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TECHNICAL REPORT
ELECTRIC FIELD IN NOT COMPLETELY SYMMETRIC SYSTEMS

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ABSTRACT

In this paper is studied theoretically the electric field in the not completely symmetric system earthed metalic sphere-uniformly charged dielectric plan, for sphere surface points situated in the plan that contains sphere's center and vertical symmetry axe of dielectric plan.

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1 Introduction

In the practic problems of electrostatic measurements we often meet the earthed metalic sphere-uniformly charged dielectric plan system [1–8], and for this reason its theoretical study is of great importance. For the first time the electric field of such a system we have studied in [9, 10], where is taken into consideration a completely symmetric system between the sphere and plan. But in the multiplexity of electrostatic measurements we can be encountered with some cases in which such a system is not completely symmetric. The theoretical study of this case is the main subject of this paper.

2 Electric field intensity in the not completely symmetric system earthed sphere – uniformly charged dielectric plan

Let us consider the not completely symmetric system that is composed by an earthed metalic sphere with radius r, and a rectangular dielectric plan with sides a and b, uniformly charged by a surface density σ, placed in the distance D from the sphere (Fig.1). We are going to study the case when the perpendicular from the centre O of the sphere, to the plan, intersects the last one in O_0, that is situated only in one of the symmetry axes (in our case in the vertical axe O_0y_0, Fig.1). For the calculations let us select two parallel coordinate systems (xyz) and $(x_0y_0z_0)$ with respective centers in O and O_0. The fact that O_0 is not situated in the centre of the dielectric plan bears witness that the system is not completely symmetric.

The electric field potential created in the point $P(xyz)$ from sphere–plan system, [9, 10] is:

$$\varphi = \frac{\sigma}{4\pi\varepsilon_0}\left\{\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0\;dz_0}{\sqrt{(x_Q-x)^2+(y_Q-y)^2+(z_Q-z)^2}}-\right.$$
$$\left.-r\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0\;dz_0}{\ell\sqrt{(x-x')^2+(y-y')^2+(z-z)^2}}\right\} \tag{1}$$

Let us note:

$$\left.\begin{aligned}\varphi_1 &= \frac{-\sigma r}{4\pi\varepsilon_0}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0\;dz_0}{\sqrt{(x_Q-x)^2+(y_Q-y)^2+(z_Q-z)^2}}\\ \varphi_2 &= \frac{\sigma}{4\pi\varepsilon_0}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0\;dz_0}{\ell\sqrt{(x-x')^2+(y-y')^2+(z-z)^2}}\end{aligned}\right\} \tag{2}$$

The (1) could be presented in the form

$$\varphi = \varphi_1 + \varphi_2 \tag{3}$$

To determine the surface charge density, we are interested in determining of electric field intensity at the points of spherical surface which are situated in the plan that contains the center O of sphere and the vertical symmetry axe y_0, of the uniformly charged plan. In this case the field components intensity for these points are:

$$E_x \neq 0, \quad E_y \neq 0, \quad E_z = 0 \tag{4}$$

To obtain the field intensity at the points of our interest, is enough to find E_x and E_y components.

2.1 Determination of E_x

For the determination of E_x, we must be restricted be in the points that satisfy the relation $y = z = 0$. In this case taking into account (1), (2), (3), the potential of x axe points can be presented:

$$\varphi_x = \frac{\sigma}{4\pi\varepsilon_0}\left\{\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0\ dz_0}{\sqrt{(D-x)^2+y_0^2+z_0^2}} - r\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0\ dz_0}{\ell\sqrt{(x-x')^2+y'^2+z'^2}}\right\}, \quad (5)$$

because of $x_Q = D, y_Q = y_0, z_Q = z_0$ (Fig.1).

Let us note:

$$\begin{aligned}\varphi_{1x} = & \frac{\sigma}{4\pi\varepsilon_0}\left\{\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0\ dz_0}{\sqrt{(D-x)^2+y_0^2+z_0^2}},\right. \\ \varphi_{2x} = & \left.\frac{-\sigma r}{4\pi\varepsilon_0}\int_{-a/2}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0\ dz_0}{\ell\sqrt{(x-x')^2+y'^2+z'^2}}\right\},\end{aligned} \quad (6)$$

then

$$\varphi_x = \varphi_{1x} + \varphi_{2x} \quad (7)$$

and

$$E_x = \frac{-\partial\varphi_x}{\partial x} = \frac{-\partial\varphi_{1x}}{\partial x} - \frac{\partial\varphi_{2x}}{\partial x}$$

2.1.1 Determination of $-\frac{\partial\varphi_{1x}}{\partial x}$

According to (6) we can write:

$$-\frac{\partial\varphi_{1x}}{\partial x} = -\frac{\sigma}{4\pi\varepsilon_0}\frac{\partial}{\partial x}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0\ dz_0}{\sqrt{(D-x)^2+y_0^2+z_0^2}},$$

while according to [9]:

$$\begin{aligned}\int_{b_1}^{b_2}\frac{dy_0}{\sqrt{(D-x)^2+z_0^2+y_0^2}} &= \ell n\left(b_2+\sqrt{(D-x)^2+z_0^2+b_2^2}\right) - \\ &-\ell n\left(b_1+\sqrt{(D-x)^2+z_0^2+b_1^2}\right)\end{aligned}$$

Then

$$\begin{aligned}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0dz_0}{\sqrt{(D-x)^2+z_0^2+y_0^2}} = 2\left[\int_0^{\frac{a}{2}}\ell n\left(b_2+\sqrt{(D-x)^2+b_2^2+z_0^2}\right)dz_0 - \right. \\ \left. -\int_0^{\frac{a}{2}}\ell n\left(b_1+\sqrt{(D-x)^2+b_1^2+z_0^2}\right)dz_0\right]\end{aligned}$$

Taking into consideration the Appendix, we can write:

$$\int_0^{\frac{a}{2}} \ell n\left(\alpha+\sqrt{\beta+\gamma z_0^2}\right) dz_0 = \frac{a}{2}\ell n\left(\alpha+\sqrt{\beta+\frac{\gamma a^2}{4}}\right) - \frac{a}{2} + \frac{\alpha}{\sqrt{\gamma}}\cdot$$

$$\ell n\frac{\frac{a}{2}\sqrt{\gamma}+\sqrt{\beta+\gamma\frac{a^2}{4}}}{\sqrt{\beta}} + \sqrt{\frac{\beta-\alpha^2}{\gamma}}\cdot\left[\frac{\pi}{2} - arcsin\,\frac{\alpha-\frac{\alpha^2-\beta}{\alpha+\sqrt{\beta+\frac{\gamma a^2}{4}}}}{\sqrt{\beta}}\right] \tag{8}$$

The relation (8) true for $\alpha^2-\beta<0$ (see the Appendix).
According to (8):

$$\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0dz_0}{\sqrt{(D-x)^2+z_0^2+y_0^2}} = a\ \ell n\ \frac{b_2+\sqrt{(D-x)^2+b_2^2+\frac{a^2}{4}}}{b_1+\sqrt{(D-x)^2+b_1^2+\frac{a^2}{4}}}+$$

$$+2\left(b_2\ \ell n\ \frac{\frac{a}{2}+\sqrt{(D-x)^2+b_2^2+\frac{a^2}{4}}}{\sqrt{(D-x)^2+b_2^2}} - b_1\ \ell_n\ \frac{\frac{a}{2}+\sqrt{(D-x)^2+b_1^2+\frac{a^2}{4}}}{\sqrt{(D-x)^2+b_1^2}}\right)+$$

$$+2(D-x)\left[arcsin\ \delta_1(b_1) - arcsin\ \delta_1(b_2)\right], \tag{9}$$

where

$$\left.\begin{aligned} \delta_1(b_1) &= \frac{b_1+\frac{(D-x)^2}{b_1+\sqrt{(D-x)^2+b_1^2+\frac{a^2}{4}}}}{\sqrt{(D-x)^2+b_1^2}} \\ \delta_1(b_2) &= \frac{b_2+\frac{(D-x)^2}{b_2+\sqrt{(D-x)^2+b_2^2+\frac{a^2}{4}}}}{\sqrt{(D-x)^2+b_2^2}} \end{aligned}\right\} \tag{10}$$

Finally, taking into consideration the relations (6), (9), (10) and by noting

$$k_1(x) = \frac{\partial}{\partial x}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0\ dz_0}{\sqrt{(D-x)^2+z_0^2+y_0^2}}$$

we can write

$$\frac{-\partial\varphi_{1x}}{\partial x} = \frac{-\sigma}{4\pi\varepsilon_0}\ k_1(x) \tag{11}$$

where

$$k_1(x) = 2(D-x)\left\{\frac{a}{2}\left[\frac{1}{\sqrt{(D-x)^2+b_1^2+\frac{a^2}{4}}\ \left(b_1+\sqrt{(D-x)^2+b_1^2+\frac{a^2}{4}}\right)}-\right.\right.$$

$$\left.-\frac{1}{\sqrt{(D-x)^2+b_2^2+\frac{a^2}{4}}\ \left(b_2+\sqrt{(D-x)^2+b_2^2+\frac{a^2}{4}}\right)}\right] + \left(\frac{\partial}{\partial x} - \frac{1}{D-x}\right)\left[arcsin\ \delta_1(b_1)-\right.$$

$$-arcsin\ \delta_1(b_2)]+b_2\left[\frac{1}{(D-x)^2+b_2^2}-\frac{1}{\sqrt{(D-x)^2+b_2^2+\frac{a^2}{4}}\left(\frac{a}{2}+\sqrt{(D-x)^2+b_2^2+\frac{a^2}{4}}\right)}\right]-$$

$$-b_1\left[\frac{1}{(D-x)^2+b_1^2}-\frac{1}{\sqrt{(D-x)^2+b_1^2+\frac{a^2}{4}}\left(\frac{a}{2}+\sqrt{(D-x)^2+\frac{a^2}{4}+b_1^2}\right)}\right]\Bigg\}\qquad(12)$$

2.1.2 Determination of $-\frac{\partial\varphi_2}{\partial x}$

According to (2) and [9]

$$-\frac{\partial\varphi_{2x}}{\partial x}=\frac{-\sigma r}{4\pi\varepsilon_0}\left\{x\int_{-a/2}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{(y_0^2+z_0^2)dy_0\,dz_0}{[(Dx-r^2)^2+(y_0^2+z_0^2)x^2]^{3/2}}-D(r^2-Dx)\cdot\right.$$
$$\left.\cdot\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0\,dz_0}{[(Dx-r^2)+(y_0^2+z_0^2)x^2]^{3/2}}\right\}\qquad(13)$$

Let us study at the beginning:

$$\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0\,dz_0}{[(Dx-r^2)^2+(y_0^2+z_0^2)x^2]^{3/2}}$$
$$\int_{b_1}^{b_2}\frac{dy_0}{[(Dx-r^2)^2+x^2z_0^2+x^2y_0^2]^{3/2}}=\int_{b_1}^{b_2}\frac{dy_0}{(\alpha_1+\beta_1y_0^2)^{3/2}},$$

where $\alpha_1=(Dx-r^2)^2+x^2z_0^2$; $\beta_1=x^2$.

According to [11]

$$\int_{b_1}^{b_2}\frac{dy_0}{(\alpha_1+\beta_1y_0^2)^{3/2}}=\frac{1}{\alpha_1}\left\{\frac{b_2}{\sqrt{\alpha_1+\beta_1b_2^2}}-\frac{b_1}{\sqrt{\alpha_1+\beta_1b_1^2}}\right\},$$

so

$$\int_{b_1}^{b_2}\frac{dy_0}{[(Dx-r^2)^2+x^2z_0^2+x^2y_0^2]^{3/2}}=$$
$$=\frac{b_2}{[(Dx-r^2)^2+x^2z_0^2]\sqrt{(Dx-r^2)^2+x^2b_2^2+x^2z_0^2}}-$$
$$-\frac{b_1}{[(Dx-r^2)^2+x^2z_0^2]\sqrt{(Dx-r^2)^2+x^2b_1^2+x^2z_0^2}},$$

and

$$\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0\,dz_0}{[(Dx-r^2)^2+(y_0^2+z_0^2)x^2]^{3/2}}=$$
$$=2\left[b_2\int_0^{\frac{a}{2}}\frac{dz_0}{[(Dx-r^2)^2+x^2z_0^2]\sqrt{(Dx-r^2)^2+x^2b_2^2+x^2z_0^2}}\right.$$
$$\left.-\int_0^{\frac{a}{2}}\frac{dz_0}{[(Dx-r^2)^2+x^2z_0^2]\sqrt{(Dx-r^2)^2+x^2b_1^2+x^2z_0^2}}\right]$$

Let us note: $\alpha_2 = (Dx - r^2)^2 + x^2 b_2^2$, then

$$\int_0^{\frac{a}{2}} \frac{dz_0}{[(Dx-r^2)^2 + x^2 z_0^2]\sqrt{(Dx-r^2)^2 + x^2 b_2^2 + x^2 z_0^2}} = \int_0^{\frac{a}{2}} \frac{dz_0}{\alpha_1 \sqrt{\alpha_2 + \beta_1 z_0^2}} =$$

$$= \int_0^{\frac{a}{2}} \frac{dz_0}{[p + (\alpha_2 + \beta_1 z_0^2)]\sqrt{\alpha_2 + \beta_1 z_0^2}},$$

where $p = -x^2 b_2^2 < 0$. According to [11]

$$\int \frac{Ax + B}{(p+R)\sqrt{R}}\, dx = \frac{A}{c}\, I_1 + \frac{2Bc - Ab}{\sqrt{c^2 p[b^2 - 4(a+p)c]}} \cdot I_2,$$

where $R = a + bx + cx^2$ and in our case:

$$I_1 = 0, \quad I_2 = arctg \frac{x b_2\, \beta_1\, z_0}{\sqrt{(\alpha_2 - x^2 b_2^2)\beta_1(\alpha_2 + \beta_1 z_0^2)}}$$

So

$$\int_0^{\frac{a}{2}} \frac{dz_0}{[p + (\alpha_2 + \beta_1 z_0^2)]\sqrt{\alpha_2 + \beta_1 z_0^2}} = \frac{\beta_1}{x b_2 \sqrt{\beta_1(\alpha_2 - x^2 b_2^2)}} \cdot \cdot arctg \frac{x b_2\, \beta_1\, z_0}{\sqrt{(\alpha_2 - x^2 b_2^2)\beta_1(\alpha_2 + \beta_1 z_0^2)}} \tag{14}$$

Passing in the initial variables, there is

$$\int_0^{\frac{a}{2}} \frac{dz_0}{[(Dx-r^2)^2 + x^2 z_0^2]\sqrt{(Dx-r^2)^2 + x^2 b_2^2 + x^2 z_0^2}} = \frac{1}{x^2 b_2 (Dx - r^2)} \cdot \cdot arctg \frac{a b_2 x^2}{2(Dx-r^2)\sqrt{(Dx-r^2)^2 + x^2 b_2^2 + \frac{x^2 a^2}{4}}} \tag{15}$$

Analogically:

$$\int_0^{\frac{a}{2}} \frac{dz_0}{[(Dx-r^2)^2 + x^2 z_0^2]\sqrt{(Dx-r^2)^2 + x^2 b_1^2 + x^2 z_0^2}} = \frac{1}{x^2 b_1 (Dx - r^2)} \cdot \cdot arctg \frac{a b_1 x^2}{2(Dx-r^2)\sqrt{(Dx-r^2)^2 + x^2 b_1^2 + \frac{x^2 a^2}{4}}} \tag{16}$$

So

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{b_1}^{b_2} \frac{dy_0\, dz_0}{[(Dx-r^2)^2 + (y_0^2 + z_0^2)x^2]^{3/2}} = \frac{2}{x^2(Dx - r^2)} \cdot \cdot \left[arctg \frac{a b_2 x^2}{2(Dx-r^2)\sqrt{(Dx-r^2)^2 + x^2 b_2^2 + x^2 \frac{a^2}{4}}} - - arctg \frac{a b_1 x^2}{2(Dx-r^2)\sqrt{(Dx-r^2)^2 + x^2 b_1^2 + x^2 \frac{a^2}{4}}} \right] \tag{17}$$

Let us study the integral

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{b_1}^{b_2} \frac{(y_0^2 + z_0^2)dy_0\, dz_0}{[(Dx-r^2)^2 + (y_0^2 + z_0^2)x^2]^{3/2}}$$

Beginning with the integral

$$\int_{b_1}^{b_2} \frac{y_0^2\, dy_0}{[(Dx-r^2)^2+x^2z_0^2+x^2y_0^2]^{3/2}} = \int_{b_1}^{b_2} \frac{y_0^2\, dy_0}{(\alpha_1+\beta_1 y_0^2)^{3/2}}$$

According to [11]

$$\int \frac{x^2 dx}{(a+cx^2)^{3/2}} = \frac{-x}{c\sqrt{a+cx^2}} + \frac{1}{c}\cdot I_1,$$

and $I_1 = \frac{1}{\sqrt{c}}\, \ell n\left(x\sqrt{c}+\sqrt{a+cx^2}\right)$. In our case $c>0$, so we can write:

$$\int_{b_1}^{b_2} \frac{y_0^2 dy_0}{(\alpha_1+\beta_1 y_0^2)^{3/2}} = \frac{b_1}{\beta_1\sqrt{\alpha_1+\beta_1 b_1^2}} - \frac{b_2}{\beta_1\sqrt{\alpha_1+\beta_1 b_2^2}} +$$

$$+\frac{1}{\beta_1\sqrt{\beta_1}}\,\ell n \frac{b_2\sqrt{\beta_1}+\sqrt{\alpha_1+\beta_1 b_2^2}}{b_1\sqrt{\beta_1}+\sqrt{\alpha_1+\beta_1 b_1^2}} \tag{18}$$

Passing in the initial variables there is:

$$\int_{b_1}^{b_2} \frac{y_0^2\, dy_0}{[(Dx-r^2)^2+x^2z_0^2+x^2y_0^2]^{3/2}} = \frac{b_1}{x^2\sqrt{(Dx-r^2)^2+x^2b_1^2+x^2z_0^2}} -$$

$$-\frac{b_2}{x^2\sqrt{(Dx-r^2)^2+x^2b_2^2+x^2z_0^2}} + \frac{1}{x^3}\,\ell n \frac{b_2 x+\sqrt{(Dx-r^2)^2+x^2b_2^2+x^2z_0^2}}{b_1 x+\sqrt{(Dx-r^2)^2+x^2b_1^2+x^2z_0^2}} \tag{19}$$

Then

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{b_1}^{b_2} \frac{y_0^2 dy_0 dz_0}{[(Dx-r^2)^2+x^2(y_0^2+z_0^2)]^{3/2}} = \frac{2b_1}{x^2}\int_0^{\frac{a}{2}} \frac{dz_0}{\sqrt{(Dx-r^2)^2+x^2b_1^2+x^2z_0^2}} -$$

$$-\frac{2b_2}{x^2}\int_0^{\frac{a}{2}} \frac{dz_0}{\sqrt{(Dx-r^2)^2+x^2b_2^2+x^2z_0^2}} + \frac{2}{x^3}\left\{\int_0^{\frac{a}{2}} \ell n[b_2x+\sqrt{(Dx-r^2)^2+x^2b_2^2+x^2z_0^2}]dz_0\right.$$

$$\left. -\int_0^{\frac{a}{2}} \ell n[b_1x+\sqrt{(Dx-r^2)^2+x^2b_1^2+x^2z_0^2}]dz_0\right\} \tag{20}$$

Beginning with the integral

$$\int_0^{\frac{a}{2}} \frac{dz_0}{\sqrt{(Dx-r^2)^2+x^2b_2^2+x^2z_0^2}} = \int_0^{\frac{a}{2}} \frac{dz_0}{\sqrt{\alpha_2+\beta_1 z_0^2}} \tag{21}$$

According to [11]

$$\int \frac{dx}{\sqrt{a+cx^2}} = \frac{1}{\sqrt{c}}\ell n(x\sqrt{c}+\sqrt{a+cx^2})\ ,$$

for $c>0$. So we can write

$$\int_0^{\frac{a}{2}} \frac{dz_0}{\sqrt{(Dx-r^2)^2+x^2b_2^2+x^2z_0^2}} = \frac{1}{x}\ell n \frac{\frac{ax}{2}+\sqrt{(Dx-r^2)^2+x^2b_2^2+\frac{a^2x^2}{4}}}{\sqrt{(Dx-r^2)^2+x^2b_2^2}} \tag{22}$$

Analogically

$$\int_0^{\frac{a}{2}} \frac{dz_0}{\sqrt{(Dx-r^2)^2+x^2b_1^2+x^2z_0^2}} = \frac{1}{x}\ell n \frac{\frac{ax}{2}+\sqrt{(Dx-r^2)^2+x^2b_1^2+\frac{a^2x^2}{4}}}{\sqrt{(Dx-r^2)^2+x^2b_1^2}} \tag{23}$$

Let us study the integral:

$$\int_0^{\frac{a}{2}} \ell n[b_2x+\sqrt{(Dx-r^2)^2+x^2b_2^2+x^2z_0^2}]dz_0 = \int_0^{\frac{a}{2}} \ell n(\beta_2+\sqrt{\alpha_2+\beta_1 z_0^2})dz_0 \ ,$$

where $\beta_2 = b_2x$. According to the Appendix:

$$\int_0^{\frac{a}{2}} \ell n(\beta_2+\sqrt{(a_2+\beta_1 z_0^2})dz_0 = \frac{a}{2}\ell n(\beta_2+\sqrt{\alpha_2+\beta_1\frac{a^2}{4}}) - \frac{a}{2}+$$

$$+\frac{\beta_2}{\sqrt{\beta_1}}\ell n\frac{\frac{a}{2}\sqrt{\beta_1}+\sqrt{\alpha_2+\beta_1\frac{a^2}{4}}}{\sqrt{\alpha_2}} + \sqrt{\frac{\alpha_2-\beta_2^2}{\beta_1}}\left[\frac{\pi}{2} - \arcsin \frac{\beta_2 - \frac{\beta_2^2-\alpha_2}{\beta_2+\sqrt{\alpha_2+\beta_1\frac{a^2}{4}}}}{\sqrt{\alpha_2}}\right]$$

Passing in the initial variables there is:

$$\int_0^{\frac{a}{2}} \ell n[b_2x+\sqrt{(Dx-r^2)^2+x^2b_2^2+x^2z_0^2}]dz_0 = \frac{a}{2}\ell n[b_2x+\sqrt{(Dx-r^2)^2+b_2^2x^2+\frac{a^2x^2}{4}}]$$

$$-\frac{a}{2}+b_2\ell n\frac{\frac{ax}{2}+\sqrt{(Dx-r^2)^2+b_2^2x^2+\frac{a^2x^2}{4}}}{\sqrt{(Dx-r^2)^2+b_2^2x^2}} + \frac{Dx-r^2}{x}\left[\frac{\pi}{2} - \arcsin\delta_2(b_2)\right] \tag{24}$$

where

$$\delta_2(b_2) = \frac{b_2x + \frac{(Dx-r^2)^2}{b_2x+\sqrt{(Dx-r^2)^2+b_2^2x^2+\frac{x^2a^2}{4}}}}{\sqrt{(Dx-r^2)^2+b_2^2x^2}}$$

Analogically

$$\int_0^{\frac{a}{2}} \ell n[b_1x+\sqrt{(Dx-r^2)^2+x^2b_1^2+x^2z_0^2}]dz_0 = \frac{a}{2}\ell n[b_1x+\sqrt{(Dx-r^2)^2+b_1^2x^2+\frac{a^2x^2}{4}}]$$

$$-\frac{a}{2}+b_1\ell n\frac{\frac{ax}{2}+\sqrt{(Dx-r^2)^2+b_1^2x^2+\frac{a^2x^2}{4}}}{\sqrt{(Dx-r^2)^2+b_1^2x^2}} + \frac{Dx-r^2}{x}\left[\frac{\pi}{2} - \arcsin\delta_2(b_1)\right] \tag{25}$$

where

$$\delta_2(b_1) = \frac{b_1x + \frac{(Dx-r^2)^2}{b_1x+\sqrt{(Dx-r^2)^2+b_1^2x^2+\frac{x^2a^2}{4}}}}{\sqrt{(Dx-r^2)^2+b_1^2x^{x_2}}}$$

Then, taking into consideration the relations (20), (22), (23), (24) and (25) we can write:

$$\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{y_0^2dy_0dz_0}{[(Dx-r^2)^2+x^2(y_0^2+z_0^2)]^{3/2}}=\frac{2}{x^3}\left\{\frac{a}{2}\ell n\frac{b_2x+\sqrt{(Dx-r^2)^2+x^2(b_2^2+\frac{a^2}{4})}}{b_1x+\sqrt{(Dx-r^2)^2+x^2(b_1^2+\frac{a^2}{4})}}\right.$$

$$\left.+\frac{Dx-r^2}{x}[\arcsin\delta_2(b_1)-\arcsin\delta_2(b_2)]\right\} \qquad (26)$$

Let us study the integral:

$$\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{z_0^2dy_0dz_0}{[(Dx-r^2)^2+x^2(y_0^2+z_0^2)]^{3/2}}\ .$$

Analogically with the relation (19) we have:

$$\int_{-\frac{a}{2}}^{\frac{a}{2}}\frac{z_0^2dz_0}{[(Dx-r^2)^2+x^2y_0^2+x^2z_0^2]^{3/2}}=\frac{2}{x^3}\ell n\frac{\frac{ax}{2}+\sqrt{(Dx-r^2)^2+\frac{a^2x^2}{4}+x^2y_0^2}}{\sqrt{(Dx-r^2)^2+x^2y_0^2}}$$

$$-\frac{a}{x^2\sqrt{(Dx-r^2)^2+\frac{a^2x^2}{4}+x^2y_0^2}}\ .$$

Then

$$\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{z_0^2dy_0dz_0}{[(Dx-r^2)^2+x^2(y_0^2+z_0^2)]^{3/2}}=\frac{2}{x^3}\left[\int_{b_1}^{b_2}\ell n\left(\frac{ax}{2}+\sqrt{(Dx-r^2)^2+\frac{a^2x^2}{4}+x^2y_0^2}\right)dy_0\right.$$

$$-\int_{b_1}^{b_2}\ell n\sqrt{(Dx-r^2)^2+x^2y_0^2}dy_0-\frac{a}{x^2}\int_{b_1}^{b_2}\frac{dy_0}{\sqrt{(Dx-r^2)^2+\frac{a^2x^2}{4}+x^2y_0^2}} \qquad (27)$$

According to the Appendix:

$$\int_{b_1}^{b_2}\ell n\left(\frac{ax}{2}+\sqrt{(Dx-r^2)^2+\frac{a^2x^2}{4}+x^2y_0^2}\right)dy_0=\int_{b_1}^{b_2}(\alpha+\sqrt{\beta+\gamma y_0^2})dy_0=$$

$$=b_2\ell n(\alpha+\sqrt{\beta+\gamma b_2^2})-b_1\ell n(\alpha+\sqrt{\beta+\gamma b_1^2})-(b_2-b_1)+\frac{\alpha}{\sqrt{\gamma}}\ .$$

$$\cdot\ell n\frac{b_2\sqrt{\gamma}+\sqrt{\beta+\gamma b_2^2}}{b_1\sqrt{\gamma}+\sqrt{\beta+\gamma b_1^2}}-\frac{\sqrt{\beta-\alpha^2}}{\sqrt{\gamma}}\left[\arcsin\frac{\alpha-\frac{\alpha^2-\beta}{\alpha+\sqrt{\beta+\gamma b_1^2}}}{\sqrt{\beta}}-\arcsin\right.$$

$$\left.\frac{\alpha-\frac{\alpha^2-\beta}{\alpha+\sqrt{\beta+\gamma b_2^2}}}{\sqrt{\beta}}\right]$$

Passing in the initial variables there is:

$$\int_{b_1}^{b_2}\ell n\left(\frac{ax}{2}+\sqrt{(Dx-r^2)^2+\frac{a^2x^2}{4}+x^2y_0^2}\right)dy_0=b_2\ell n\left(\frac{ax}{2}+\sqrt{(Dr-r^2)^2+\frac{a^2x^2}{4}+x^2b_2^2}\right)-$$

$$-b_1\ell n\left(\frac{ax}{2}+\sqrt{(Dx-r^2)^2+\frac{a^2x^2}{4}+x^2b_1^2}\right)-(b_2-b_1)+$$

$$+\frac{a}{2}\ell n\frac{b_2x+\sqrt{(Dx-r^2)^2+x^2(\frac{a^2}{4}+b_2^2)}}{b_1x+\sqrt{(Dx-r^2)^2+x^2(\frac{a^2}{4}+b_1^2)}}+\frac{(Dx-r^2)^2}{x}\Big[\arcsin\,\delta_2(\frac{a}{2},b_1)-$$

$$-\arcsin\,\delta_2(\frac{a}{2},b_2)\Big]\,, \tag{28}$$

where

$$\delta_2(\frac{a}{2},b_1)=\frac{\frac{ax}{2}+\frac{(Dx-r^2)^2}{\frac{ax}{2}+\sqrt{(Dx-r^2)^2+\frac{x^2a^2}{4}+x^2b_1^2}}}{\sqrt{(Dx-r^2)^2+\frac{x^2a^2}{4}}}$$

$$\delta_2(\frac{a}{2},b_2)=\frac{\frac{ax}{2}+\frac{(Dx-r^2)^2}{\frac{ax}{2}+\sqrt{(Dx-r^2)^2+\frac{x^2a^2}{4}+x^2b_2^2}}}{\sqrt{(Dx-r^2)^2+\frac{x^2a^2}{4}}}$$

Let us study the integral:

$$\int_{b_1}^{b_2}\frac{dy_0}{\sqrt{(Dx-r^2)^2+\frac{x^2a^2}{4}+x^2y_0^2}}\,.$$

Analogically with the relation (21) we have:

$$\int_{b_1}^{b_2}\frac{dy_0}{\sqrt{(Dx-r^2)^2+\frac{a^2x^2}{4}+x^2y_0^2}}=\int_{b_1}^{b_2}\frac{dy_0}{\sqrt{\beta_3+\beta_1y_0^2}}=\frac{1}{\sqrt{\beta_1}}\ell n\frac{b_2\sqrt{\beta}_1+\sqrt{\beta_3+\beta_1b_2^2}}{b_1\sqrt{\beta}_1+\sqrt{\beta_3+\beta_1b_1^2}}$$

Passing in the initial variables, there is:

$$\int_{b_1}^{b_2}\frac{dy_0}{\sqrt{(Dx-r^2)^2+\frac{x^2a^2}{4}+x^2y_0^2}}=\frac{1}{x}\ell n\frac{b_2x+\sqrt{(Dx-r^2)^2+\frac{x^2a^2}{4}+x^2b_2^2}}{b_1x+\sqrt{(Dx-r^2)^2+\frac{x^2a^2}{4}+x^2b_1^2}} \tag{29}$$

Using partial integration we have:

$$\int_{b_1}^{b_2}\ell n\sqrt{(Dx-r^2)^2+x^2y_0^2}\,dy_0=b_2\ell n\sqrt{(Dx-r^2)^2+x^2b_2^2}-$$

$$-b_1\ell n\sqrt{(Dx-r^2)^2+x^2b_1^2}-(b_2-b_1)+\frac{Dx-r^2}{x}\Big[\text{arctg}\,\frac{b_2x}{Dx-r^2}-$$

$$-\text{arctg}\,\frac{b_1x}{Dx-r^2}\Big]\,. \tag{30}$$

Then, taking into consideration the relations (27), (28), (29), (30) we can write:

$$\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{z_0^2dy_0dz_0}{[(Dx-r^2)^2+x^2(y_0^2+z_0^2)]^{3/2}}=\frac{2}{x^3}\Bigg\{b_2\cdot\ell n\frac{\frac{ax}{2}+\sqrt{(Dx-r^2)^2+\frac{x^2a^2}{4}+x^2b_2^2}}{\sqrt{(Dx-r^2)^2+x^2b_2^2}}$$

$$-b_1\ell n\frac{\frac{ax}{2}+\sqrt{(Dx-r^2)^2+\frac{x^2a^2}{4}+x^2b_1^2}}{\sqrt{(Dx-r^2)^2+x^2b_1^2}}+\frac{Dx-r^2}{x}\Big[\arcsin\,\delta_2(\frac{a}{2},b_1)-$$

$$-\text{arcsin}\delta_2(\frac{a}{2},b_2)-\text{arctg}\ \frac{b_2x}{Dx-r^2}+\text{arctg}\ \frac{b_1x}{Dx-r^2}\Big]\Big\} \qquad (31)$$

Then, according to (26) and (31):

$$\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{(y_0^2+z_0^2)dy_0dz_0}{[(Dx-r^2)^2+x^2(y_0^2+z_0^2)]^{3/2}}=\frac{2}{x^3}\Big\{\ell n\frac{b_2x+\sqrt{(Dx-r^2)^2+x^2b_2^2+x^2\frac{a^2}{4}}}{b_1x+\sqrt{(Dx-r^2)^2+x^2b_1^2+x^2\frac{a^2}{4}}}+$$

$$+b_2\ell n\frac{\frac{ax}{2}+\sqrt{(Dx-r^2)+x^2b_2^2+\frac{x^2a^2}{4}}}{\sqrt{(Dx-r^2)^2+x^2b_2^2}}-$$

$$-b_1\ell n\frac{\frac{ax}{2}+\sqrt{(Dx-r^2)+x^2\frac{a^2}{4}+x^2b_1^2}}{\sqrt{(Dx-r^2)^2+x^2b_1^2}}+\frac{Dx-r^2}{x}\cdot$$

$$\cdot[\text{arcsin}\ \delta_2(b_1)-\text{arcsin}\ \delta_2\delta_2(b_2)+\text{arcsin}\ \delta_2(\frac{a}{2},b_1)-$$

$$-\text{arcsin}\ \delta_2(\frac{a}{2},b_2)-\text{arc tg}\ \frac{b_2x}{Dx-r^2}+\text{arctg}\ \frac{b_1x}{Dx-r^2}\Big]\Big\} \qquad (32)$$

Finally, taking into consideration the relations (13), (17) and (32) we can write:

$$-\frac{\partial\varphi_{2x}}{\partial x}=\frac{-\sigma r}{4\pi\varepsilon_0}\kappa_2(x)\ , \qquad (33)$$

where

$$\kappa_2(x)=\frac{2}{x^2}\Big\{\frac{a}{2}\ell n\frac{b_2x+\sqrt{(Dx-r^2)^2+x^2(\frac{a^2}{4}+b_2^2)}}{b_1x+\sqrt{(Dx-r^2)^2+x^2(\frac{a^2}{4}+b_1^2)}}+$$

$$+b_2\ell n\frac{\frac{ax}{2}+\sqrt{(Dx-r^2)+x^2(\frac{a^2}{4}+b_2^2)}}{\sqrt{(Dx-r^2)^2+x^2b_2^2}}-$$

$$-b_1\ell n\frac{\frac{ax}{2}+\sqrt{(Dx-r^2)^2+x^2(\frac{a^2}{4}+x^2b_1^2)}}{\sqrt{(Dx-r^2)^2+x^2b_1^2}}+\frac{Dx-r^2}{x}\cdot$$

$$\cdot[\text{arcsin}\ \delta_2(b_1)-\text{arcsin}\ \delta_2(b_2)+\text{arcsin}\ \delta_2(\frac{a}{2},b_1)-\text{arcsin}\delta_2(\frac{a}{2},b_2)-$$

$$-\text{arctg}\ \frac{b_2x}{Dx-r^2}+\text{arctg}\frac{b_1x}{Dx-r^2}]+D[\text{arctg}\frac{ab_2x^2}{2(Dx-r^2)\sqrt{(Dx-r^2)^2+x^2(\frac{a^2}{4}+b_2^2)}}$$

$$-\text{arctg}\frac{ab_1x^2}{2(Dx-r^2)\sqrt{(Dx-r^2)^2+x^2(\frac{a^2}{4}+b_1^2)}}\Big]\Big\} \qquad (34)$$

2.1.3 Determination of $\frac{-\partial\varphi_x}{\partial x}$

Taking into consideration the relations (7), (11) and (33) we can write:

$$E_x=\frac{-\sigma}{4\pi\varepsilon_0}[\kappa_1(x)+r\kappa_2(x)]\ , \qquad (35)$$

where $\kappa_1(x)$ and $\kappa_2(x)$ are respectively determined by (12) and (34).

2.2 Determination of Ey

For the determination of Ey, we must be restricted be in the points that satisfy the relation $x = z = 0$. In this case taking into account (1), (2), (3), the potential of y axe points can be presented:

$$\varphi_y = \frac{\sigma}{4\pi\varepsilon_0}\left\{\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0\,dz_0}{\sqrt{D^2+(y_0-y)^2+z_0^2}}\right.$$

$$\left.-r\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0\,dz_0}{\ell\sqrt{x'^2+(y-y')^2+z^2}}\right\}$$

Let us note:

$$\left.\begin{aligned}\varphi_{1y} &= \frac{\sigma}{4\pi\varepsilon_0}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0\,dz_0}{\sqrt{D^2+(y_0-y)^2+z_0^2}}\\ \varphi_{2y} &= \frac{-\sigma r}{4\pi\varepsilon_0}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0\,dz_0}{\ell\sqrt{x'^2+(y-y')^2+z'^2}}\end{aligned}\right\} \quad (36)$$

Then

$$\varphi_y = \varphi_{1y} + \varphi_{2y}$$

and

$$E_y = \frac{-\partial\varphi_y}{\partial y} = \frac{-\partial\varphi_{1y}}{\partial y} - \frac{\partial\varphi_{2y}}{\partial y} \quad (37)$$

2.2.1 Determination of $\frac{-\partial\varphi_{1y}}{\partial y}$

$$-\frac{\partial\varphi_1}{\partial y} = \frac{-\sigma}{4\pi\varepsilon_0}\frac{\partial}{\partial y}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0\,dz_0}{\sqrt{D^2+(y_0-y)^2+z_0^2}}$$

$$= \frac{-\sigma}{4\pi\varepsilon_0}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{(y_0-y)dy_0dz_0}{(D^2+z_0^2+y^2-2yy_0+y_0^2)^{3/2}}$$

Let us study at the beginning

$$\int_{b_1}^{b_2}\frac{(y_0-y)dy_0}{[D^2+z_0^2+(y_0-y)^2]^{3/2}} = \int_{b_{1-y}}^{b_{2-y}}\frac{tdt}{(\alpha_3+t^2)^{3/2}},\ \text{where}\ \left\{\begin{array}{l} t = y_0 - y\\ \alpha_3 = D^2+z_0^2\end{array}\right.$$

According to [11]

$$\int\frac{xdx}{\sqrt{a+cx^2}^{2n+1}} = \frac{-1}{(2n-1)c\sqrt{a+cx^2}^{2n-1}}$$

For

$$n=1\int\frac{xdx}{(a+cx^2)^{3/2}} = -\frac{1}{c\sqrt{a+cx^2}}$$

and

$$\int_{b_{1-y}}^{b_{2-y}}\frac{tdt}{(a_3+t^2)^{3/2}} = \frac{1}{\sqrt{D^2+(b_{1-y})^2z_0^2}} - \frac{1}{\sqrt{D^2+(b_{2-y})^2+z_0^2}}.$$

Then

$$\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{(y_0-y)dy_0dz_0}{[D^2+z_0^2+(y-y_0)^2]^{3/2}}=2\left[\int_0^{\frac{a}{2}}\frac{dz_0}{\sqrt{D^2+(b_1-y)^2+z_0^2}}-\int_0^{\frac{a}{2}}\frac{dz_0}{\sqrt{D^2+(b_2-y)^2+z_0^2}}\right]$$

Let us note $\beta_4^2 = D^2 + (b_1 - y)^2$ and according to [11]

$$\int\frac{dx}{\sqrt{1+x^2}}=\ell n(x+\sqrt{x^2+1}),\text{so }\int\frac{dz_0}{\sqrt{\beta_4^2+z_0^2}}=\ell n\left(\frac{z_0}{\beta_4}+\sqrt{1+\frac{z_0^2}{\beta_4^2}}\right)$$

Passing in the initial variables there is

$$\int_0^{\frac{a}{2}}\frac{dz_0}{\sqrt{D^2+(b_1-y)^2+z_0^2}}=\ell n\left(\frac{a}{2\sqrt{D^2+(b_1-y)^2}}+\sqrt{1+\frac{a^2}{4[D^2+(b_1-y)^2]}}\right)$$

Analogically

$$\int_0^{\frac{a}{2}}\frac{dz_0}{\sqrt{D^2+(b_2-y)^2+z_0^2}}=\ell n\left(\frac{a}{2\sqrt{D^2+(b_2-y)^2}}+\sqrt{1+\frac{a^2}{4[D^2+(b_2-y)^2]}}\right)$$

So

$$\frac{-\partial\varphi_{1y}}{\partial y}=\frac{-\sigma}{4\pi\varepsilon_0}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{(y_0-y)dy_0dz_0}{[D^2+z_0^2+(y-y_0)^2]^{3/2}}=\frac{-\sigma}{4\pi\varepsilon_0}\kappa_3(y)\ , \tag{38}$$

where

$$\kappa_3(y)=2\left\{\ell n\left[\frac{a}{2\sqrt{D^2+(b_1-y)^2}}+\sqrt{1+\frac{a^2}{4[D^2+(b_1-y)^2]}}\right]-\right.$$
$$\left.-\ell n\left[\frac{a}{2\sqrt{D^2+(b_2-y)^2}}+\sqrt{1+\frac{a^2}{4[D^2+(b_2-y)^2]}}\right]\right. \tag{39}$$

2.2.2 Determination of $\frac{-\partial\varphi_{2y}}{\partial y}$

According to (36)

$$\varphi_{2y}=\frac{-\sigma r}{4\pi\varepsilon_0}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0dz_0}{\ell\sqrt{x'^2+(y-y')^2+z'^2}}$$

Taking into consideration [9] $\ell' = \frac{r^2}{\ell} = \sqrt{x'^2 + y'^2 + z'^2}$, and $\frac{y'}{y_0} = \frac{\ell'}{\sqrt{D^2+y_0^2+z_0^2}}$, so we can write:

$$\varphi_{2y}=\frac{-\sigma r}{4\pi\varepsilon_0}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0dz_0}{\sqrt{(y^2(D^2+y_0^2+z_0^2)-r^2(2yy_0-r^2)}}$$

$$\frac{-\partial\varphi_{2y}}{\partial y}=\frac{-\sigma r}{4\pi\varepsilon_0}\left[-\frac{\partial}{\partial y}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0dz_0}{\sqrt{y^2(D^2+y_0^2+z_0^2)-r^2(2yy_0-r^2)}}\right]$$

Let us note $y^2D^2 + r^4 + y^2z_0^2 = a$; $-2r^2y = b$; $y^2 = c$.

Let us study the integral:

$$\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{b_1}^{b_2}\frac{dy_0dz_0}{\sqrt{a+by_0+cy_0^2}}$$

According to [11] for $c>0$

$$\int\frac{dx}{R}=\frac{1}{\sqrt{c}}\ell n(2\sqrt{cR}+2cx+b),\ \text{where } R=a+bx+cx^2,\ \text{so}$$

$$\int_{b_1}^{b_2}\frac{dy_0}{\sqrt{a+by_0+cy_0^2}}=\frac{1}{\sqrt{c}}\left\{\ell n\left[2\sqrt{c(a+bb_2+cb_2^2)}+2cb_2+b\right]-\right.$$

$$\left.-\ell n[2\sqrt{c(a+bb_1+cb_1^2)}+2cb_1+b]\right\}=$$

$$=\frac{1}{y}\ell n\frac{\sqrt{y^2(y^2D^2+r^4+y^2z_0^2-2r^2yb_2+y^2b_2^2}+y^2b_2-r^2y}{\sqrt{y^2(y^2D^2+r^4+y^2z_0^2-2r^2yb_1+y^2b_1^2)}+y^2b_1-r^2y}$$

Let us note $\alpha_4=y^2b_2-r^2y$; $\beta_5=y^4D^2+y^2r^4-2r^2y^3b_2+y^4b_2^2$; $y^4=\gamma_1$; and according to the Appendix we can write:

$$\int_{-\frac{a}{2}}^{\frac{a}{2}}\ell n(\alpha_4+\sqrt{\beta_5+\gamma_1z_0^2})dz_0=a\ell n\left(\alpha_4+\sqrt{\beta_5+\frac{\gamma_1a^2}{4}}\right)-a+$$

$$+\frac{2\alpha_4}{\sqrt{\gamma_1}}\ell n\frac{\frac{a}{2}\sqrt{\gamma_1}+\sqrt{\beta_5+\gamma_1\frac{a^2}{4}}}{\sqrt{\beta_5}}+2\sqrt{\frac{\beta_5-\alpha_4^2}{\gamma_1}}\left[\frac{\pi}{2}-\right.$$

$$\left.-\arcsin\frac{\alpha_4-\frac{\alpha_4^2-\beta_5}{\alpha_4+\sqrt{\beta_5+\gamma_1\frac{a^2}{4}}}}{\sqrt{\beta_5}}\right]=a\ell ny[(b_2y-r^2)+$$

$$+\sqrt{D^2y^2+r^4-2r^2b_2y+b_2^2y^2+\frac{a^2y^2}{4}}]-a+\frac{2(b_2y-r^2)}{y}\cdot$$

$$\cdot\ell n\frac{\frac{ay}{2}+\sqrt{D^2y^2+r^4-2r^2b_2y+b_2^2y^2+\frac{a^2y^2}{4}}}{\sqrt{D^2y^2+r^4-2r^2b_2y+b_2^2y^2}}+2D\left[\frac{\pi}{2}-\right.$$

$$\left.-\arcsin\frac{b_2y-r^2+\frac{D^2y^2}{b_2^2y-r^2+\sqrt{D^2y^2+r^4-2r^2b_2y+b_2^2y^2+\frac{a^2y^2}{4}}}}{\sqrt{D^2y^2+r^4-2r^2b_2y+b_2^2y^2}}\right.$$

The problem in the case including the term b_1 is treated analogically. So, finally:

$$\frac{-\partial\varphi_{2y}}{\partial y}=\frac{-\sigma_r}{4\pi\varepsilon_0}\kappa_4(y)\ , \qquad (40)$$

where

$$\kappa_4(y) = \frac{-\partial}{\partial y}\left\{\left\{\frac{1}{y}a\ell n\frac{b_2y - r^2 + \sqrt{(D^2+\frac{a^2}{4}+b_2^2)y^2 - 2r^2b_2y + r^4}}{b_1y - r^2 + \sqrt{(D^2+\frac{a^2}{4}+b_1^2)y^2 - 2r^2b_1y + r^4}} + \right.\right.$$

$$+\frac{2(b_2y-r^2)}{y}\cdot \ell n\frac{\frac{ay}{2} + \sqrt{(D^2+\frac{a^2}{4}+b_2^2)y^2 - 2r^2b_2y + r^4}}{\sqrt{(D^2+b_2^2)y^2 - 2r^2b_2 + r^4}} -$$

$$-\frac{2(b_1y-r^2)}{y}\cdot \ell n\frac{\frac{ay}{2} + \sqrt{(D^2+\frac{a^2}{4}+b_1^2)y^2 - 2r^2b_1y + r^4}}{\sqrt{(D^2+b_1^2)y^2 - 2r^2b_1y + r^4}} +$$

$$2D\left[\arcsin\frac{(b_1y-r^2) + \frac{D^2y^2}{(b_1y-r^2)+\sqrt{(D^2+\frac{a^2}{4}+b_1^2)y^2-2r^2b_1y+r^4}}}{\sqrt{(D^2+b_1^2)y^2 - 2r^2b_1y + r^4}} - \right.$$

$$\left.\left.\left.-\arcsin\frac{(b_2y-r^2) + \frac{D^2y^2}{(b_2y-r^2)+\sqrt{(D^2+\frac{a^2}{4}+b_2^2)y^2-2r^2b_2y+r^4}}}{\sqrt{(D^2+b_2^2)y^2 - 2r^2b_2y + r^4}}\right]\right\}\right\} \qquad (41)$$

2.2.3 Determination of $\frac{-\partial\varphi_y}{\partial y}$

According to (37), (38) and (40)

$$E_y = \frac{-\sigma}{4\pi\varepsilon_0}[\kappa_3(y) + r\kappa_4(y)] \,, \qquad (42)$$

where $\kappa_3(y)$ and $\kappa_4(y)$ are respectively determined by (39) and (41).

3 Conclusions

In this paper is studied theoretically the electric field in the not completely symmetric system, earthed metallic sphere uniformly charged dielectric plan, for sphere surface points situated in the plan that contains sphere's center and vertical symmetry axe of dielectric plan. The above system is of a great importance in electrostatics, because of its direct relation with the evaluation of electrostatic charges, as well as with the resulting risks.

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Appendix

Integral calculation $\int_{b_1}^{b_2} \ell n(\alpha + \sqrt{\beta + \gamma y_0^2}) dy_0$

Using partial integration we have:

$$\int_{b_1}^{b_2} \ell n(\alpha + \sqrt{\beta + \gamma y_0^2}) dy_0 = y_0 \ \ell n(\alpha + \sqrt{\beta + \gamma y_0^2}) \Big|_{b_1}^{b_2} - \int_{b_1}^{b_2} - \frac{\gamma y_0^2 dy_0}{\sqrt{\beta + \gamma y_0^2}(\alpha + \sqrt{\beta + \gamma y_0^2})}$$

Let us note $p = \sqrt{\beta + \gamma y_0^2}$

$$\int_{b_1}^{b_2} \frac{\gamma y_0^2 dy_0}{\sqrt{\beta + \gamma y_0^2}(\alpha + \sqrt{\beta + \gamma y_0^2})} = \frac{1}{\sqrt{\gamma}} \int_{\sqrt{\beta+\gamma b_1^2}}^{\sqrt{\beta+\gamma b_2^2}} \frac{\sqrt{p^2 - \beta}}{\alpha + p} dp \ .$$

According to [11]

$$\int \frac{\sqrt{R} dx}{x + p_0} = c \int \frac{x dx}{\sqrt{R}} + (b - c p_0) \int \frac{dx}{\sqrt{R}} +$$

$$+ (a - b p_0 + c p_0^2) \int \frac{dx}{(x + p_0)\sqrt{R}} \text{ where } R = a + bx + cx^2.$$

So

$$\int_{\sqrt{\beta+\gamma b_1^2}}^{\sqrt{\beta+\gamma b_2^2}} \sqrt{\frac{p^2 - \beta}{\alpha + p}} dp = \int_{\sqrt{\beta+\gamma b_1^2}}^{\sqrt{\beta+\gamma b_2^2}} \frac{p dp}{\sqrt{p^2 - \beta}}$$

$$-\alpha \int_{\sqrt{\beta+\gamma b_1^2}}^{\sqrt{\beta+\gamma b_2^2}} \frac{dp}{\sqrt{p^2 - \beta}} + (\alpha^2 - \beta) \int_{\sqrt{\beta+\gamma b_1^2}}^{\sqrt{\beta+\gamma b_2^2}} \frac{dp}{(\alpha + p)\sqrt{p^2 - \beta}};$$

According to [11]

$$\int \frac{dx}{u} = \frac{1}{\sqrt{c}} \ell n(x\sqrt{c} + u),$$

where $u = \sqrt{a + cx^2}$.

So

$$\int_{\sqrt{\beta+\gamma b_1^2}}^{\sqrt{\beta+\gamma b_2^2}} \frac{dp}{\sqrt{p^2 - \beta}} = \ell n \frac{\sqrt{\beta + \gamma b_2^2} + b_2\sqrt{\gamma}}{\sqrt{\beta + \gamma b_1^2} + b_1\sqrt{\gamma}}$$

and

$$\int_{\sqrt{\beta+\gamma b_1^2}}^{\sqrt{\beta+\gamma b_2^2}} \frac{p dp}{\sqrt{p^2 - \beta}} = \sqrt{\gamma}$$

According to [11]

$$\int \frac{dx}{(x + p_0)\sqrt{R}} = - \int \frac{dt}{c + (b - 2p_0 c)t + (a - b p_0 + c p_0^2)t^2},$$

where $t = \frac{1}{x + p_0}$ and $\int \frac{dx}{\sqrt{R}} = \frac{-1}{\sqrt{-c} \arcsin \frac{2cx+b}{\sqrt{-\Delta}}}$, for $c < 0$ and $\Delta = 4ac - b^2 < 0$. So in our case we can write:

$$\int_{\sqrt{\beta+\gamma b_1^2}}^{\sqrt{\beta+\gamma b_2^2}} \frac{dp}{(\alpha + p)\sqrt{p^2 - \beta}} = - \int_{\frac{1}{\alpha+\sqrt{\beta+\gamma b_1^2}}}^{\frac{1}{\alpha+\sqrt{\beta+\gamma b_2^2}}} \frac{dt}{\sqrt{1 - 2\alpha t + (\alpha^2 - \beta)t^2}} =$$

$$= \frac{1}{\sqrt{\beta - \alpha^2}} \left[\frac{\arcsin\alpha - \frac{\alpha^2-\beta}{\alpha+\sqrt{\beta+\gamma b_1^2}}}{\sqrt{\beta}-} \right.$$

$$\left. -\arcsin\frac{\alpha - \frac{\alpha^2-\beta}{\alpha+\sqrt{\beta+\gamma b_2^2}}}{\sqrt{\beta}} \right] .$$

So

$$\int_{b_1}^{b_2} \frac{\gamma y_0^2 dy_0}{\sqrt{\beta+\gamma y_0^2}(\alpha+\sqrt{\beta+\gamma y_0^2})} = b_2 - b_1 - \frac{\alpha}{\sqrt{\gamma}} \cdot \ell n \frac{b_2\sqrt{\gamma}+\sqrt{\beta+\gamma b_2^2}}{b_1\sqrt{\gamma}+\sqrt{\beta+\gamma b_1^2}} -$$

$$-\sqrt{\frac{\beta-\alpha^2}{\gamma}} \cdot \left[\arcsin\frac{\alpha - \frac{\alpha^2-\beta}{\alpha+\sqrt{\beta+\gamma b_1^2}}}{\sqrt{\beta}} - \arcsin\frac{\alpha - \frac{\alpha^2-\beta}{\alpha+\sqrt{\beta+\gamma b_2^2}}}{\sqrt{\beta}} \right] ,$$

and finally:

$$\int_{b_1}^{b_2} \ell n(\alpha+\sqrt{\beta+\gamma y_0^2}) dy_0 = b_2 \ell n(\alpha+\sqrt{\beta+\gamma b_2^2}) - b_1 \ell n(\alpha+\sqrt{\beta+\gamma b_1^2}) -$$

$$-(b_2 - b_1) + \frac{\alpha}{\sqrt{\gamma}} \cdot \ell n \frac{b_2\sqrt{\gamma}+\sqrt{\beta+\gamma b_2^2}}{b_1\sqrt{\gamma}+\sqrt{\beta+\gamma b_1^2}} + \sqrt{\frac{\beta-\alpha^2}{\gamma}} \; .$$

$$\cdot \left[\arcsin\frac{\alpha_3 - \frac{\alpha^2-\beta}{\alpha+\sqrt{\beta+\gamma b_1^2}}}{\sqrt{\beta}} - \arcsin\frac{\alpha - \frac{\alpha^2-\beta}{\alpha+\sqrt{\beta+\gamma b_2^2}}}{\sqrt{\beta}} \right]$$

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Figure Captions

Fig. 1 - Not completely symmetric system: earthed metalic sphere-uniformly charged dielectric plan.