega_1 = c_1(\mathcal{E})/[S^1]$  and  $\omega_i = a_i(\mathbf{P}(\mathcal{E}))/[S^1]$ ,  $i = 2, \dots, n$ . Since  $H^1(\mathcal{B}) = \mathbf{Q}\theta_1$ , we can write  $\omega_1 = \lambda\theta_1$  for some  $\lambda \in \mathbf{Q}$ . Choose a vector bundle  $V$  on  $S^1 \times S^1$  such that  $c_1(V) \neq 0$ . By Proposition 3.2, there exists a map  $\psi : S^1 \rightarrow \mathcal{B}$  such that  $(\psi \times 1)^*\mathcal{E} = V$ , hence  $c_1(V)/[S^1] = \psi^*\omega_1 = \lambda\psi^*\theta_1$ . Since  $c_1(V) \neq 0$ , this implies that  $\lambda \neq 0$ , and  $\theta_1 = \lambda^{-1}\omega_1$ . Now let  $2 \leq k \leq n$  and assume that for each  $i = 1, \dots, k-1$ ,  $\theta_i$  is a polynomial in  $\omega_1, \dots, \omega_i$ . Thus  $H^*(\mathcal{B})$  is generated by  $\omega_1, \dots, \omega_{k-1}, \theta_k, \dots, \theta_n$ . Write

$$\omega_k = P(\omega_1, \dots, \omega_{k-1}) + \mu\theta_k, \quad (2)$$

where  $P$  is some polynomial and  $\mu \in \mathbf{Q}$ . By Lemma 2.7, there exists a vector bundle  $V$  on  $S^{2k-1} \times S^1$  satisfying the conditions of Proposition 3.2, such that  $a_k(\mathbf{P}(V)) \neq 0$ . Choose a map  $\psi : S^{2k-1} \rightarrow \mathcal{B}$  such that  $(\psi \times 1)^*\mathcal{E} = V$ . We see then that  $\psi^*\omega_k = a_k(\mathbf{P}(V))/[S^1]$ ; moreover, for  $1 \leq i \leq k-1$ ,  $\psi^*\omega_i \in H^{2i-1}(S^{2k-1}) = 0$ . Therefore, pulling back equation (1) by  $\psi$ , we get  $a_k(\mathbf{P}(V))/[S^1] = \mu\psi^*\theta_k$ . Since  $a_k(\mathbf{P}(V)) \neq 0$ , we conclude that  $\mu \neq 0$ , and dividing by  $\mu$ , we express  $\theta_k$  as a polynomial in  $\omega_1, \dots, \omega_k$ .  $\square$

**Proposition 3.4** *If  $E$  is a complex vector bundle of rank  $n$  on  $S^2$ , then the cohomology algebra of  $\mathcal{B}(S^2, E)$  is generated by  $a_i(\mathbf{P}(\mathcal{E}))/[S^2]$ ,  $2 \leq i \leq n$ .*

*Proof.* By definition,  $\mathcal{B}(S^2, E)$  is a connected component of  $\Omega^2 BU(n)$ , which is homotopically equivalent to  $\Omega U(n)$ . Therefore (see [10], p.68),  $H^*(\mathcal{B})$  is generated by  $n-1$  elements  $\theta_i \in H^{2i}(\mathcal{B})$ ,  $1 \leq i \leq n-1$ . If  $\omega_i = a_{i+1}(\mathbf{P}(\mathcal{E}))/[S^2]$ ,  $1 \leq i \leq n-1$ , then using Lemma 2.7 and Proposition 3.2, we see, as in the proof of Proposition 3.3, that each  $\theta_k$  is a polynomial in  $\omega_1, \dots, \omega_k$ .  $\square$

We now use these results to obtain generators for  $\mathcal{B}(X, E)$  when  $X$  is a 2-manifold.

**Proposition 3.5** *Let  $X$  be a pointed, compact, connected and oriented surface, and let  $E$  be a complex vector bundle of rank  $n$  on  $X$ . Then the cohomology algebra of  $\mathcal{B}(X, E)$  is generated by the images of*

$$\sigma(c_1(\mathcal{E})) : H_1(X) \rightarrow H^1(\mathcal{B}) \quad \text{and}$$

$$\sigma(a_i(\mathbf{P}(\mathcal{E}))) : H_r(X) \rightarrow H^{2i-r}(\mathcal{B}) \quad (2 \leq i \leq n, 1 \leq r \leq 2),$$

where  $\sigma$  denotes, as usual, the slant product.

*Proof.* Write  $X$  as a cofibration  $B \xrightarrow{i} X \xrightarrow{\pi} S^2$ , where  $B$  is a wedge of  $2g$  circles, and assume, without loss of generality, that the centre of the wedge is the base point  $x_0$  of  $X$ . Denote  $G = i^*E$ . Let  $z_0 = \pi(x_0)$ , and let  $F$  be a vector bundle over  $S^2$  such that  $\pi^*F \cong E$ . Since the mapping functor transforms cofibrations into fibrations, we get a fibration

$$\mathcal{B}(S^2, F) \xrightarrow{\pi^*} \mathcal{B}(X, E) \xrightarrow{i^*} \mathcal{B}(B, G).$$



Proposition 3.1 implies that the pull-back of  $a_i(\mathbf{PE}(X, E))/[X]$  by  $\pi^\#$  equals  $a_i(\mathbf{PE}(S^2, F))/\pi_*[X]$ . Since  $H_2(\pi)$  is an isomorphism, Proposition 3.4 now tells us that the Leray-Hirsch theorem applies. Thus the cohomology algebra of  $\mathcal{B}(X, E)$  is generated by the  $a_i(\mathbf{PE}(X, E))/[X]$  together with the image of  $H^*(i^\#)$ . Let  $B = \bigvee_{\alpha=1}^{2g} S_\alpha$ , where each  $S_\alpha$  is a circle; then  $\gamma_\alpha = [S_\alpha]$  form a basis of  $H_1(B)$ . Since  $\mathcal{B}(B, G) = \prod_{\alpha=1}^{2g} \mathcal{B}(S_\alpha, G)$ , and since  $H_1(i)$  is an isomorphism, Propositions 3.1 and 3.3, applied as above, lead us to the finish.  $\square$

**Theorem 3.6** *Let  $X$  be a 2-manifold as in Proposition 3.5. Then, the cohomology algebra of  $\mathcal{BG}$  is generated by the images of*

$$\sigma(c_1(\mathcal{E})) : H_r(X) \longrightarrow H^{2-r}(\mathcal{G}) \quad (0 \leq r \leq 1) \quad \text{and}$$

$$\sigma(a_i(\mathbf{P}(\mathcal{E}))) : H_r(X) \longrightarrow H^{2i-r}(\mathcal{G}) \quad (0 \leq r \leq 2, 2 \leq i \leq n).$$

*Proof.* Let  $x_0$  be the base point of  $X$ , and consider the fibration  $\mathcal{B} \hookrightarrow \mathcal{BG} \xrightarrow{\varepsilon_{x_0}} BU(n)$ , where  $\varepsilon_{x_0}(f) = f(x_0)$ . Since the images of  $H_1(X)$  and  $H_2(X)$  under the various slant products restrict, by Proposition 3.5, to generators of  $H^*(\mathcal{B})$ , the Leray-Hirsch theorem applies. By Lemma 2.6,  $c_1(EU(n))$  and  $a_i(\mathbf{PEU}(n))$ ,  $2 \leq i \leq n$ , generate  $H^*(BU(n))$ . Since  $\varepsilon_{x_0}^* d(EU(n)) = d(\mathcal{E})/[x_0]$  for  $d = c_1(\cdot)$  or  $d = a_i(\mathbf{P}(\cdot))$ , the result follows.  $\square$

**Remark 3.7** In view of Lemma 2.6, Theorem 2.6 implies the assertion concerning rational cohomology in Proposition 2.20 of [1]. Actually Lemma 2.6 may give one the impression that the above theorem can be deduced from Proposition 2.20 of [1], but this impression is hard to substantiate; the difficulty is due to the fact that the slant product does not behave well with the cup product.

We now apply the above results in the context of parabolic bundles over a curve. The standard reference for parabolic bundles is Mehta

and Seshadri [7], and we refer to Nitsure [9] for the gauge theoretic aspects of parabolic bundles.

Let  $X$  be a compact, connected and oriented surface, fix a positive integer  $n$ , and let  $\Delta$  be a parabolic datum of rank  $n$  on  $X$ . Thus,  $\Delta$  consists of:

- a finite subset  $J$  of  $X$ ; and
- for each  $x \in J$ , a sequence  $(n_{x,1}, \dots, n_{x,k_x})$  of positive integers such that  $\sum_{i=1}^{k_x} n_{x,i} = n$ , and a sequence  $0 \leq \alpha_{x,1} < \dots < \alpha_{x,k_x} < 1$  of real numbers.

Fix a quasi-parabolic vector bundle of rank  $n$  and type  $\Delta$  on  $X$ , and let  $\mathcal{G}_{\text{par}}$  denote the subgroup of the gauge group  $\mathcal{G}$  of  $E$ , consisting of parabolic gauge transformations.

Let  $\mathcal{E}$  denote the universal bundle on  $\mathcal{BG} \times X$ , and for each  $x \in J$ , let  $\mathcal{F}_x$  denote the bundle of flags of type  $\Delta$  in  $j_x^* \mathcal{E}$ , where  $j_x : \mathcal{BG} \rightarrow \mathcal{BG} \times X$  is the map  $f \mapsto (f, x)$ . Define  $\phi : \mathcal{F} \rightarrow \mathcal{BG}$  to be the fibre product of  $\mathcal{F}_x$  ( $x \in J$ ) over  $\mathcal{BG}$ . With these preparations out of the way, we can identify  $\mathcal{BG}_{\text{par}}$ .

**Lemma 3.8** *The space  $\mathcal{F}$  is a classifying space for  $\mathcal{G}_{\text{par}}$ .*

*Proof.* General considerations give  $\mathcal{BG}_{\text{par}}$  as  $E\mathcal{G}/\mathcal{G}_{\text{par}}$ ; further, by [1],  $E\mathcal{G}$  is the space of all maps  $\tilde{f} : E \rightarrow EU(n)$  which carry each fibre of  $E$  isomorphically to some fibre of  $EU(n)$ . Note that such an  $\tilde{f}$  defines an element  $f \in \mathcal{BG}$  such that  $\tilde{f}$  is an isomorphism of  $E$  with  $f^* EU(n)$ . On the other hand, the fibre of  $\mathcal{F}$  at  $f \in \mathcal{BG}$  is the product of certain flag manifolds of  $EU(n)_{f(x)}$  ( $x \in J$ ). Define  $\alpha : E\mathcal{G} \rightarrow \mathcal{F}$  by  $\alpha(\tilde{f}) = (\tilde{f}(F^i E_x) \subset EU(n)_{f(x)})$ , where  $f \in \mathcal{BG}$  is induced by  $\tilde{f}$ , and  $F^i E_x$  are given by the quasi-parabolic structure of  $E$ . By the definition of  $\mathcal{G}_{\text{par}}$ ,  $\alpha$  factors through a map  $\tilde{\alpha} : E\mathcal{G}/\mathcal{G}_{\text{par}} \rightarrow \mathcal{F}$ , which is easily seen to be a homeomorphism.  $\square$

So, denote  $\mathcal{F}$  by  $B\mathcal{G}_{\text{par}}$ . The pull-back  $\mathcal{E}_{\text{par}}$  of  $\mathcal{E}$  by  $\phi \times 1 : B\mathcal{G}_{\text{par}} \times X \rightarrow B\mathcal{G} \times X$  is a family of quasi-parabolic bundles, i.e., for each  $x \in J$ , there is a decreasing flag  $\mathcal{E}_{\text{par}}^{x,1} \supset \mathcal{E}_{\text{par}}^{x,2} \supset \dots$  of type  $\Delta$  in  $j_x^* \mathcal{E}_{\text{par}}$ , where, as usual,  $j_x(t) = (t, x)$  for  $t \in B\mathcal{G}_{\text{par}}$ . We call  $\mathcal{E}_{\text{par}}$  the universal bundle on  $B\mathcal{G}_{\text{par}} \times X$ .

**Theorem 3.9** *With notation as above, the cohomology algebra of  $B\mathcal{G}_{\text{par}}$  is generated by  $c_j(\mathcal{H}om(\mathcal{E}_{\text{par}}^{x,i}, \mathcal{E}_{\text{par}}^{x,i-1}))$  ( $x \in J$ ) and the images of*

$$\sigma(c_1(\mathcal{E}_{\text{par}})) : H_r(X) \rightarrow H^{2-r}(B\mathcal{G}_{\text{par}}) \quad (0 \leq r \leq 1) \quad \text{and}$$

$$\sigma(a_i(\mathcal{P}\mathcal{E}_{\text{par}})) : H_r(X) \rightarrow H^{2i-r}(B\mathcal{G}_{\text{par}}) \quad (0 \leq r \leq 2, 2 \leq i \leq n).$$

*Proof.* As already remarked, the fibre of  $\phi : B\mathcal{G}_{\text{par}} \rightarrow B\mathcal{G}$  over  $f$  is a product of flag manifolds  $M_f^x$  of the vector spaces  $\mathcal{E}_{(f,x)}$  ( $x \in J$ ). Each of these flag manifolds carries a tautological flag  $F_f^{x,i}$  of vector bundles, and the flag  $\mathcal{E}_{\text{par}}^{x,i}$  on  $B\mathcal{G}_{\text{par}}$ , in fact, restricts to the tautological flag on each factor  $M_f^x$  of the fibre. Now, in general, if  $M$  is a flag manifold, and if  $F^1 \supset F^2 \supset \dots$  is its tautological flag of vector bundles, then  $c_j(\mathcal{H}om(F^i, F^{i-1}))$  generate the cohomology algebra of  $M$ . In our context, this fact implies that the Leray-Hirsch theorem holds for the fibration  $\phi$ , and the result follows from Theorem 3.6.  $\square$

## 4 Proofs

This section brings together the results of the previous sections to prove Theorems 1.4 and 1.5. The notation is the same as before.

Let  $X$  be a compact Riemann surface, and let  $n$  and  $d$  be integers with  $n$  positive, and let  $\Delta$  be a parabolic datum of rank  $n$  on  $X$ . Suppose that Assumptions 1.1 and 1.2 are satisfied.

Let  $E$  be a  $C^\infty$  quasi-parabolic bundle on  $X$  of rank  $n$ , degree  $d$  and parabolic type  $\Delta$ . Let  $\mathcal{A}$  be the space of holomorphic structures in  $E$ , and  $\mathcal{A}_{\text{par}}^s$  the open subset of  $\mathcal{A}$  consisting of holomorphic structures

which are parabolic stable with respect to the datum  $\Delta$ . Let  $\mathcal{G}$  denote the gauge group of  $E$ , and denote  $\bar{\mathcal{G}} = \mathcal{G}/\mathbf{C}^*$  and  $\bar{\mathcal{G}}_{\text{par}} = \mathcal{G}_{\text{par}}/\mathbf{C}^*$ , where  $\mathbf{C}^*$  is the constant scalar subgroup of  $\mathcal{G}$ . There is a natural action of  $\bar{\mathcal{G}}$  on  $\mathcal{A}$ , which induces a free action of  $\bar{\mathcal{G}}_{\text{par}}$  on  $\mathcal{A}_{\text{par}}^s$ , and there is a canonical homeomorphism of  $\mathcal{A}_{\text{par}}^s/\bar{\mathcal{G}}_{\text{par}}$  with  $\mathcal{U}_X(n, d, \Delta)$ , hence we will identify them with each other from now on.

**Notation 4.1** If  $G$  is a topological group, and  $T$  is a  $G$ -space, then  $T(G)$  denotes the homotopy quotient  $EG \times_G T$ . (We write  $T(G)$  instead of the standard notation  $T_G$  for reasons of convenience.)

**Remark 4.2** Note that for any  $G$ -space  $T$ , there are two canonical maps  $T(G) \rightarrow BG$  and  $T(G) \rightarrow T/G$ . The first map is a fibre bundle over  $BG$  with fibre  $T$ , and is a homotopy equivalence if  $T$  is contractible. The second map is a homotopy equivalence of  $T(G)$  with  $T/G$  if  $T$  is a free  $G$ -space.

Consider the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{C}^* & \longrightarrow & \mathcal{G}_{\text{par}} & \xrightarrow{f} & \bar{\mathcal{G}}_{\text{par}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{C}^* & \longrightarrow & \mathcal{G} & \xrightarrow{f'} & \bar{\mathcal{G}} & \longrightarrow & 1 \end{array}$$

where the vertical maps are the canonical inclusions, and  $f$  and  $f'$  denote the canonical projections. This induces a diagram

$$\begin{array}{ccc} B\mathcal{G}_{\text{par}} & \xrightarrow{\pi} & B\bar{\mathcal{G}}_{\text{par}} \\ \downarrow & & \downarrow \\ B\mathcal{G} & \xrightarrow{\pi'} & B\bar{\mathcal{G}} \end{array}$$

of fibrations; the fibres of  $\pi$  and  $\pi'$  are homeomorphic to  $BU(1)$ .

Since  $\mathcal{A}$  is contractible, by Remark 4.2, the natural map  $\phi : \mathcal{A}(\mathcal{G}_{\text{par}}) \rightarrow B\mathcal{G}_{\text{par}}$  is a homotopy equivalence. Let  $\mathcal{E}_{\text{par}}$  denote the universal bundle on  $B\mathcal{G}_{\text{par}} \times X$ , and let  $V$  denote the bundle on  $\mathcal{A}_{\text{par}}^s(\mathcal{G}_{\text{par}}) \times X$  obtained by pulling back  $\mathcal{E}_{\text{par}}$  via the composition

$$\mathcal{A}_{\text{par}}^s(\mathcal{G}_{\text{par}}) \times X \xrightarrow{\lambda \times 1} \mathcal{A}(\mathcal{G}_{\text{par}}) \times X \xrightarrow{\phi \times 1} B\mathcal{G}_{\text{par}} \times X,$$

where  $\lambda : \mathcal{A}_{\text{par}}^s(\mathcal{G}_{\text{par}}) \hookrightarrow \mathcal{A}(\mathcal{G}_{\text{par}})$  denotes the inclusion map.

**Remark 4.3** The  $\mathcal{G}_{\text{par}}$ -equivariant perfectness of a certain stratification (see Nitsure [9]) implies that the inclusion  $\lambda$  induces a surjection in rational cohomology. Thus, by Theorem 3.9, the Chern classes  $c_j(\mathcal{H}om(V^{x,i}, V^{x,i-1}))$  and the slant products  $c_1(V)/z$  ( $z \in H_1(X)$ ),  $c_1(V)/[x_0]$  ( $x_0$  a fixed base point in  $X$ ) and  $a_i(\mathbf{P}(V))/[y]$  ( $y \in H_r(X)$ ,  $0 \leq r \leq 2$ ,  $2 \leq i \leq n$ ) generate the algebra  $H^*(\mathcal{A}_{\text{par}}^s(\mathcal{G}_{\text{par}}))$ .

**Proof of Theorem 1.4:** Let notation be as above, and as in Theorem 1.4. Since the action of  $\bar{\mathcal{G}}_{\text{par}}$  on  $\mathcal{A}_{\text{par}}^s$  is free, the canonical map  $\psi : \mathcal{A}_{\text{par}}^s(\bar{\mathcal{G}}_{\text{par}}) \rightarrow \mathcal{U}_X(n, d, \Delta)$  is a homotopy equivalence. Let  $V'$  denote the bundle on  $\mathcal{A}_{\text{par}}^s(\mathcal{G}_{\text{par}}) \times X$  obtained by pulling back the universal bundle

$U \rightarrow \mathcal{U}_X(n, d, \Delta) \times X$  by the composition

$$\mathcal{A}_{\text{par}}^s(\mathcal{G}_{\text{par}}) \times X \xrightarrow{\pi \times 1} \mathcal{A}_{\text{par}}^s(\bar{\mathcal{G}}_{\text{par}}) \times X \xrightarrow{\psi \times 1} \mathcal{U}_X(n, d, \Delta) \times X$$

where  $\pi$  is induced by  $\pi : B\mathcal{G}_{\text{par}} \rightarrow B\bar{\mathcal{G}}_{\text{par}}$  above. Recall now that we have constructed above another family  $V$  on  $\mathcal{A}_{\text{par}}^s(\mathcal{G}_{\text{par}}) \times X$  using  $B\mathcal{G}_{\text{par}}$ . Now  $V$  and  $V'$  are families of parabolic stable bundles parametrized by  $\mathcal{A}_{\text{par}}^s(\mathcal{G}_{\text{par}})$  such that for each  $t \in \mathcal{A}_{\text{par}}^s(\mathcal{G}_{\text{par}})$ ,  $V_t \cong V'_t$ . Therefore, there exists a line bundle  $\xi$  on  $\mathcal{A}_{\text{par}}^s(\mathcal{G}_{\text{par}})$  such that  $V' \cong V \otimes p^*\xi$ , where  $p : \mathcal{A}_{\text{par}}^s(\mathcal{G}_{\text{par}}) \times X \rightarrow \mathcal{A}_{\text{par}}^s(\mathcal{G}_{\text{par}})$  is the canonical projection. This implies that  $\mathbf{P}(V) \cong \mathbf{P}(V')$ ,  $\mathcal{H}om(V^{x,i}, V^{x,i-1}) \cong \mathcal{H}om((V')^{x,i}, (V')^{x,i-1})$ , and  $c_1(V)/z = c_1(V')/z$  for all  $z \in H_1(X)$ .

Thus, if  $W = (\psi \times 1)^*U$ , then  $\mathbf{P}(V) = (\pi \times 1)^*\mathbf{P}(W)$ ,  $\mathcal{H}om(V^{x,i}, V^{x,i-1}) = \pi^*(\mathcal{H}om(W^{x,i}, W^{x,i-1}))$ , and  $c_1(V)/z = \pi^*(c_1(W)/z)$  for all  $z \in H_1(X)$ . Further, we easily see that under the inclusion  $BU(1) \hookrightarrow \mathcal{A}_{\text{par}}^s(\mathcal{G})$  as a fibre of  $\pi$ ,  $c_1(V)/[x_0]$  restricts to a generator of  $H^2(BU(1))$ . Now in general, if  $\pi : E \rightarrow B$  is a fibration with fibre  $F$  such that: (a) the algebra  $H^*(E)$  is generated by certain classes  $\alpha, \beta_1, \dots, \beta_k$ ; (b) the classes  $\beta_i$  are pull-backs of certain classes  $\theta_i \in H^*(B)$  by  $\pi$ ; and (c)  $H(F)$  is a polynomial algebra on  $\alpha_F$ , where  $\alpha_F$  denotes the restriction of  $\alpha$  to  $F$ ; then,  $H^*(B)$  is generated by  $\theta_1, \dots, \theta_k$ . This fact applies in our situation because of the above observations and because of Remark 4.3, and implies that the Chern classes  $c_j(\mathcal{H}om(W^{x,i}, W^{x,i-1}))$  and the slant products  $c_1(W)/z$  ( $z \in H_1(X)$ ) and  $a_i(\mathbf{P}(W))/y$  ( $y \in H_r(X)$ ,  $0 \leq r \leq 2$ ,  $2 \leq i \leq n$ ) generate  $H^*(\mathcal{A}_{\text{par}}^s(\bar{\mathcal{G}}_{\text{par}}))$ . Since  $\psi : \mathcal{A}_{\text{par}}^s(\bar{\mathcal{G}}_{\text{par}}) \rightarrow \mathcal{U}_X(n, d, \Delta)$  is a homotopy equivalence, we are done.  $\square$

**Proof of Theorem 1.5:** Suppose  $U$  is a universal bundle on  $SU_X(n, L, \Delta) \times X$ . Consider the right action of the  $n$ -torsion subgroup  $\Gamma_X(n)$  of the Jacobian  $J_X$  on  $SU_X(n, L, \Delta) \times X$ , defined by  $(E, \alpha) \cdot \zeta = (E \otimes \zeta, \zeta^{-1} \otimes \alpha)$ , where  $E \in SU_X(n, L, \Delta)$ ,  $\alpha \in J_X$  and  $\zeta \in \Gamma_X(n)$ . Then the map

$$\pi : SU_X(n, L, \Delta) \times J_X \rightarrow \mathcal{U}_X(n, d, \Delta), \quad (E, \alpha) \mapsto E \otimes \alpha$$

is a principal  $\Gamma_X(n)$ -bundle, i.e., a Galois covering with Galois group  $\Gamma_X(n)$ . On the other hand, the Poincaré polynomials of  $\mathcal{U}_X(n, d, \Delta)$  and

$SU_X(n, L, \Delta) \times J_X$  are equal (see Nitsure [9], Remark 3.11). Since  $\Gamma_X(n)$  is a finite group, this means that the action of  $\Gamma_X(n)$  on  $SU_X(n, L, \Delta) \times J_X$  induces a trivial action on the cohomology of  $SU_X(n, L, \Delta) \times J_X$ , or equivalently that the map  $\pi$  induces an isomorphism in rational cohomology. (In the case of usual vector bundles, the triviality of the action of  $\Gamma_X(n)$  on the rational cohomology of  $SU_X(n, L)$ , where  $n$  and the degree of  $L$  are coprime, is a theorem

of Harder and Narasimhan [6], who proved it using arithmetic techniques. It was reproved by Atiyah and Bott [1] using gauge theory. The methods of Nitsure [9] generalize the approach of Atiyah and Bott [1] to parabolic bundles.) Now, if

$$i : SU_X(n, L, \Delta) \longrightarrow SU_X(n, L, \Delta) \times J_X$$

denotes the map  $E \mapsto (E, \mathcal{O}_X)$ , and if  $j : SU_X(n, L, \Delta) \rightarrow \mathcal{U}_X(n, d, \Delta)$  denotes the inclusion, then  $j = \pi \circ i$ . Since  $\pi^*$  is an isomorphism and  $i^*$  is surjective, we see that

$$j^* : H^*(\mathcal{U}_X(n, d, \Delta)) \longrightarrow H^*(SU_X(n, L, \Delta))$$

is surjective. Now, let  $\tilde{V}$  be an arbitrary universal bundle on  $\mathcal{U}_X(n, d, \Delta) \times X$ , and denote the restriction of  $\tilde{V}$  to  $SU_X(n, L, \Delta) \times X$  by  $V$ . Then, Theorem 1.4 applied to  $\tilde{V}$ , and the surjectivity of  $j^*$  imply that the Chern classes  $c_j(\text{Hom}(V^{x,i}, V^{x,i-1}))$  and the slant products  $c_1(V)/z$  ( $z \in H_1(X)$ ) and  $a_i(\mathbf{P}(V))/y$  ( $y \in H_r(X)$ ,  $0 \leq r \leq 2$ ,  $2 \leq i \leq n$ ) generate  $H^*(SU_X(n, L, \Delta))$ . But  $SU_X(n, L, \Delta)$  is simply connected, so the classes  $c_1(V)/z$  ( $z \in H_1(X)$ ) are all zero. Finally, since  $U$  and  $V$  are both universal bundles on

$SU_X(n, L, \Delta) \times X$ , they differ by a line bundle coming from  $SU_X(n, L, \Delta)$ , hence

$$\text{Hom}(V^{x,i}, V^{x,i-1}) \cong \text{Hom}(U^{x,i}, U^{x,i-1}) \text{ and } \mathbf{P}(V) \cong \mathbf{P}(U). \quad \square$$

**Proof of Corollary 1.6:** If  $U^x = j_x^* U$  ( $x \in J$ ), then the exact sequence

$$0 \longrightarrow S^x \longrightarrow U^x \longrightarrow Q^x \longrightarrow 0$$

implies that  $U^x \cong S^x \oplus Q^x$  topologically. Since  $S^x$  is either zero or a line bundle,  $S_x^* \otimes S_x$  is either zero or a trivial line bundle, and hence  $c_j(\text{Hom}(S^x, S^x)) = c_j(\text{Hom}(S_x^*, S_x))$ . Finally, Example 2.5 (1) implies that

$$a_2(\mathbf{P}(U)) = c_2(\mathcal{E}nd U). \quad \square$$

**Proof of Proposition 1.7:** As in Atiyah and Bott (see [1], Section 9), the crux of the proof consists in finding a holomorphic  $\mathcal{G}_{\text{par}}$ -line bundle  $\xi$  on  $\mathcal{A}_{\text{par}}^s$  on which  $\mathbf{C}^* \subset \mathcal{G}_{\text{par}}$  acts via the identity homomorphism  $\mathbf{C}^* \rightarrow \mathbf{C}^*$ ,  $t \mapsto t$ . Let  $U = \mathcal{A}_{\text{par}}^s \times E$  and  $U^{x,i} = \mathcal{A}_{\text{par}}^s \times F^i E_x$  ( $x \in J$ ), where  $E$  is the fixed  $C^\infty$  quasi-parabolic bundle under consideration. If we let  $\mathcal{G}_{\text{par}}$  act trivially on  $X$ , then  $U$  and  $U^{x,i}$  are naturally  $\mathcal{G}_{\text{par}}$ -vector bundles on which  $\mathbf{C}^*$  acts by the identity homomorphism. Fix a line bundle  $\mathcal{O}_X(1)$  of degree 1 on  $X$ , and for each  $k \in \mathbf{Z}$ , let  $U(k) = U \otimes q^* \mathcal{O}_X(k)$ , where  $q : \mathcal{A}_{\text{par}}^s \times X \rightarrow X$  denotes the canonical projection. Denote by  $\text{Det } U(k)$  the determinant line bundle of  $U(k)$  in the sense of Quillen [11]. Then  $\text{Det } U(k)$  is a holomorphic  $\mathcal{G}_{\text{par}}$ -line bundle over  $\mathcal{A}_{\text{par}}^s$  on which  $\mathbf{C}^*$  acts by the homomorphism  $t \mapsto t^{N+kn}$ , where  $N = d + n(1 - g)$ ,  $g$  being the genus of  $X$ . If  $(n, d) = 1$ , let  $a, b \in \mathbf{Z}$  be such that  $an + bN = 1$ , and take

$$\xi = (\text{Det } U(1))^a \otimes (\text{Det } U)^{b-a};$$

then  $\mathbf{C}^*$  acts by the identity homomorphism on  $\xi$ . If  $\sum_{i=j}^{k_x} n_{x,i}$  and  $n$  are coprime for some  $x$  and  $j$ , then the rank  $m$  of  $U^{x,j}$  and  $n$  are coprime; let  $a, b \in \mathbf{Z}$  be such that  $am + bn = 1$ , and take

$$\xi = (\det U^{x,j})^a \otimes (\text{Det } U)^{-b} \otimes (\text{Det } U(1))^b;$$

then  $\xi$  has the required property. Lastly, if  $\sum_{i=j}^{k_x} n_{x,i}$  and  $n + d$  are coprime for some  $x$  and  $j$ , let  $a, b \in \mathbf{Z}$  be such that  $am + b(d+n) = 1$ , where  $m$  is the rank of  $U^{x,i}$ ; then

$$\xi = (\det U^{x,j})^a \otimes (\text{Det } U(1))^b \otimes (\det U^x)^{b(g-1)}$$

will do, where  $U^x = j_x^* U$ .  $\square$

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## References

- [1] Atiyah, M.F., Bott, R.: The Yang-Mills equations on Riemann surfaces. *Philos. Trans. R. Soc. London A* **308**, 523-615 (1982)
- [2] Beauville, A.: Sur les cohomologie de certains espaces du modules de fibrés vectoriels. To appear in the Proceedings of the Int. Coll. on Geometry and Analysis, Bombay, 1992.
- [3] Borel, A., Hirzebruch, F.: Characteristic classes and homogeneous spaces I. *Amer. J. Math.* **80**, 458-538 (1958)
- [4] Donaldson, S.K., Kronheimer, P.B.: The geometry of four-manifolds. Oxford: Clarendon Press 1990
- [5] Drezet, J.M.: Cohomologie du groupe de jauge. In: Verdier, J-L., Le Potier, J. (eds.): *Module des fibrés stable sur les courbes algébriques*, Notes de l'Ecole Normale Supérieure, Printemps, 1983 (Progress in Math., vol. 54, pp. 51-80) Boston Basel Stuttgart : Birkhauser 1985
- [6] Harder, G., Narasimhan, M.S.: On the cohomology groups of moduli spaces. *Math. Ann.* **212**, 215-248 (1975)
- [7] Mehta, V.B., Seshadri, C.S.: Moduli of vector bundles on curves with parabolic structures. *Math. Ann.* **248**, 205-239 (1980)
- [8] Newstead, P.E.: Characteristic classes of stable bundles of rank 2 over an algebraic curve. *Trans. Am. Math. Soc.* **169**, 337-345 (1972)
- [9] Nitsure, N.: Cohomology of the moduli of parabolic vector bundles. *Proc. Indian Acad. Sci. (Math. Sci)* **95**, 61-77 (1986)
- [10] Pressley, A., Segal, G.: *Loop Groups*. Oxford: Clarendon Press 1986
- [11] Quillen, D.: Determinants of Cauchy-Riemann operators over a Riemann surface. *Funct. Anal. Appl.* **19**, 31-34 (1985)

