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ON THE OVERLAP FORMULATION OF CHIRAL GAUGE THEORY

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ABSTRACT

The overlap formula proposed by Narayanan and Neuberger in chiral gauge theories is examined. The free chiral and Dirac Green's functions are constructed in this formalism. Four dimensional anomalies are calculated and the usual anomaly cancellation for one standard family of quarks and leptons is verified.

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1 Introduction

Non–perturbative study of the standard electroweak model has long been obstructed by difficulties with lattice regularization of theories involving chiral fermions. Recently, however, there has been some progress in meeting these difficulties. In particular, following an earlier work of Kaplan [1], an interesting new interpretation of the chirality problem has been proposed by Narayanan and Neuberger [2]. These authors claim that their approach circumvents the well known no–go theorems [3] and thus has the potential for application to numerical study of chiral symmetry breaking and other important problems. Their basic idea is to represent the vacuum amplitude for a chiral fermion as the overlap of two well–defined vectors that can be interpreted as ground states of massive fermions in an auxiliary 5–dimensional spacetime. They show that this overlap transforms anomalously, as it should. On the other hand, it can be regularized by any of the standard methods, including the usual lattice regularization for massive, non–chiral fermions. This reconciliation of features long thought to be incompatible is a remarkable achievement.

Our purpose in this note is to explore this mechanism by constructing the free fermion Green's function in the auxiliary spacetime to see how the chiral Green's function emerges when the auxiliary mass is taken to infinity. We also verify that the 4–dimensional vacuum amplitude expresses the expected perturbative anomalies. Finally, we show how a single anomaly free family of leptons and quarks can be represented in this scheme.

2 Green's functions for free chiral fermions

The idea is that the Green's functions for a quantum field theory in 4–dimensional Euclidean spacetime can be given an operator realization in the Hilbert space associated with a 4+1–dimensional (Minkowskian) spacetime. This Hilbert space contains many states that are not relevant to the 4–dimensional physics and these are projected out by taking the limit $|\Lambda| \to \infty$, where Λ is a "regulator" mass introduced into the Hamiltonian of the 4+1–dimensional parent theory. In taking this limit the fields of the parent theory are scaled to zero in such a way that the Green's functions of 4–dimensional Euclidean theory are obtained. We shall verify this firstly for a 1–flavour chiral fermion and then for a 3–flavour example in which two of the chiral fields combine to give a Dirac fermion. Further generalization is then straightforward.

The basic recipe is to represent every Weyl doublet of the 4–dimensional theory by a 4–component Dirac spinor in the 4+1–dimensional theory. For one such field the parent

Hamiltonian (in the Schroedinger picture) is

$$H = \int d^4x\ \hat{\bar{\psi}}(x)\ (\partial\!\!\!/ + \Lambda)\hat{\psi}(x)$$

where $\hat{\psi}(x)$ is a 4–component spinor and $\hat{\bar{\psi}}(x) = \psi(x)^\dagger\gamma_5$. The Dirac matrices γ_μ and γ_5 are hermitian. The Schroedinger picture operators satisfy the usual anticommutation rules,

$$\{\hat{\psi}(x), \hat{\psi}(x')\} = 0, \quad \{\hat{\psi}(x), \hat{\psi}(x')^\dagger\} = \delta_4(x-x'), \quad \{\hat{\psi}(x)^\dagger, \hat{\psi}(x')^\dagger\} = 0$$

They can be realized on a Fock space with vacuum state, $|0\rangle$, annihilated by $\psi(x)$. The eigenstates of H are generated in the usual way by solving the 1–body problem,

$$H(k)\ u(k,\lambda) = \omega(k)\ u(k,\lambda), \quad H(k)\ v(k,\lambda) = -\omega(k)\ v(k,\lambda)$$

where $H(k) = \gamma_5(ik\!\!\!/ + \Lambda)$ and $\omega(k) = \sqrt{k^2+\Lambda^2}$ is positive. The eigenspinors, u and v, are orthonormal and complete since $H(k)$ is hermitian. We shall write $u_\pm$ and $v_\pm$ to distinguish the cases $\Lambda = \pm|\Lambda|$. Correspondingly, there are two independent plane wave expansions

$$\hat{\psi}(x) = \frac{1}{\Omega}\sum_{k,\lambda}\left(b_\pm(k,\lambda)u_\pm(k,\lambda) + d_\pm^\dagger(k,\lambda)v_\pm(k,\lambda)\right)e^{ikx}$$

where Ω is the volume of a 4–box. The operators b_+ and d_+ annihilate the Dirac vacuum, $|+\rangle$, defined by

$$|+\rangle\ = \prod_{k,\lambda}\ \Omega^{-1/2}\ d_+(k,\lambda)\ |0\rangle$$

but, of course, they do not annihilate the other Dirac vacuum,

$$|-\rangle\ = \prod_{k,\lambda}\ \Omega^{-1/2}\ d_-(k,\lambda)\ |0\rangle$$

The two Dirac vacua are normalized and their overlap is given by

$$\begin{aligned}\langle +|-\rangle &= \prod_k\ \det{}_{\lambda\lambda'}(v_+^\dagger(k,\lambda)v_-(k,\lambda')) \\ &= \prod_k\ (k^2/\omega^2)\end{aligned}$$

In order for this to be non–vanishing we must suppose that the fermions are subject to antiperiodic boundary conditions.

Consider the Green's function defined by

$$\begin{aligned}G(x-x') &= \frac{\langle +|\hat{\psi}(x)\hat{\bar{\psi}}(x')|-\rangle}{\langle +|-\rangle} \\ &= \frac{1}{\Omega^2}\ \Sigma\ u_+(k,\lambda)\ \frac{\langle +|b_+(k,\lambda)b_-^\dagger(k',\lambda')|-\rangle}{\langle +|-\rangle}\ \bar{u}_-(k',\lambda')e^{ikx-ik'x'} \\ &= \frac{1}{\Omega}\sum_k\ \tilde{G}(k)\ e^{ik(x-x')}\end{aligned}$$

To compute $\tilde{G}(k)$ one needs the commutators $\{b_+, b_-^\dagger\}$, $\{b_+, d_-\}$, $\{d_+^\dagger, b_-^\dagger\}$ and $\{d_+^\dagger, d_-\}$ which can all be expressed in terms of the eigenspinors $u_\pm$ and $v_\pm$. The result is

$$\tilde{G}(k) = \frac{1}{2}(\omega + \gamma_5(i\not{k} + \Lambda))\ \frac{1}{i\not{k}}$$

Scaling with $1/|\Lambda|$ and taking the limit $\Lambda \to \pm\infty$ gives

$$\frac{1}{|\Lambda|}\ \tilde{G}(k) \to \frac{1}{2}(1 \pm \gamma_5)\ \frac{1}{i\not{k}}$$

i.e. the Euclidean Green's function for a chiral fermion. In effect,

$$\lim_{\Lambda\to\pm\infty} \left(|\Lambda|^{-1/2}\ \hat{\psi}(x)\right) = \frac{1}{2}(1 \pm \gamma_5)\psi(x)$$

where $\psi(x)$ represents a massless fermion in 4 dimensions.

This very elementary discussion can easily be extended to the many–flavour case. With a view towards the Standard Model, consider the 3–flavour neutrino–electron system, ν_L, e_L and e_R. These fields are represented in the parent theory by three 4–component spinors $\hat{\nu}_L, \hat{e}_L$ and $\hat{e}_R$. The Hamiltonian is

$$H = \int d^4x\ \bar{\hat{\psi}}\,(\not{\partial} + \Lambda T_c + \phi\cdot T)\,\hat{\psi}$$

where

$$\hat{\psi} = \begin{pmatrix} \hat{\nu}_L \\ \hat{e}_L \\ \hat{e}_R \end{pmatrix} \quad \text{and} \quad \Lambda T_c + \phi\cdot T = \begin{pmatrix} \Lambda & 0 & \phi_1 \\ 0 & \Lambda & \phi_2 \\ \phi_1^* & \phi_2^* & -\Lambda \end{pmatrix}$$

with ϕ_1, ϕ_2 representing the Higgs doublet: for the following discussion we can replace these fields by vacuum values, $\langle\phi_1\rangle = 0, \langle\phi_2\rangle = m$. Note the $-\Lambda$ associated with e_R.

Eigenspinors of the 1–body Hamiltonian are to be computed and the vacuum overlap is

$$\langle +|-\rangle = \prod_k\ \det_{\sigma\sigma'}(v_+^\dagger(k,\sigma)v_-(k,\sigma'))$$

where σ and σ' are 6–valued labels comprising helicity and flavour. The eigenspinors can be computed approximately by expanding in powers of $1/\Lambda$,

$$\begin{aligned} u_\pm(k,\sigma) &= u_\pm(\sigma) + \frac{1}{4|\Lambda|}(1 \mp \gamma_5 T_c)\ V(k)u_\pm(\sigma) + \ldots \\ v_\pm(k,\sigma) &= v_\pm(\sigma) + \frac{1}{4|\Lambda|}(1 \mp \gamma_5 T_c)\ V(k)v_\pm(\sigma) + \ldots \end{aligned}$$

where $V(k) = \gamma_5(i\not{k} + \phi\cdot T)$ and $u_\pm(\sigma) = v_\mp(\sigma)$ are constant spinors normalized such that

$$\sum_\sigma\ u_\pm(\sigma)\ u_\pm^\dagger(\sigma) = \frac{1}{2}(1 \pm \gamma_5 T_c)$$

The simple perturbative structure of the eigenspinors depends crucially on the fact that $\gamma_5 T_c$ anticommutes with $V(k)$.

The Green's function for this system is given by

$$\begin{aligned}\tilde{G}(k) &= \frac{1}{\Omega}\sum_{\sigma,\sigma'} u_+(k,\sigma)\,\frac{\langle +|b_+(k,\sigma)b_-^\dagger(k,\sigma')|-\rangle}{\langle +|-\rangle}\,\bar{u}_-(k,\sigma') \\ &= \sum_{\sigma\sigma'} u_+(\sigma)\,(K^{-1})_{\sigma\sigma'}\,\bar{u}_-(\sigma)+\ldots\end{aligned}$$

to leading order in Λ, where the matrix K is given by

$$\begin{aligned}K_{\sigma\sigma'} &= \frac{1}{|\Lambda|}\,u_-^\dagger(\sigma)\,V(k)\,u_+(\sigma)+\ldots \\ &= \frac{1}{|\Lambda|}\,u_-^\dagger(\sigma)\,\gamma_5(i\not{k}+\phi\cdot T)\,u_+(\sigma)+\ldots\end{aligned}$$

Its inverse can be written

$$\begin{aligned}K^{-1}_{\sigma\sigma'} &= |\Lambda|\,u_+^\dagger(\sigma)\,V(k)^{-1}\,u_-(\sigma')+\ldots \\ &= |\Lambda|\,u_+^\dagger(\sigma)\,(i\not{k}+\phi\cdot T)^{-1}\,\gamma_5\,u_-(\sigma')+\ldots\end{aligned}$$

so that, finally,

$$\tilde{G}(k)=\frac{|\Lambda|}{2}\,(1+\gamma_5 T_c)\,(i\not{k}+\phi\cdot T)^{-1}+\ldots \qquad (*)$$

To interpret this we should rescale the fields to remove the factor $|\Lambda|$ and then take the limit $\Lambda\to\infty$,

$$\lim_{\Lambda\to\infty}\left(\Lambda^{-1/2}\,\hat{\psi}(x)\right)=\frac{1}{2}(1+\gamma_5 T_c)\,\psi(x)$$

The "wrong chirality" parts of the fields, $\gamma_5 T_c=-1$, do not contribute to the renormalized Green's functions in this limit. The effective fields satisfy $\gamma_5 T_c=1$, i.e. ν_L and e_L have $\gamma_5=1$ while e_R has $\gamma_5=-1$. If the Higgs doublet acquires the usual vacuum value, $\langle\phi_1\rangle=0$ and $\langle\phi_2\rangle=m$, then the neutrino is massless and the electron is massive.

The structure $(*)$ will be valid for any number of flavours provided $\gamma_5 T_c$ anticommutes with $V(k)$ i.e. if the chirality matrix, T_c, anticommutes with the mass matrix, $\phi\cdot T$.

3 External gauge field and anomalies

Having examined the free fermion Green's function we now include a vector background by the minimal prescription $\partial_\mu\to\nabla_\mu=\partial_\mu-i\,A_\mu(x)$, writing $H(A)=H(0)+V$. The Dirac vacua are perturbed.

$$|\pm\rangle\to|A\pm\rangle=\alpha_\pm(A)\left[1-\frac{1}{E_0-H_\pm(0)}\,\Pi_\pm(V-\Delta E_\pm)\right]^{-1}|\pm\rangle$$

where, again, the $\pm$ notation indicates the sign of Λ. The operators $\Pi_\pm$ project out the unperturbed vacua,

$$\Pi_\pm = 1 - |\pm\rangle\langle\pm|$$

and the normalization factors $\alpha_\pm(A)$ are chosen to be real and positive. The vacuum shifts, $\Delta E_\pm$, are determined by the consistency conditions

$$0 = \langle\pm|(V - \Delta E_\pm)|A\pm\rangle$$

The perturbed vacuum amplitude $\langle A+|A-\rangle$ depends on the vector background and we must test its response to gauge transformations. Acting on the Schroedinger picture fields these transformations are realized by unitary operators, U_θ,

$$U_\theta^{-1}\ \hat\psi(x)\ U_\theta = e^{i\theta(x)}\ \hat\psi(x)$$

where $\theta(x)$ is an hermitian matrix acting in flavour space. If the free Hamiltonian $H(0)$ is invariant under constant gauge transformations then

$$U_\theta^{-1}\ H(A)\ U_\theta = H(A^\theta)$$

where $A^\theta_\mu = e^{-i\theta}(A_\mu + i\partial_\mu)e^{i\theta}$. To keep the anomaly discussion as simple as possible we discard the mass term, $\phi\cdot T$ and require only that $\theta(x)$ commute with the chirality matrix, T_c.

If $A_\mu(x)$ is a weak, topologically trivial field then the perturbed Dirac vacua must be non–degenerate. It follows that

$$U_\theta|A\pm\rangle = |A^\theta\pm\rangle\ e^{i\Phi_\pm(\theta,A)}$$

where the angles $\Phi_\pm$ are real. The group property, $U_{\theta_1}\ U_{\theta_2} = U_{\theta_{12}}$ implies the composition rule,

$$\Phi(\theta_{12}, A) = \Phi(\theta_1, A^{\theta_2}) + \Phi(\theta_2, A)$$

identically in θ_1, θ_2 and A. It is possible to compute $\Phi_\pm(\theta, A)$ by applying time independent perturbation theory. To first order in θ this gives

$$\Phi_\pm(\theta, A) = \int_\Omega dx\ \langle\pm|\hat\psi^\dagger\ \theta\ \hat\psi \left[1 - \frac{1}{E_0 - H_\pm(0)}\ \Pi_\pm(V - \Delta E_\pm)\right]^{-1} |\pm\rangle$$

with the understanding that $\alpha_\pm(A)$ is real.

Since U_θ is unitary it follows that the vacuum amplitude must satisfy

$$\langle A^\theta + |A^\theta -\rangle = \langle A + |A-\rangle\ e^{i\Phi_+(\theta,A) - i\Phi_-(\theta,A)}$$

which implies an anomaly if $\Phi_+ - \Phi_- \neq 2\pi n, n \in \mathbb{Z}$. We compute this difference perturbatively and show that, in the limit $\Lambda \to \infty$, it contains the usual (consistent) anomalies. (The abelian anomaly in 2–dimensions was computed in an analogous way in Ref.4.)

A complete discussion would require regularization of the theory but we shall merely pick out the terms that contribute to the anomaly since they will be expressible as convergent integrals. In effect, we look for terms of second order in A that contain the tensor $\varepsilon_{\kappa\lambda\mu\nu}$. The second order part of Φ_+ is given by

$$\Phi_+^{(2)} = \sum_{n,m>0} \int_\Omega dx \ \langle +|\hat{\psi}^\dagger \ \theta \ \hat{\psi}|n\rangle \ \frac{1}{E_0 - E_n} \ \langle n|V|m\rangle \ \frac{1}{E_0 - E_m} \ \langle m|V|+\rangle$$

where the sums are restricted to 2–particle intermediate states,

$$\sum_n |n\rangle\langle n| = \frac{1}{\Omega^2} \ \Sigma \ |k\sigma, \overline{k'\sigma'}\rangle\langle k\sigma, \overline{k'\sigma'}|$$

where

$$|k\sigma, \overline{k'\sigma'}\rangle = b_+^\dagger(k,\sigma) \ d_+^\dagger(k',\sigma')|+\rangle$$

In the infinite volume limit this reduces to

$$\begin{aligned}\Phi_+^{(2)} \ &= \ \int dx \int \left(\frac{dp_1}{2\pi} \ \frac{dp_2}{2\pi} \ \frac{dk}{2\pi}\right)^4 \left[\frac{\mathrm{tr}(\theta(x)U(k)i\tilde{A\!\!\!/}(p_1)\gamma_5 U(k-p_1)i\tilde{A\!\!\!/}(p_2)\gamma_5 V(k-p_1-p_2))}{(\omega(k)+\omega(k-p_1-p_2))(\omega(k-p_1)+\omega(k-p_1-p_2))} - \right.\\ &\left. - \frac{\mathrm{tr}(\theta(x)U(k)i\tilde{A\!\!\!/}(p_1)\gamma_5 V(k-p_1)i\tilde{A\!\!\!/}(p_2)\gamma_5 V(k-p_1-p_2))}{(\omega(k)+\omega(k-p_1-p_2))(\omega(k)+\omega(k-p_1))}\right] e^{i(p_1+p_2)x}\end{aligned}$$

where $\omega(k) = \sqrt{k^2+\Lambda^2}$ and

$$U(k) = \frac{\omega_k + \gamma_5(i k\!\!\!/ + \Lambda)}{2\omega_k} = 1 - V(k)$$

To obtain $\Phi_-^{(2)}$, reverse the sign of Λ. Evalutating the Dirac trace and discarding terms that do not contain the antisymmetric tensor one finds

$$\begin{aligned}\Phi_\pm^{(2)} \ &= \ \pm\frac{|\Lambda|}{2} \int dx \int \left(\frac{dp_1}{2\pi} \ \frac{dp_2}{2\pi} \ \frac{dk}{2\pi}\right)^4 \frac{1}{\omega(k)\omega(k-p_1)\omega(k-p_1-p_2)} \ \frac{1}{\omega(k)+\omega(k-p_1-p_2)} \cdot \\ &\cdot \left(\frac{1}{\omega(k-p_1)+\omega(k-p_2)} + \frac{1}{\omega(k)+\omega(k-p_1)}\right) \varepsilon_{\kappa\lambda\mu\nu} \ p_{1\kappa}p_{2\lambda} \ \mathrm{tr}\left(\theta(x)\tilde{A}_\mu(p_1)\tilde{A}_\nu(p_2)\right)\end{aligned}$$

where the trace here is restricted to flavour space. The integral over k converges and can be estimated at large $|\Lambda|$ by setting $p_1 = p_2 = 0$ to obtain

$$\int \left(\frac{dk}{2\pi}\right)^4 \ \frac{1}{\omega(k)^5} = \frac{1}{12\pi^2|\Lambda|}$$

Hence, for $|\Lambda| \to \infty$,

$$\Phi_{\pm}^{(2)}(\theta, A) \to \pm \frac{1}{48\pi^2} \int dx \; \varepsilon_{\kappa\mu\lambda\nu} \; \mathrm{tr}\,(\theta(x)\; \partial_\kappa \; A_\mu \; \partial_\lambda \; A_\nu)$$

and we see that the difference, $\Phi_+ - \Phi_-$, indeed contains the usual consistent anomaly.

To construct anomaly–free models it is necessary to combine chiral multiplets in a suitably way. In effect this means choose a chirality matrix, T_c, that commutes with $\theta(x)$ and satisfies

$$\varepsilon_{\kappa\mu\lambda\nu} \; \mathrm{tr}\,(T_c \; \theta(x)\; \partial_\kappa \; A_\mu \; \partial_\lambda \; A_\nu) = 0$$

For example for one family of quarks and leptons in the standard $SU(2)_L \times U(1)_Y$ model choose $T_c^{lep} = diag(1, 1, -1)$ and $T_c^{quark} = diag(1, 1, -1, -1)$ for each colour.

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