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PROOF OF POLYAKOV CONJECTURE ON SUPERCOMPLEX PLANE

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ABSTRACT

Using Neumann series, we solve iteratively SBE to arbitrary order. Then applying this, we compute the energy momentum tensor and n points functions for generic n starting from WZP action on the supercomplex plane. We solve the superconformal Ward identity and we show that the iterative solution to arbitrary order is resums by WZP action. This proves the Polyakov conjecture on supercomplex plane.

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1 Introduction

At the quantum level, in superconformal field theory one has either a super Weyl rescalings of the metric or a superdiffeomorphism anomaly. However, in two dimensions, there is a local functional Γ_{loc} whose BRS variation $s\Gamma_{loc}$ relates these two anomalies [1]. This result in holomorphic factorization of partition functions (i.e. chirally split) of the vacuum as functionals of superBeltrami differentials $\hat{\mu}$, $\hat{\bar{\mu}}$

$$Z_v(\hat{\mu},\hat{\bar{\mu}}) = Z_v(\hat{\mu},0) + Z_v(0,\hat{\bar{\mu}})$$

The chiral part $Z_v(\hat{\mu},0)$ satisfies the Ward identity [2,3]

$$(\bar{\partial} - \hat{\mu}\partial - \frac{3}{2}\partial\hat{\mu} - \frac{1}{2}D\hat{\mu}D)\frac{\delta Z_v(\hat{\mu},0)}{\delta\hat{\mu}} = k\partial^2 D\hat{\mu} \tag{1}$$

where k is the central charge of the model. It measures the strength of the superdiffeomorphism anomaly and

$$\partial \equiv \frac{\partial}{\partial z},\ \bar{\partial} \equiv \frac{\partial}{\partial \bar{z}};\ D \equiv \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z};\ D^2 = \partial$$

The Wess-Zumino-Polyakov action Γ_{WZP} corresponding to the super diffeomorphism anomaly on the supercomplex plane $S\mathbb{C}$, writes [2,3]

$$\Gamma_{WZP} = k\pi \int d\tau_1 \partial_1 D_1 \wedge (a_1)\hat{\mu}(a_1) \tag{2}$$

where $\wedge = \ell n(D\hat{\theta})$, with $\hat{\theta}$ is the isothermal coordinates. It satisfies the superBeltrami equation (SBE) [3,4] given below. And where a_1 and $d\tau_1$ will be specified in section [3].

Polyakov conjecture that the action (2) resums the normalized perturbative series of $Z_v(\hat{\mu},0)$ solution of the superconformal Ward identity (1). Thus for a suitable choice of $\hat{\theta}$ one can make

$$\Gamma_{WZP} = Z_v(\hat{\mu})$$

In bosonic case S. Lazzarini [5] has shown that on the complex plane, Polyakov conjecture is true for the second order and the third order in $\hat{\mu}$. On the supercomplex plane $S\mathbb{C}$, this was dealt to the third order. The purpose of this paper is to generalize the work of [3] and to show that Polyakov conjecture is true to any order in $\hat{\mu}$. However, it would be worthwhile to emphasize that our result for the third order does not seem to fit with that given in [3].

The paper is organized as follows.

In section 2, we will give some preliminaries of SBE on supercomplex plane by considering some restrictions of the superspace geometry. In section 3, we will solve the SBE iteratively to arbitrary order, using a method based on Neumann series. The solution will be inserted in Γ_{WZP}, in order to compute the energy momentum tensor and the OPE. This will be achieved in section 4. Section 5 will be devoted to solve the anomalous superconformal Ward identity (1). Afterwards, we will show that the solution, independent of the fields of the model, agrees with that given in the previous section. This will conclude Polyakov conjecture. Section 6 contains conclusion and discussions. The appendix contains an outline of the calculation related to section 4.

2 Complex Superplane and SuperBeltrami Equations

A super Riemann surface SRS is a complex supermanifold of dimension (1,1) endowed with coordinates (z,θ) and provided locally by the Homomorphic coordinates $(z,\hat{\theta})$ referred to as th isothermal or projectives coordinates, they are related to the former ones by a quasisuperconformal transformation dependant on superBeltrami coefficients $\hat{\mu},\nu$ and σ which parametrize the superconformal structure on the supermanifold. These superBeltrami are only known in supper upper half plane SU. One extend these coefficients to take value into the whole supercomplex plane by defining $\mu=\sigma=0$, $\nu=\theta$ in the lower half plane [6].

The isothermal coordinates $\hat{z}$ and $\hat{\theta}$ are diffeomorphism with respect to the reference system (z,θ) and verify the following SBE [4,7]

$$\begin{aligned} \bar{\partial}\hat{z}+\hat{\theta}\bar{\partial}\hat{\theta} &= \hat{\mu}(\partial\hat{z}+\hat{\theta}\partial\hat{\theta}), \\ -\partial_\theta\hat{z}+\hat{\theta}\partial_\theta\hat{\theta} &= \nu(\partial\hat{z}+\hat{\theta}\partial\hat{\theta}), \\ -\partial\bar{\theta}\hat{z}+\hat{\theta}\partial_{\bar{\theta}}\hat{\theta} &= \sigma(\partial\hat{z}+\hat{\theta}\partial\hat{\theta}) \ . \end{aligned} \tag{3}$$

Here we adopt a special gauge [3] $(\nu=\theta,\sigma=0)$ which simplifies the description of the deformation of SRS's considerably. It correspond to Martinec's formulation [8].

The SBE (3) write as

$$\bar{\partial}\hat{z}+\hat{\theta}\bar{\partial}\hat{\theta}=\hat{\mu}(\partial\hat{z}+\hat{\theta}\partial\hat{\theta}),\ D\hat{z}=\hat{\theta}D\hat{\theta},\ \partial_{\bar{\theta}}\hat{z}=\hat{\theta}\partial_{\bar{\theta}}\hat{\theta} \tag{4}$$

The last equation imply there is no $\bar{\theta}$ dependence, while the second is known as the superconformal constraint. it can be derived by requiring that the derivative D transforms

[3]In [6] Rabin showed that this special gauge choice is always possible for all infinitesimal and some finite deformations of split SRS's, but almost certainly false for arbitrary deformation. However in [4] Takama proved that this is always possible.

homogeneously [6,7]. The decoupled form of SBE, for general case, was given in [4]. For this special gauge it writes

$$\bar{\partial}\hat{z} = \hat{\mu}\partial\hat{z} + \frac{1}{2}D\hat{\mu}D\hat{z} \tag{5a}$$

$$\bar{\partial}\hat{\theta} = \hat{\mu}\partial\hat{\theta} + \frac{1}{2}D\hat{\mu}D\hat{\theta} \tag{5b}$$

One sees that for $\hat{\mu} = 0$, the solution of SBE are superconformal maps. In the sequel, we will be interested only in $\hat{\theta}$ as the action Γ_{WZP} is constructed in terms of $\frac{1}{2}$ superdifferential $D\hat{\theta}$.

3 Resolution of superBeltrami equation

In this section, we solve the SBE (5b) iteratively to arbitrary order using a method based on Neumann series without discussing the convergence of this series for $|\mu| < \varepsilon$, where ε is sufficiently small quantity. This a generalization to supersymmetric case of Lazzarini's work [5]. For this purpose we write (5b) as

$$\bar{\partial}\Lambda = \frac{1}{2}\partial\hat{\mu} + BD\Lambda \tag{6}$$

where $B \equiv \hat{\mu}D + \frac{1}{2}D\hat{\mu}$ and Λ defined above.

We fix the following normalization: For $\hat{\mu} = 0$, $\hat{\theta} = \theta(\Lambda = 0)$. The resolution of equation (6) iteratively in power of $\hat{\mu}$ will be accomplished by inverting the operator $\bar{\partial}$ by Cauchy kernel.

By generalized Cauchy formula, we define [3]

$$(\bar{\partial}^{-1}F)(z_1,\theta_1) = \int_{S\mathbb{C}} \frac{dz_2 \wedge d\bar{z}_2}{2\pi i} d\theta_2 \left(\frac{\theta_1 - \theta_2}{z_1 - z_2 - \theta_1\theta_2)}\right) F(z_2,\theta_2) \tag{7}$$

where $F(z,\theta)$ is some function defined on $S\mathbb{C}$ and $\frac{\theta_1-\theta_2}{z_1-z_2-\theta_1\theta_2} = \frac{\theta_1-\theta_2}{z_1-z_2}$ is the Cauchy kernel on $S\mathbb{C}$, that is we have [4]

$$\bar{\partial}_{z_1}\left(\frac{\theta_1 - \theta_2}{z_1 - z_2}\right) = -\pi\delta(z_1 - z_2)\delta(\bar{z}_1 - \bar{z}_2)(\theta_1 - \theta_2) \equiv \delta^3(a_1 - a_2)$$

The formal series solution of equation (6) write

$$\Lambda = \sum_{n=1}^{\infty} \bar{\partial}^{-1}\lambda_n(z,\theta)$$

with $\lambda_1 = \frac{1}{2}\partial\hat{\mu}$ and $\lambda_n = BD\bar{\partial}^{-1}\lambda_{n-1}$

[4]this definition implies $-\int \frac{dz_2\wedge d\bar{z}_2}{2i} d\theta_2\delta^3(a_1 - a_2)F(a_2) = F(a_1)$

For $n > 1$ one gets

$$\bar{\partial}^{-1}\lambda_n = (-1)^{\frac{n(n-1)}{2}} \int \prod_{j=2}^{n+1} d\tau_j \prod_{i=1}^{n-1} (c_{ii+1} B_{i+1} D_{i+1}) C_{nn+1} \lambda_1(a_{n+1}) \tag{8}$$

Here and in the following we will be using the notation

$$d\tau_i = \frac{dz_i \wedge d\bar{z}_i}{2\pi i} d\theta_i,\ Di \equiv \partial_{\theta_i} + \theta_i \partial_i;\ C_{ij} = \frac{\theta_i - \theta_j}{z_i - z_j}$$

The subscription on Bi means B is evaluated at the point $a_i \equiv (z_i, \bar{z}_i, \theta_i)$. The sign in front of the integral $(-1)^{\frac{n(n-1)}{2}}$ arise from the commutation of Cauchy kernel with the product of the measures $\pi d\tau$.

By adopting the convention $\Pi_{i=1}^{0}(\quad) = 1$ we extend the formula (8) to all values of n.

The formula (8) contains a power of superBeltrami differentials $\hat{\mu}$ and its derivatives. In order to get it developed only in power of $\hat{\mu}$, we proceed as follow. First we write the equation (8) as

$$\bar{\partial}^{-1}\lambda_n(a_1) = (-1)^{\frac{n(n-1)}{2}} \int \Pi_{j=2}^{n+1} d\tau_j f_{1,k-1} c_{kk+1} \partial_{k+1} f_{k+1,n-1} g$$

$$+(-1)^{\frac{n(n-1)}{2}} \int \Pi_{j=2}^{n+1} d\tau_j f_{1,k-1} c_{kk+1} (D_{k+1}\hat{\mu}(a_{k+1})) D_{k+1} f_{k+1,n-1} g \tag{9}$$

where we denote $f_{\ell,m} = \Pi_{i=\ell}^{m}(c_{ii+1} B_{i+1} D_{i+1})$ and $g = c_{nn+1}\lambda_1(a_{n+1})$ and where we developed the k-th term of the product $(c_{kk+1} B_{k+1} D_{k+1})$.

After integrating by part, the second term in r.h.s. of equations g, we get

$$\bar{\partial}^{-1}\lambda_n = (-1)^{\frac{n(n-1)}{2}} \int \Pi_{j=2}^{n+1} d\tau_j f_{1,k-1} (c_{kk+1}\partial_{k+1} - D_k c_{kk+1} D_{k+1}) f_{k+1,n-1} g$$

This means that each term in the product of equation (8) can be written as above. Finally, after integrating by part the term g, one has

$$\bar{\partial}^{-1}\lambda_n(a_1) = \frac{(-1)^{\frac{n(n-1)}{2}}}{2^n} \int \Pi_{j=2}^{n+1} d\tau_j \left[\Pi_{i+1}^{n-1}(c_{ii+1}\partial_{i+1} - D_i c_{ii+1} D_{i+1}) \partial_n c_{nn+1}\right] \Pi_{\ell=2}^{n+1} \mu(a_\ell) \tag{10}$$

The summation over n of this quantity gives Λ.

For instance, the solution of Λ to the second order in $\hat{\mu}$, writes

$$\Lambda(a_1) = \sum_{n=1}^{2} \bar{\partial}^{-1}\lambda_n = \frac{1}{2}\int d\tau_2 \partial_1 c_{12} \mu(2) - \frac{1}{4}\int d\tau_{23}[(c_{12}\partial_2 - D_1 c_{12} D_2)\partial_2 c_{23}]\mu(a_2)\mu(a_3)$$

We have thus obtained the solution $(\hat{z}, \hat{\bar{z}}, \hat{\theta})$ as a perturbation parametrized by $\hat{\mu}$ and $\hat{\bar{\mu}}$ of the initial complex structure $(z, \bar{z}, \theta)$.

4 OPE and Ward Identities

As was pointed out earlier, we insert the result obtained in the previous section (Eq.10) in the action Γ_{WZP} given in (2). Thus

$$\Gamma_{WZP} = k\pi \sum_{n=1}^{\infty} \frac{(-1)^{\frac{n(n+1)}{2}}}{2^n} \int \Pi_{j=1}^{n+1} d\tau_j [\partial_1 D_1 \Pi_{i=1}^{n-1} (c_{ii+1}\partial_{i+1} - D_i c_{ii+1} D_{i+1}) \partial_n c_{nn+1} \times \Pi_{\ell=1}^{n+1} \mu(\ell) \tag{11}$$

The variation of this action with respect to $\hat{\mu}$, yields the energy momentum tensor.

$$\mathcal{T}(a_1) = \frac{\delta \Gamma_{WZP}}{\delta \mu(a_1)} = -2k\pi(\partial_1 D_1 \wedge (a_1) - D_1 \wedge (a_1)\partial_1 \wedge (a_1)) = -2k\pi \mathcal{S} \tag{12}$$

Where $\mathcal{S}$ denotes the superSchwarzian derivative

$$\mathcal{S} = \frac{D^4\hat{\theta}}{D\hat{\theta}} - 2\frac{D^3\hat{\theta} D^2\hat{\theta}}{(D(\hat{\theta})^2}$$

Note that $\mathcal{T}(a_1)$ vanishes at $\hat{\mu} = 0$, which means that $\hat{\mu}$ is an exterior source. The n points functions can be easily derived.

$$\langle T(1)T(2)\dots T(n)\rangle = (-1)^n \frac{\delta^n \Gamma_{WZP}}{\delta\hat{\mu}(n)\dots\delta\hat{\mu}(1)}\Big|_{\hat{\mu}(i)=0} =$$

$$k\frac{(-1)^{\frac{n(n+1)}{2}}}{(2\pi)^{n-1}} \sum_{perm} (-1)^P \partial_1 D_1 \Pi_{i=1}^{n-2} (c_{ii+1}\partial_{i+1} - D_i c_{ii+1} D_{i+1}) \partial_{n-1} c_{n-1n} \tag{13}$$

The sum is over all possible permutations and $(-1)^P$ stands for the sign of the permutation. In the appendix, we will give an outline of how the n-points functions can be reexpressed in terms of $n-1$ points functions according to the formula

$$\langle T(1)T(2) - T(n)\rangle = \frac{(-1)^n}{2\pi} \sum_{\substack{perm \\ \neq 1}} (-1)^P (2c_{12}\partial_2 + D_1 c_{12} D_2 - 3\partial_1 c_{12}) < \langle T(2)T(3) - T(n)\rangle \tag{14}$$

Here the sum is restricted to all permutation different than 1.

Using this result as much as necessary, one can show that the n-points functions write as

$$\langle T(1)\dots T(n)\rangle = \frac{(-1)^{\frac{n(n+1)}{2}}}{\pi(2\pi)^{n-2}} \sum_{\substack{perm \\ \neq 1}} (-1)^P \Pi_{i=1}^{n-2} (2c_{ii+1}\partial_{i+1} + D_i c_{ii+1} D_{i+1} - 3\partial_i c_{ii+1}) \times \partial_{n-1}^2 D_{n-1} C_{n-1,n} \tag{15}$$

With the expression (15), we can now apply $\bar{\partial}_1$ operator, which yields the Ward identity.

$$\bar{\partial}_1 \langle T(1)\dots T(n)\rangle = \frac{(-1)^{\frac{n(n+1)}{2}} k}{2(2\pi)^{n-1}} \sum_{\substack{perm \\ \neq 1}} (-1)^P (2\delta^3(1-2)\partial_2 + D_1\delta^3(1-2)D_2 - 3\partial_1\delta^3(a-2))$$

$$\Pi_{i=2}^{n-2}(2c_{ii+1}\partial_{i+1} + D_i c_{ii+2} D_{i+1} - 3\partial\delta_i c_{ii+1}) D_{n-1}\partial_{n-1}^2 c_{n-1,n}\,. \tag{16}$$

As example, the 3-points functions reads

$$\langle T(1)T(2)T(3)\rangle = \frac{k}{2(2\pi)^2}\sum_{\substack{perm\\ \neq 1}}(-1)^P(2c_{12}\partial_2 + D_1 c_{12} D_2 - 3\partial_1 c_{12}) D_2\partial_2^2 C_{23} \tag{16.a}$$

for which the Ward identity is:

$$\bar{\partial}_1\langle T(1)T(2)T(3)\rangle = \frac{k}{2(2\pi)^2}\{[2\delta^3(1-2)\partial_2 + D_1\delta^3(1-2)D_2 - 3\partial_1\delta^3(1-2)]D_2\partial_2^2 c_{23} - (2\leftrightarrow 3)$$

As we pointed out earlier, we do not recover the result given in [3].

5 Iterative Resolution of the Anomalous Ward Identity

We apply the same method used in section (3) in order to get the iterative solution of superconformal Ward identity given by Eq.(1). This is written as

$$\bar{\partial}\left(\frac{\delta Z}{\delta\hat{\mu}}\right) = p_n + K\frac{\delta z}{\delta\mu}$$

With $p_n = k\partial^2 D\hat{\mu}$ and $K = \hat{\mu}\partial + \frac{3}{2}\partial\hat{\mu} + \frac{1}{2}D\hat{\mu}D$ Iteratively one gets

$$\bar{\partial}\left(\frac{\delta Z}{\delta\hat{\mu}}\right) = \sum_{n=1}^{\infty} p_n \quad \text{or} \quad \frac{\delta Z}{\delta\hat{\mu}} = \sum_{n=1}^{\infty}\bar{\partial}^{-1}p_n \tag{17}$$

With $p_n = \kappa\bar{\partial}^{-1}p_{n-1}$ and p_n given above.

After performing the calculation one finds

$$\bar{\partial}^{-1}p_n = (-1)^{\frac{n(n-1)}{2}} k\int \Pi_{j=2}^{n+1}d\tau_j c_{12}\Pi_{i=2}^{n}(k_i c_{ii+1})\partial_{n+1}^2 D_{n+1}\mu(n+1)$$

Here again the sign $(-1)^{\frac{n(n-1)}{2}}$ in front of the equation arises from the commutation of Cauchy kernel c_{ij} with the product of the measures $\pi d\tau$. Explicitly this writes

$$\partial^{-1}p_n = (-1)^{\frac{n(n-1)}{2}} k\int \Pi_{j=2}^{n+1}d\tau_j c_{12}\Pi_{i=2}^{n}(\hat{\mu}(i)\partial_i c_{ii+1} + \frac{3}{2}(\partial_i\hat{\mu}(i))c_{ii+1} + \frac{1}{2}(D_i\mu(i))D_i c_{ii+1}$$

$$\times\partial_{n+1}^2 D_{n+1}\mu(n+1)$$

As in section (3), we express the above formula in terms of power of $\hat{\mu}$ only (i.e. without derivatives of $\hat{\mu}$) by integrating by part. The result is

$$\bar{\partial}^{-1}p_n = k\frac{(-1)^{\frac{n(n+1)}{2}}}{2^{n-1}}\int \Pi_{j=2}^{n+1}d\tau_j\Pi_{i=1}^{n-1}(2c_{ii+1}\partial_{i+1} + D_i c_{ii+1} D_{i+1} - 3\partial_i c_{ii+1}) D_n\partial_n^2 c_{nn+1}\Pi_{\ell=2}^{n+1}\mu(\ell)$$

The summation over n of this quantity gives $\frac{\delta Z}{\delta \mu(1)}$.

The n-points function can thus be easily derived. It coincides with the Eq.(15). This means that the perturbative series solution of Ward identity resums by Polyakov action. Thus Polyakov conjecture concerning uniqueness of the solution and universalities which stands that the solution is independent of the model, is proved.

6 Conclusion

In summary, we have solved the superBeltrami equations for the projective coordinates $(\hat{z}, \hat{\theta})$ iteratively to arbitrary order in power of superBeltrami differentials. Then using this result in WZP action, we derive the energy momentum tensor and n points functions for arbitrary n. We solve the anomalous superconformal Ward identity iteratively using Neumann series and show that the n-points functions calculated by means of this solution coincides with the one derived from WZP-action.

Polyakov conjecture states that first the solution of the Ward identity is unique and second is independent of the fields of the model. This is resums by WZP action. Here we showed that this conjecture is true in $\delta\mathbb{C}$.

It would be interesting to generalize this work to the case of supertorus and superRiemannian surfaces with higher genus with non trivial spin structure. This would require appropriate Cauchy kernel and corresponding Plyamov actions.

At the end, we emphasize that to the third order on supertorus, this was done in [3] where they show that the Polyakov action resums the iterative series solving the Ward identity.

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Appendix

Here we use the following notation for the sake of simplicity.

$$A_{ij} = c_{ii+1}\partial_{i+1} - D_i c_{ii+1} D_{i+1}$$

$$B_{1i} = 2c_{1i} + D_1 c_{1i} D_i - 3\partial_1 c_{1i}$$

To derive the relation (14), first we write the n-points functions (Eq.13) as a sum over all permutations different than one. For $n \geq 4$, one has

$$\langle T(1)\dots T(n)\rangle = k\frac{(-1)^{\frac{n(n+1)}{2}}}{(2\pi)^{n-1}} \sum_{\substack{perm \\ \neq 1}} (-1)^P [\partial_n D_1 A_{12} A_{23} \dots A_{n-2,n-1}\partial_{n-1} c_{n-1,n}$$

$$+ \sum_{j=2}^{n-2} (-1)^{j-1} D_2 \partial_1 A_{23} \dots A_{j-1,j} A_{j1} A_{1j+1} \dots A_{n-2,n-1} \partial_{n-1} c_{n-1,n}$$

$$+(-1)^{n-2} D_2 \partial_2 A_{23} A_{34} \dots A_{n-1,1} \partial_1 c_{1n}$$

$$+(-1)^{n-1} D_2 \partial_2 A_{23} \dots A_{n-n,n} \partial_n c_{n1}] \qquad (I)$$

Note that for $n = 3$, this formula holds with the term with summation over j is absent.

Second, by acting by appropriate derivatives as much as necessary on the relation verified by Cauchy kernels on supercomplex plane $S\mathbb{C}$.

$$c_{ni} c_{ij} - c_{i1} c_{1j} = c_{nj} c_{ij}$$

One shows that $A_{i1} A_{1j} A_{jk} - A_{ij} A_{j1} A_{1k}$ gives $B_{1j} A_{ij} A_{jk}$ plus other terms which, when we add the following term in I, combine with the following term to give a term with B and so on. After that, we perform the permutation in I, one gets the relation (14).

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