IC/95/6

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

CHIRAL FERMIONS ON THE LATTICE

S. Randjbar–Daemi

and

J. Strathdee

INTERNATIONAL
ATOMIC ENERGY
AGENCY

UNITED NATIONS
EDUCATIONAL,
SCIENTIFIC
AND CULTURAL
ORGANIZATION

MIRAMARE-TRIESTE

IC/95/6

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

CHIRAL FERMIONS ON THE LATTICE

S. Randjbar–Daemi and J. Strathdee
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

The overlap approach to chiral gauge theories on arbitrary D–dimensional lattices is studied. The doubling problem and its relation to chiral anomalies for $D = 2$ and 4 is examined. In each case it is shown that the doublers can be eliminated and the well known perturbative results for chiral anomalies can be recovered. We also consider the multi–flavour case and give the general criteria for the construction of anomaly free chiral gauge theories on arbitrary lattices. We calculate the second order terms in a continuum approximation to the overlap formula in D dimensions and show that they coincide with the bilinear part of the effective action of D–dimensional Weyl fermions coupled to a background gauge field. Finally, using the same formalism we reproduce the correct Lorentz, diffeomorphism and gauge anomalies in the coupling of a Weyl fermion to 2–dimensional gravitational and Maxwell fields.

MIRAMARE – TRIESTE

January 1995

1 Introduction

A problem of long standing in particle physics has been to invent a lattice regularization scheme for chiral fermions. A solution would be desirable because it could be employed in conjunction with the well developed methods of lattice gauge theory to study non–perturbative phenomena such as chiral symmetry breaking in the standard model. Some progress seems to have been made recently. Narayanan and Neuberger [1], influenced by earlier work of Kaplan [2] and others [3], have constructed a lattice representation for the vacuum amplitude of chiral fermions coupled to a background gauge field that seems to meet all the physical requirements. In particular, it has been shown that in the continuum framework it carries the expected chiral anomalies in two [4] and four [5] dimensions. A lattice analysis of the 2–dimensional chiral Schwinger model has also been performed [6]. Whether this representation will turn out to be useful for numerical studies is a separate question.

The proposal of Narayanan and Neuberger is to embed the D–dimensional Euclidean chiral theory in a $D+1$–dimensional auxiliary system involving twice as many fields. For example, each 2–component Weyl spinor in 4 dimensions is represented by a 4–component Dirac spinor in $4+1$–dimensions. The embedding theory involves a mass parameter, Λ, which is eventually taken to infinity in such a way that all the physically irrelevant auxiliary states are projected out. The surviving states, zero modes, so to speak, of the embedding theory, represent chiral fermions in D–dimensions. The mechanism is somewhat obscure, to us at least, but it seems to work. It may prove useful and, therefore, we believe that it deserves to be better understood. Our aim in this paper is to describe the mechanism as we understand it and to present some calculations that we hope will clarify its function.

By way of historical introduction we begin with a brief description of the chirality problem for fermions on a lattice. In 4–dimensional Euclidean spacetime the Green's function for a 2–component Weyl spinor is given, in the continuum, by

$$\begin{aligned} G(k)^{-1} &= k_\mu\, \sigma_\mu \\ &= ik_4 + \underline{k}\cdot\underline{\sigma} \end{aligned} \tag{1.1}$$

where the momenta, k_μ, take values on the Euclidean 4–plane. Physical states are associated with poles at $k_4 = \pm i|\underline{k}|$. To put this on a lattice one must replace k_μ by a vector function $C_\mu(k)$ defined over a torus. In effect, the momenta must be viewed as angular variables, $k_\mu \sim k_\mu + 2\pi/a$ where a represents the lattice spacing. Physically meaningful states must have $k_\mu a \ll 1$ and the Green's function should reduce to (1.1) near $k_\mu a = 0$, i.e. $C_\mu(k) \sim k_\mu$.

There is a topological theorem due to Poincaré and Hopf [7] according to which the zeroes of a vector function on a torus do not occur singly. The occurrence of zeroes is conditioned by the requirement

$$\sum_{zeroes} \frac{\det C}{|\det C|} = 0 \tag{1.2}$$

where $\det C = \det_{\mu\nu}(\partial C_\mu/\partial k_\nu)$, evaluated at $C_\mu = 0$. Since physics requires $C_\mu(0) = 0$, the theorem indicates at least one more zero at some finite value of k_μ. The Green's function, $(C_\mu(k)\sigma_\mu)^{-1}$, therefore has additional poles corresponding to additional massless fermions. This is the origin of the problem [8].

To eliminate the unwanted states one might consider the modified Green's function

$$G(k)^{-1} = C_\mu(k)\sigma_\mu + B(k) \tag{1.3}$$

where the scalar function, $B(k)$, is chosen to vanish at $k = 0$ but nowhere else on the torus. For example, if $B(k) \sim rk^2$ then (1.3) would approximate (1.1) near $k = 0$ but there would be no other light fermions. Unfortunately, this is not an acceptable solution. It leads indirectly to the violation of Lorentz invariance in the low energy sector of the theory. Loop effects in theories with interactions give contributions involving integrals over the torus and these are, of course, quite generally not Lorentz invariant. However, Lorentz invariance (or $SO(4)$ in the Euclidean version) is recovered, approximately, in the low energy sector if the lattice theory possesses a sufficiently strong discrete symmetry. One of the 4–dimensional hypercubic crystal structures is generally adequate for this purpose. To recover chirality in the low energy sector it is further necessary that the crystal group itself admits chiral spinor representations (an example of this is the F_4 weight lattice whose Weyl group has such 2–component spinor representations [9]). The scalar term, $B(k)$, in (1.3) is not allowed by this kind of crystal symmetry. With 2–component fermions there is nothing else to be done, the problem is insoluble.

The approach of Narayanan and Neuberger uses 4–component fermions. Each Weyl spinor is paired with an auxiliary spinor of opposite chirality. Instead of working directly in 4–dimensional Euclidean spacetime they consider, following Kaplan, an auxiliary quantum mechanics problem in $4+1$–dimensions. They define two Hamiltonians,

$$H_{\pm} = \int d^4x \; \psi(x)^\dagger \; \gamma_5(\gamma_\mu \; \partial_\mu \pm |\Lambda|)\psi(x) \tag{1.4}$$

where γ_μ and γ_5 are hermitian Dirac matrices and Λ is an auxiliary parameter. The Schrödinger picture fields (from the $4+1$–dimensional point of view) satisfy appropriate anticommutation rules,

$$\{\psi(x), \psi^\dagger(x')\} = \delta_4(x - x')$$

etc. The eigenvalue spectra of H_+ and H_- are identical but their eigenstates are not. In particular, there are two distinct Dirac vacuum states, $|+\rangle$ and $|-\rangle$, whose overlap, $\langle +|-\rangle$, becomes an interesting quantity. Specifically, if the fermions are coupled to an external gauge potential in the usual way,

$$\partial_\mu \; \psi(x) \rightarrow (\partial_\mu - i \; A_\mu(x)) \; \psi(x)$$

then the overlap becomes a functional of A,

$$\langle A + |A - \rangle \;\; = \;\; \langle +|-\rangle \; e^{-\Gamma(A)} \tag{1.5}$$

This functional will be interpreted, in the limit $|\Lambda| \rightarrow \infty$, as the vacuum amplitude for a chiral fermion. Symbolically,

$$\ell n \;\det \left[\frac{1+\gamma_5}{2} \; (\partial\!\!\!/ - i \; A\!\!\!/) \right] \equiv - \lim_{|\Lambda| \rightarrow \infty} \; \Gamma(A) \tag{1.6}$$

Is this a sensible interpretation? The answer is not yet clear. At present all we can do is try to compute $\Gamma(A)$ and verify that it has the features to be expected of a chiral vacuum amplitude, including the anomaly.

Some plausibility is found, we believe, by defining an appropriate free fermion Green's function,

$$G_\Lambda(x-x') = \frac{1}{|\Lambda|} \frac{\langle +|\psi(x)\ \bar{\psi}(x')|-\rangle}{\langle +|-\rangle}$$

where $\bar{\psi} = \psi^\dagger \gamma_5$. In the limit this gives [5]

$$\lim_{|\Lambda|\to\infty} \widetilde{G}_\Lambda(k) = \frac{1+\gamma_5}{2} \frac{1}{i\not{k}} \tag{1.7}$$

which is encouraging. It suggests that the positive chirality components of the rescaled field $|\Lambda|^{-1/2}\ \psi(x)$ generate massless fermions while the negative chirality components do not. Further plausibility is added by perturbative computation of Γ, assuming A_μ to be weak. In Sec.5 we examine the terms of second order and show that the expected structure emerges. In Sec.4 we compute chiral anomalies in the 2– and 4–dimensional cases.

Most of our computations are of course expressed in the lattice regularized version of the theory since that is the real motivation for this work. The 1–body Hamiltonians, $\widetilde{H}_\pm = \gamma_5(i\not{k} \pm |\Lambda|)$, are replaced by

$$\widetilde{H}_\pm(k) = \gamma_5\left[i\ \gamma^\mu\ C_\mu(k) + (B(k) \pm |\Lambda|)T_c\right] \tag{1.8}$$

where T_c is a diagonal matrix with N_L eigenvalues -1, and N_R eigenvalues $+1$. It commutes with the Dirac matrices and serves to label the chiral flavours. The Hamiltonians (1.8) are invariant with respect to the global group $U(N_L) \times U(N_R)$. (The breaking of this symmetry by a mass term is considered in Sec.2.)

It should be emphasized at this point that Λ has nothing to do with ultraviolet regularization. The lattice takes care of that problem. Scales are generally in the relation

$$k \ll \Lambda \ll \frac{1}{a}$$

where k is a typical external momentum and a is the lattice spacing. While Λ goes to infinity relative to k, it goes to zero relative to the lattice scale $1/a$. We shall for the most part use a natural system of units with $a = 1$ in this paper. In these units Λ goes to zero but k/Λ is understood always to be small. In particular, the chiral anomaly, which shows up as a relative phase in the gauge transformation of the ground states $|A+\rangle$ and $|A-\rangle$, will be computed as the discontinuity at $\Lambda = 0$ of the angle functional. Of course, it is also possible to obtain the anomaly by setting $a = 0$, ignoring the ultraviolet divergences, keeping k finite and letting Λ go infinity. This procedure while not very satisfying from a mathematical point of view, is much simpler to apply. It will be used for the evaluation of the bilinear terms in Sec.5 and the 2–dimensional gravitational anomalies in Sec.6.

Throughout this paper the lattice, i.e. its geometry and for the most part its dimensionality will be arbitrary. Our Hamiltonians (1.8) are restricted only by the crystal symmetry of the lattice. In particular no further assumptions are made about the nature of couplings, i.e. nearest neighbour next nearest, etc.

In this paper we have exclusively concentrated on a particular approach to the problem of chiral fermions on the lattice. We should, however, emphasize that there are other interesting studies of this difficult problem in the literature [10].

The plan of this paper is as follows. Sec.2 treats the kinematics of free fermions, Green functions and chiral symmetry breaking on the lattice. Sec.3 introduces the coupling of lattice fermions to slowly varying weak external gauge fields and some perturbative formulae are derived. Sec.4 is devoted to the chiral anomalies. The cases of 2 and 4 dimensional lattices are discussed in detail. In this section the criteria for the cancellation of anomalies on an arbitrary 4–dimensional lattice are derived. In Sec.5 we calculate the bilinear part of the overlap amplitude in a continuum approximation and show that it coincides with the second order terms of the effective action of D–dimensional Weyl fermions coupled to an external gauge field. In Sec.6 we apply the continuum version of the overlap formalism to the computation of the Lorentz, diffeomorphism and Maxwell anomalies in the coupling of the Weyl fermions to 2–dimensional gravitational and electromagnetic fields. Some technical details of analysis of the relation between the doublers and chiral anomalies have been relegated to an appendix.

2 Free fermions

As described above the idea is to represent chiral fermions in D–dimensional Euclidean spacetime in terms of Dirac fermions in a $D+1$–dimensional Minkowskian space. We shall deal mainly with arbitrary D but, for simplicity, some of the features will be illustrated in $D=2$ and 4. To each chiral spinor in D–dimensions is associated a Dirac spinor in $D+1$–dimensions. The number of field components is doubled. Moreover, in the $D+1$–dimensional "parent" theory the fields depend on the extra unphysical coordinate and the number of degrees of freedom is thereby enlarged. To make a precise association of $D+1$–dimensional fields with D–dimensional degrees of freedom a regulator mass, Λ, is introduced in the parent Hamiltonian and a prescription is found for scaling the fields and taking the limit $p/\Lambda \to 0$ that projects onto chiral states with 4–momentum p_μ.

Although the required prescription could be formulated in continuum field theory we shall deal with fermions on a lattice since this is the real motivation behind this formalism. In the lattice formulation fermionic variables are associated with the sites of an integer lattice in D dimensions, $\psi = \psi(n), n \in \mathbb{Z}^D$. The fields have $2^{D/2}$ components and they satisfy the usual anticommutation rules,

$$\begin{aligned} \{\psi_\alpha(n), \psi_\beta(m)\} &= 0 , \\ \{\psi_\alpha(n), \psi_\beta(m)^*\} &= \delta_{nm}\,\delta_{\alpha\beta} \\ \{\psi_\alpha(n)^*, \psi_\beta(m)^*\} &= 0 \end{aligned} \tag{2.1}$$

where $\psi_\alpha(n)^*$ denotes the hermitian conjugate of $\psi_\alpha(n)$. The Fock vacuum is defined by

$$\psi_\alpha(n)|0\rangle = 0 \tag{2.2}$$

The free Hamiltonian is bilinear in ψ and ψ^*,

$$H = \sum_{n,m} \psi(n)^\dagger\; H(n-m)\; \psi(m) \tag{2.3}$$

where the 1–body Hamiltonian, $H(n-m)$, is a matrix in Dirac space. Translation invariance is assumed. The Fourier transform of the 1–body Hamiltonian is given by

$$\widetilde{H}(k) = \sum_n \; H(n)\; e^{-ikn} \tag{2.4}$$

where $kn = k_\mu\, n^\mu$ and the sum ranges over all sites of the integer lattice. Conversely,

$$\begin{aligned} H(n) &= \frac{1}{\Omega} \sum_k \widetilde{H}(k)\, e^{ikn} \\ &\rightarrow \int_{BZ} \left(\frac{dk}{2\pi}\right)^D \widetilde{H}(k)\, e^{ikn} \end{aligned} \tag{2.5}$$

where Ω denotes the number of sites on the lattice and the latter expression corresponds to the limit $\Omega \rightarrow \infty$. If Ω is finite then the sum over k_μ contains Ω terms. When Ω goes to infinity the resulting integral ranges over a cell of volume $(2\pi)^D$. Since $\widetilde{H}(k)$ is periodic it is defined, in effect, on a torus. The momenta range from $-\pi$ to π or, equivalently, over a suitably constructed Brillouin zone.

In addition to translation invariance it is necessary in practice to assume a crystal symmetry. Although it may not be the most general structure allowed by the symmetry, we shall adopt the form

$$\widetilde{H}(k) = \gamma_5 \Big(i\gamma^\mu\, C_\mu(k) + B(k) + \Lambda \Big) \tag{2.6}$$

where γ_μ and γ_5 are hermitian Dirac matrices.

$$\begin{aligned} \{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu} \\ \{\gamma^\mu, \gamma_5\} &= 0 \\ \gamma_5^2 &= 1 \end{aligned} \tag{2.7}$$

The metric tensor $g^{\mu\nu}$ is an invariant tensor in the sense that $g_{\mu\nu}\, n^\mu\, m^\nu$ is invariant with respect to the point group of the lattice. The coefficients, $C_\mu(k)$, $B(k)$ and Λ are real [1].

In the natural units employed here, the lattice spacing is equal to unity and the momentum components are angular variables. In these units the physically interesting momenta are concentrated around $k = 0$. The crystal symmetry forces the vector C_μ to vanish at the origin and it can be normalized such that

$$C_\mu(k) \simeq k_\mu + \ldots \tag{2.8a}$$

[1]For example, a simple cubic lattice with nearest neighbour couplings would have $g_{\mu\nu} = \delta_{\mu\nu}$, $C_\mu(k) = \sin k_\mu$ and $B(k) = r \sum_\mu (1 - \cos k_\mu)$.

near $k = 0$. By separating out the constant Λ we may assume

$$B(k) \simeq rk^2 + \dots \tag{2.8b}$$

where $k^2 = g^{\mu\nu}\, k_\mu\, k_\nu$ and r is a constant. The $SO(D)$ invariant structure (2.8) around $k = 0$ is ensured by imposing a sufficiently potent crystal symmetry.

The constant Λ will play an auxiliary role in projecting out the zero modes. It will be taken to be large compared to the typical momentum k_μ, but small relative to the ultraviolet cutoff,

$$k_\mu \ll |\Lambda| \ll 1 \tag{2.9}$$

The sign of Λ is significant and in the following we shall often indicate it explicitly, for example by writing

$$\widetilde{H}_\pm(k) = \gamma_5\Big(i\gamma^\mu\, C_\mu(k) + B(k) \pm |\Lambda|\Big) \tag{2.10}$$

To discuss the Hilbert space of the $D + 1$-dimensional theory it is useful to define creation and annihilation operators for the free fermions and their antiparticles by making a plane wave expansion in the Schrödinger picture,

$$\psi(n) = \frac{1}{\Omega} \sum_{k,\sigma} \Big(b_\pm(k,\sigma)\; u_\pm(k,\sigma) + d^\dagger_\pm(k,\sigma)\; v_\pm(k,\sigma)\Big)\, e^{ikn} \tag{2.11}$$

where $u_\pm$ and $v_\pm$ denote the positive and negative energy eigenspinors of the two Hamiltonians (2.10),

$$\widetilde{H}_\pm(k)\; u_\pm(k,\sigma) = \omega_\pm(k)\; u_\pm(k,\sigma), \quad \widetilde{H}_\pm(k)\; v_\pm(k,\sigma) = -\omega_\pm\; v_\pm(k,\sigma)$$

where

$$\omega_\pm(k) = \sqrt{C_\mu(k)\; C^\mu(k) + (B(k) \pm |\Lambda|)^2} \tag{2.12}$$

Since $\widetilde{H}_+$ and $\widetilde{H}_-$ are hermitian matrices the two sets (u_+, v_+) and (u_-, v_-) are orthonormal and complete. A convenient choice is given by

$$\begin{aligned} u_\pm(k,\sigma) &= \frac{\omega_\pm + B \pm |\Lambda| - i\not{C}}{\sqrt{2\omega_\pm(\omega_\pm + B \pm |\Lambda|)}}\; \chi(\sigma) \\ v_\pm(k,\sigma) &= \frac{\omega_\pm - B \mp |\Lambda| + i\not{C}}{\sqrt{2\omega_\pm(\omega_\pm - B \mp |\Lambda|)}}\; \chi(\sigma) \end{aligned} \tag{2.13}$$

where $\gamma_5\chi(\sigma) = \chi(\sigma)$, $\chi^\dagger(\sigma)\chi(\sigma') = \delta_{\sigma\sigma'}$, and the spin lable, σ, takes the values $1, 2, \ldots, 2^{\frac{D}{2}-1}$. It is straightforward to find the relation between the two sets of eigenspinors,

$$\begin{aligned} u_-(k,\sigma) &= \cos\beta\; u_+(k,\sigma) - \sin\beta\; v_+(k,\sigma) \\ v_-(k,\sigma) &= \sin\beta\; u_+(k,\sigma) + \cos\beta\; v_+(k,\sigma) \end{aligned} \tag{2.14}$$

where the angle $\beta(k)$ is given by

$$\begin{aligned} \sin\beta(k) &= \sqrt{\frac{\omega_+ + B + |\Lambda|}{2\omega_+}\,\frac{\omega_- - B + |\Lambda|}{2\omega_-}} - \sqrt{\frac{\omega_+ - B - |\Lambda|}{2\omega_+}\,\frac{\omega_- + B - |\Lambda|}{2\omega_-}} \\ \cos\beta(k) &= \sqrt{\frac{\omega_+ - B - |\Lambda|}{2\omega_+}\,\frac{\omega_- - B + |\Lambda|}{2\omega_-}} + \sqrt{\frac{\omega_+ + B + |\Lambda|}{2\omega_+}\,\frac{\omega_- + B - |\Lambda|}{2\omega_-}} \end{aligned} \tag{2.15}$$

This angle lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. It is independent of σ.

The anticommutors of b_+, d_+ and their hermitian conjugates take the usual form. The only non–vanishing ones are

$$\{b_+(k,\sigma), b_+^\dagger(k',\sigma')\} = \Omega\;\delta_{kk'}\;\delta_{\sigma\sigma'} = \{d_+(k,\sigma), d_+^\dagger(k',\sigma')\}$$

Likewise for b_-, d_-. The two sets are related by a Bogoliubov transformation,

$$\begin{aligned} b_-(k,\sigma) &= \cos\beta\; b_+(k,\sigma) - \sin\beta\; d_+^\dagger(k,\sigma) \\ d_-^\dagger(k,\sigma) &= \sin\beta\; b_+(k,\sigma) + \cos\beta\; d_+^\dagger(k,\sigma) \end{aligned} \tag{2.16}$$

so that, for example,

$$\begin{aligned} \{d_+(k,\sigma), d_-^\dagger(k',\sigma')\} &= \Omega\cos\beta\;\delta_{kk'}\;\delta_{\sigma\sigma'} \\ \{d_+(k,\sigma), b_-(k',\sigma')\} &= -\Omega\sin\beta\;\delta_{kk'}\;\delta_{\sigma\sigma'} \end{aligned} \tag{2.17}$$

The Fock vacuum, $|0\rangle$, is annihilated by $b_\pm$ and $d_\pm^\dagger$. By filling the negative energy states one defines two Dirac vacua,

$$|\pm\rangle = \prod_{k,\sigma}\;\Omega^{-1/2}\;d_\pm(k,\sigma)|0\rangle \tag{2.18}$$

These vacuum states are annihilated by $d_\pm(k,\sigma)$, respectively. They are normalized and their overlap is given by

$$\langle +|-\rangle = \prod_k (\cos\beta(k))^\nu \tag{2.19}$$

where $\nu = 2^{\frac{D}{2}-1}$ is the number of spin states. Zeroes of the overlap, due to the vanishing of one or more of the factors $\cos\beta(k)$, will be interpreted as a signal that fermion zero–modes are present and these will be associated with physical states of D–dimensional chiral theory. To make this concrete we construct a 2–point Green's function.

The fermion Green's function is defined by the ratio

$$G(n-m) = \frac{\langle +|\psi(n)\ \psi(m)^\dagger|-\rangle}{\langle +|-\rangle} \tag{2.20}$$

It can be evaluated by substituting the plane wave expansions (2.11), using the commutation rules and the vacuum conditions, $d_-|-\rangle = 0 = \langle +|d_+^\dagger$,

$$\begin{aligned} G(n-m) &= \frac{1}{\Omega^2} \sum u_+(k,\sigma)\ \frac{\langle +|b_+(k,\sigma)\ b_-^\dagger(k'\sigma')|-\rangle}{\langle +|-\rangle}\ u_-^\dagger(k'\sigma')\ e^{ikn-ik'm} \\ &= \frac{1}{\Omega} \sum_k \tilde{G}(k)\ e^{ik(n-m)} \end{aligned}$$

where, to obtain $\tilde{G}(k)$, one needs the matrix element

$$\begin{aligned} \frac{\langle +|b_+(k,\sigma)\ b_-^\dagger(k',\sigma)|-\rangle}{\langle +|-\rangle} &= \Omega\ \delta_{kk'}\ \delta_{\sigma\sigma'}\ \cos\beta - \frac{\langle +|b_-^\dagger(k',\sigma')\ b_+(k,\sigma)|-\rangle}{\langle +|-\rangle} \\ &= \Omega\ \delta_{kk'}\ \delta_{\sigma\sigma'}\ \cos\beta + \sin\beta\ \sin\beta'\ \frac{\langle +|d_+(k',\sigma')\ d_-^\dagger(k,\sigma)|-\rangle}{\langle +|-\rangle} \\ &= \Omega\ \delta_{kk'}\ \delta_{\sigma\sigma'}\ \left(\cos\beta + \frac{\sin^2\beta}{\cos\beta}\right) \end{aligned}$$

Hence,

$$\tilde{G}(k) = \frac{1}{\cos\beta(k)} \sum_\sigma u_+(k,\sigma)\ u_-^\dagger(k,\sigma) \tag{2.21}$$

The polarization sum could be expressed in terms of C_μ, B and $|\Lambda|$ using the explicit formulae (2.13) but the result is not very informative. Instead we shall explore the vicinity of $k=0$. From (2.8) and (2.12) we have

$$\omega_\pm \simeq |\Lambda| + \frac{k^2}{2|\Lambda|} \pm rk^2 + \ldots$$

for $k \ll |\Lambda| \ll 1$. It follows from (2.13) and (2.15) that

$$\begin{aligned}
\cos\beta &\simeq \sqrt{\frac{k^2}{\Lambda^2}}\,(1+\ldots) \\
u_+(k,\sigma) &\simeq \left(1-\frac{i\not{k}}{2|\Lambda|}+\ldots\right)\chi(\sigma) \\
u_-(k,\sigma) &\simeq \left(-\frac{i\not{k}}{\sqrt{k^2}}+\frac{\sqrt{k^2}}{2|\Lambda|}+\ldots\right)\chi(\sigma)
\end{aligned}$$

which can be substituted into (2.21) to give

$$\tilde{G}(k) \simeq |\Lambda|\,\frac{1+\gamma_5}{2}\,\frac{i\not{k}}{k^2}+\ldots \tag{2.22}$$

The vanishing of $\cos\beta$ at $k=0$ is thereby seen to mimic the contribution of a massless fermion with chirality, $\gamma_5=+1$. To normalize this contribution it would be necessary to scale the fields, $\psi(n)\to|\Lambda|^{-1/2}\,\psi(n)$.

One may ask whether there are any other points on the momentum torus where $\cos\beta$ vanishes. This is the famous doubling problem for chiral fermions on a lattice. Since, for real k_μ the square roots in the expression (2.15) are non–negative, they must both vanish if $\cos\beta$ is to vanish. This implies

$$\omega_+ = \Big|B+|\Lambda|\Big| \quad \text{and} \quad \omega_- = \Big|B-|\Lambda|\Big|$$

or, according to (2.12), $C_\mu=0$. However, this is not sufficient. It is also necessary that $B+|\Lambda|$ and $B-|\Lambda|$ should have opposite signs. Necessary and sufficient conditions for the vanishing of $\cos\beta$ are

$$C_\mu(k)=0 \quad \text{and} \quad B(k)^2<\Lambda^2 \tag{2.23}$$

These conditions are satisfied at $k=0$. To avoid the unwelcome doubling of massless fermions they must be violated for $k\neq0$. According to the Poincaré–Hopf theorem there are other points on the momentum torus where $C_\mu=0$. However, if $B^2>\Lambda^2$ at these points then $\cos\beta$ will not vanish. Hence, to exclude the unwanted states one has only to choose a function $B(k)$ that vanishes at $k=0$ but nowhere else, and take $|\Lambda|$ sufficiently small. If this is done then the infrared structure of the theory will be strictly confined to the neighborhood of $k=0$.

Up to this point we have been considering a single Dirac fermion, $\psi_\alpha(n)$, on the lattice. Generalization is straightforward. Replace the Hamiltonians (2.10) by the matrices

$$\widetilde{H}_\pm(k)^{ij} = \gamma_5 \left\{ i\gamma^\mu C_\mu(k)\ \delta^{ij} + (B(k) \pm |\Lambda|)\ T_c^{ij} \right\} \tag{2.24}$$

where the (flavour) indices i, j run from 1 to $N_L + N_R$ and T_c is a diagonal matrix with N_L entries -1 and N_R entries $+1$. Define the Dirac vacua and the Green's function exactly as before. Most of the above formulae can be adapted by making the replacement $B \pm |\Lambda| \to (B \pm |\Lambda|)T_c$ and it is understood that the spinors $\chi(\sigma)$ now carry a flavour index and they are required to satisfy

$$\gamma_5\ T_c\ \chi(\sigma) = \chi(\sigma) \tag{2.25}$$

Equations (2.13) are replaced by

$$\begin{aligned} u_\pm(k,\sigma) &= \frac{\omega_\pm + B \pm |\Lambda| - i\not{C}\,T_c}{\sqrt{2\omega_\pm(\omega_\pm + B \pm |\Lambda|)}}\ \chi(\sigma) \\ v_\pm(k,\sigma) &= \frac{\omega_\pm - B \mp |\Lambda| + i\not{C}\,T_c}{\sqrt{2\omega_\pm(\omega_\pm - B \mp |\Lambda|)}}\ \chi(\sigma) \end{aligned} \tag{2.26}$$

and, near $k = 0$, (2.22) is replaced by

$$\widetilde{G}(k) \simeq |\Lambda|\ \frac{1 + \gamma_5 T_c}{2}\ \frac{i\not{k}}{k^2}\ T_c \tag{2.27}$$

The pole at $k^2 = 0$ now corresponds to a massless fermion with chirality $\gamma_5\ T_c = 1$. The peculiar factor, T_c, on the right of (2.27) can be removed if we define the adjoint field

$$\bar{\psi}(n) = \psi(n)^\dagger\ \gamma_5 \tag{2.28}$$

This is appropriate because γ_5 plays a role in the $4 + 1$–dimensional theory analogous to γ_0 in $3 + 1$–dimensional theory. With this definition the result (2.27) would be replaced by the form appropriate to chiral fermions in Euclidean spacetime,

$$\widetilde{G}(k)\gamma_5 \simeq |\Lambda|\ \frac{1 + \gamma_5 T_c}{2}\ \frac{1}{i\not{k}} \tag{2.29}$$

near $k = 0$. The left (right) flavours are distinguished by $T_c = -1(+1)$.

Up to this point we have described a system that represents massless chiral fermions in D–dimensional spacetime with a global chiral symmetry $U(N_L)\times U(N_R)$. To complete this section we consider the breaking of chiral symmetry by mass terms. We can suppose that the symmetry breaking is spontaneous, i.e. represent the mass by the expectation value of a Higgs field, although we shall not discuss the Higgs dynamics. Let the Hamiltonian (2.3) be modified by the addition of a Yukawa term.

$$\sum_n g\ \psi(n)^\dagger\ \gamma_5\ \phi(n)\ \psi(n) \tag{2.30}$$

where $\phi(n)$ belongs to the representation $(N_L, \bar{N}_R)\oplus(\bar{N}_L, N_R)$, i.e. an N_L+N_R hermitian matrix with zeroes in the $N_L\times N_L$ and $N_R\times N_R$ blocks. Suppose it acquires a constant vacuum expectation value

$$g\langle\phi(n)\rangle = m \tag{2.31}$$

The 1–body Hamiltonian (2.24) is thereby modified to read

$$\widetilde{H}(k) = \gamma_5(i\not{k}+m) \pm |\Lambda|\gamma_5\ T_c + \dots \tag{2.32}$$

near $k=0$. We take $m \ll |\Lambda|$ and assume that $B(k)$ has been chosen to eliminate any other light fermions. Our aim is to compute the Green's function near $k=0$ by expanding in powers of k/Λ and m/Λ, treating k and m as comparable. For this perturbative calculation it is convenient to introduce a new set of eigenspinors in place of (2.13).

Write (2.32) as the sum of zero and first order terms

$$\begin{aligned} \widetilde{H}_\pm(k) &= H_{0\pm} + V(k) \\ H_{0\pm} &= \pm|\Lambda|\gamma_5\ T_c \\ V(k) &= \gamma_5(i\not{k}+m) \end{aligned} \tag{2.33}$$

Note that $H_{0\pm}$ anticommutes with $V(k)$ because γ_5 anticommutes with $\not{k}$ and T_c anticommutes with m. Define the zeroth order eigenspinors $\chi_\pm(\sigma)$ such that

$$\gamma_5\ T_c\ \chi_\pm(\sigma) = \pm\chi_\pm(\sigma) \tag{2.34}$$

where σ represents spin and flavour. It is easy to verify that, to first order in $V/|\Lambda|$,

$$\begin{aligned} u_\pm(k,\sigma) &= \left(1+\frac{1}{2|\Lambda|}\,V+\ldots\right)\chi_\pm(\sigma) \\ v_\pm(k,\sigma) &= \left(1-\frac{1}{2|\Lambda|}\,V+\ldots\right)\chi_\mp(\sigma) \end{aligned} \tag{2.35}$$

where $u_\pm$ correspond to the eigenvalue $|\Lambda|$ of $\widetilde{H}_\pm$ and $v_\pm$ correspond to $-|\Lambda|$. The various overlaps are

$$\begin{aligned} u_+^\dagger(k,\sigma)\,u_-(k,\sigma') &= \frac{1}{|\Lambda|}\,\chi_+^\dagger(\sigma)\,V(k)\chi_-(\sigma')+\ldots \\ u_+^\dagger(k,\sigma)\,v_-(k,\sigma') &= \delta_{\sigma\sigma'}+\ldots \\ v_+^\dagger(k,\sigma)\,u_-(k,\sigma') &= \delta_{\sigma\sigma'}+\ldots \\ v_+^\dagger(k,\sigma)\,v_-(k,\sigma') &= -\frac{1}{|\Lambda|}\,\chi_-^\dagger(\sigma)\,V(k)\,\chi_+(\sigma')+\ldots \end{aligned} \tag{2.36}$$

where the dots represent terms of second order.

To compute the Green's function one needs the matrix element

$$\begin{aligned} \frac{\langle+|b_+(k,\sigma)\,b_-^\dagger(k',\sigma')|-\rangle}{\langle+|-\rangle} &= \Omega\,\delta_{kk'}\,u_+^\dagger(k,\sigma)\,u_-(k,\sigma') - \frac{\langle+|b_-^\dagger(k',\sigma')\,b_+(k,\sigma)|-\rangle}{\langle+|-\rangle} \\ &= \Omega\,\delta_{kk'}\,u_+^\dagger(k,\sigma)\,u_-(k,\sigma') - \\ -\sum_{\sigma_1,\sigma_1'} u_+^\dagger(k,\sigma)\,v_-(k,\sigma_1)\, &\frac{\langle+|d_+(k',\sigma_1')d_-^\dagger(k,\sigma_1)|-\rangle}{\langle+|-\rangle}\,v_+^\dagger(k',\sigma_1')\,u_-(k',\sigma') \end{aligned} \tag{2.37}$$

To evaluate the latter term use the commutation rules,

$$\begin{aligned} \{d_\pm^\dagger(k,\sigma),d_\pm(k',\sigma')\} &= \Omega\,\delta_{kk'}\,\delta_{\sigma\sigma'} \\ \{d_+^\dagger(k,\sigma),d_-(k',\sigma')\} &= \Omega\,\delta_{kk'}\,K_{\sigma\sigma'} \end{aligned}$$

where

$$K_{\sigma\sigma'} = v_+^\dagger(k,\sigma)\,v_-(k,\sigma') = -\frac{1}{|\Lambda|}\,\chi_-^\dagger(\sigma)\,V(k)\,\chi_+(\sigma')+\ldots$$

One obtains a ratio of determinants that expresses the inverse of K,

$$\begin{aligned} \frac{\langle+|d_+(k',\sigma_1')\,d_-^\dagger(k,\sigma_1)|-\rangle}{\langle+|-\rangle} &= \Omega\,\delta_{kk'}\,K^{-1}_{\sigma_1\sigma_1'} \\ &= -\Omega\,\delta_{kk'}\,|\Lambda|\,\chi_+^\dagger(\sigma_1)\,V(k)^{-1}\,\chi_-(\sigma_1')+\ldots \end{aligned} \tag{2.38}$$

where the dots indicate terms of zeroth order. Using this result together with the formulae (2.36) one finds the leading term in (2.37),

$$\frac{\langle +|b_+(k,\sigma)\ b_-^\dagger(k',\sigma')|-\rangle}{\langle +|-\rangle} = \Omega\ \delta_{kk'}\ |\Lambda|\ \chi_+^\dagger(\sigma)\ V(k)^{-1}\ \chi_-(\sigma') + \ldots$$

The Green's function follows immediately [5]

$$\begin{aligned}\tilde{G}(k) &= \frac{1}{\Omega} \sum_{\sigma,\sigma'} u_+(k,\sigma)\ \frac{\langle +|b_+(k,\sigma)\ b_-^\dagger(k,\sigma')|-\rangle}{\langle +|-\rangle}\ u_-^\dagger(k,\sigma') \\ &= |\Lambda|\ \frac{1+\gamma_5\ T_c}{2}\ (i\not{k}+m)^{-1}\ \gamma_5 + \ldots\end{aligned}$$

The chiral symmetry breaking mass matrix, anticommuting with T_c, connects left and right components as it should.

3 Yang–Mills coupling

The Hamiltonians $H_\pm$ discussed in Sec.2, for $N_L + N_R$ Dirac fermions on the lattice, are invariant with respect to global $U(N_L) \times U(N_R)$. This can be extended to local transformations by introducing Yang–Mills fields. To couple an external gauge field we adopt a lattice version of the standard minimal prescription. Define

$$H_\pm(A) = \sum_{n,m} \psi(n)^\dagger\ H_\pm(n-m)\ U(n,m)\ \psi(m) \tag{3.1}$$

where $H_\pm(n-m)$ is the free 1–body Hamiltonian defined as the Fourier transform of

$$\widetilde{H}_\pm(k) = \gamma_5\Big(i\gamma^\mu C_\mu(k) + \big(B(k) \pm |\Lambda|\big)T_c\Big) \tag{3.2}$$

and $U(n,m)$ is a unitary matrix in flavour space that commutes with the chirality matrix, T_c. The matrix $U(n,m)$ is a functional of the external gauge field, $A_\mu(x)$ which we assume to be smooth. It is defined by the path–ordered integral,

$$\begin{aligned}U(n,m) &= T\left(\exp\ i\int_m^n A\right) \\ &= T\left(\exp\ i\int_0^1 dt\ \dot{\xi}^\mu(t)\ A_\mu(\xi(t))\right)\end{aligned} \tag{3.3}$$

$$\xi^\mu(t) = t\ n^\mu + (1-t)m^\mu \tag{3.4}$$

The path is a straight line joining the lattice sites n and m. In the following we shall always assume that $A_\mu(x)$ is weak and slowly varying. The aim is to compute perturbative corrections to the vacuum amplitude $\langle +|-\rangle$.

Under local transformations,

$$A_\mu(x) \to A^\theta_\mu(x) = e^{i\theta(x)}\ (A_\mu(x) + i\partial_\mu)e^{-i\theta(x)} \tag{3.5}$$

where $\theta(x)$ belongs to the algebra of $U(N_L)\times U(N_R)$, the matrices U transform according to

$$U(n,m) \to U^\theta(n,m) = e^{i\theta(n)}\ U(n,m)\ e^{-i\theta(m)} \tag{3.6}$$

It follows that the Hamiltonian (3.1) transforms according to

$$\begin{aligned} H_\pm(A^\theta) &= \sum_{n,m} \psi(n)^\dagger\ H_\pm(n-m)\ e^{i\theta(n)}\ U(n,m)\ e^{-i\theta(m)}\ \psi(m) \\ &= U_\theta\ H_\pm(A)\ U_\theta^{-1} \end{aligned} \tag{3.7}$$

where U_θ is a unitary operator that acts on the fermions,

$$\begin{aligned} U_\theta\ \psi(n)\ U_\theta^{-1} &= e^{-i\theta(n)}\ \psi(n) \\ U_\theta\ \psi(n)^\dagger\ U_\theta^{-1} &= \psi(n)^\dagger\ e^{i\theta(n)} \end{aligned} \tag{3.8}$$

It can be expressed in terms of these fields,

$$U_\theta = \exp\left(i\ \sum_n\ \psi(n)^\dagger\ \theta(n)\ \psi(n)\right) \tag{3.9}$$

The perturbed vacuum states $|A\pm\rangle$ are obtained by solving the eigenvalue problem

$$H_\pm(A)|A\pm\rangle\ = |A\pm\rangle\ E_\pm(A) \tag{3.10}$$

or, writing $H_\pm(A) = H_\pm(0) + V$ and $E_\pm(A) = E_\pm(0) + \Delta E_\pm$,

$$(E_\pm(0) - H_\pm(0))|A\pm\rangle\ = (V - \Delta E_\pm)|A\pm\rangle$$

This can be expressed as an integral equation,

$$|A\pm\rangle\ = |\pm\rangle\ \alpha_\pm(A) + G_\pm(V - \Delta E_\pm)|A\pm\rangle \tag{3.11}$$

where $|\pm\rangle$ denotes the Dirac vacua of Sec.2 and

$$G_{\pm} = \frac{1 - |\pm\rangle\langle\pm|}{E_{\pm}(0) - H_{\pm}(0)} \tag{3.12}$$

The vacuum energy shifts $\Delta E_{\pm}$ are determined by the consistency requirement,

$$0 = \langle\pm|(V - \Delta E_{\pm})|A\pm\rangle \tag{3.13}$$

The Dirac vacua $|\pm\rangle$ are non–degenerate ground states of the free Hamiltonians $H_{\pm}(0)$ and, in perturbation theory at least, the interacting ground states $|A\pm\rangle$ must also be non–degenerate. They are expressed formally as solutions of (3.11),

$$|A\pm\rangle = \alpha_{\pm}(A)\left[1 - G_{\pm}(V - \Delta E_{\pm})\right]^{-1}|\pm\rangle \tag{3.14}$$

where the numerical factors $\alpha_{\pm} = \langle\pm|A\pm\rangle$ are determined up to a phase by the normalization condition $\langle A\pm|A\pm\rangle = 1$. To fix the phase we choose $\alpha_{\pm}$ to be real and positive.

In order to see to what extent the amplitude $\langle A+|A-\rangle$ resembles the vacuum amplitude for chiral fermions coupled to a weak external gauge potential we shall calculate it in a continuum approximation to second order in Sec.5. This is already a fairly cumbersome exercise and, although one could extend it to higher orders, one could not expect to learn much from it (except for possible anomalies to be considered in Sec.4).

Various matrix elements of the perturbation V are needed. In particular,

$$\begin{aligned}
\langle 1\bar{2}+|V|+\rangle &= \langle +|\; d_{+}(2)\; b_{+}(1)\; V|+\rangle \\
&= \sum_{n,m} \langle +|d_{+}(2)\; b_{+}(1)\; \psi(n)^{\dagger}\; H(n-m)(U(n,m)-1)\psi(m)|+\rangle \\
&= \sum_{n,m} e^{-ik_1 n + ik_2 m}\; u_{+}^{\dagger}(1)\; H(n-m)(U(n,m)-1)v_{+}(2)
\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
U(n,m) &= 1 + i\int_0^1 dt_1\; \dot{\xi}^{\mu}(t_1)\; A_{\mu}(\xi(t_1)) + \\
&\quad + \frac{i^2}{2}\int_0^1 dt_1\; dt_2\; \dot{\xi}^{\mu}(t_1)\; \dot{\xi}^{\nu}(t_2)\; T\left(A_{\mu}(\xi(t_1)\; A_{\nu}(\xi(t_2))\right) + \ldots
\end{aligned} \tag{3.16}$$

To simplify the calculations we consider the infinite volume limit where the momenta are continuous variables and

$$H(n-m) = \int_{BZ} \left(\frac{dk}{2\pi}\right)^D \widetilde{H}(k)\ e^{ik(n-m)}$$

The integral extends over a Brillouin zone of volume $(2\pi)^D$. The external field is smooth and slowly varying so that

$$A_\mu(x) = \int \left(\frac{dk}{2\pi}\right)^D \widetilde{A}_\mu(k)\ e^{ikx}$$

where the integrand is concentrated around $k=0$. With these assumptions the first order contribution reduces to

$$\begin{aligned} \langle 1\bar{2}+|V^{(1)}|+\rangle &= -\sum_n \int_0^1 dt\ u_+^\dagger(1)\ \widetilde{H}_+\left((1-t)k_1+tk_2-2\pi nt\right)^{,\mu} \cdot \\ &\quad \cdot \widetilde{A}_\mu(k_1-k_2+2\pi n)\ v_+(2) \end{aligned} \tag{3.17}$$

where the lattice sum is needed to ensure periodicity in k_1 and k_2, and $\widetilde{H}(p)^{,\mu} = \partial\widetilde{H}(p)/\partial p_\mu$. Since $\widetilde{A}$ is concentrated at the origin only one term from this sum will contribute for given values of k_1 and k_2. Also, since $k_1-k_2+2\pi n$ is small it is practical to expand $\widetilde{H}$ in powers of t and integrate this parameter. One may also express (3.17) in the form

$$\begin{aligned} \langle 1\bar{2}+|\ \frac{\delta V}{\delta \widetilde{A}_\mu^\alpha(p)}\ |+\rangle &= -(2\pi)^D\ \delta_{2\pi}(k_2-k_1+p)\ \cdot \\ &\quad \cdot \int_0^1 dt\ u_+^\dagger(1)\ \widetilde{H}_+(k_1-tp)^{,\mu}\ T_\alpha\ v_+(2) \end{aligned} \tag{3.18}$$

where the matrices T_α provide a basis for the algebra and $\delta_{2\pi}$ denotes the periodic delta function,

$$\delta_{2\pi}(k_2-k_1+p) = \sum_n \delta(k_2-k_1+p+2\pi n)$$

In the same notation the second order contribution is given by

$$\begin{aligned} &\langle 1\bar{2}+|\ \frac{\delta^2 V}{\delta\widetilde{A}_\mu^\alpha(p_1)\ \delta\widetilde{A}_\nu^\beta(p_2)}|+\rangle = \\ &= (2\pi)^D\ \delta_{2\pi}(k_2-k_1+p_1+p_2) \int_0^1 dt_1\ dt_2\ u_+^\dagger(1)\ \widetilde{H}(k_1-t_1p_1-t_2p_2)^{,\mu\nu} \cdot \\ &\quad \cdot \{\theta(t_1-t_2)\ T_\alpha T_\beta + \theta(t_2-t_1) T_\beta T_\alpha\}\ v_+(2) \end{aligned} \tag{3.19}$$

Again the integrals over t_1 and t_2 can be evaluated by expanding in powers of the small momenta p_1 and p_2.

Other 2–particle matrix elements of V are obtained from the expressions (3.18) and (3.19) by using the appropriate eigenspinors in place of $u_+(1)$ and $v_+(2)$. For example, the matrix element $\langle 1+|V|2+\rangle$ involves the replacement $v_+(2) \to u_+(2)$, whereas $\langle\bar{2}+|V|\bar{1}+\rangle$ requires $u_+^\dagger(1) \to -v_+^\dagger(1)$, etc. In these particular examples, however, if $k_1 = k_2$ there will be a vacuum contribution,

$$\langle +|\frac{\delta V}{\delta \widetilde{A}^\alpha_\mu(p)}|+\rangle = -(2\pi)^D\ \delta_{2\pi}(p) \int_{BZ} \left(\frac{dk}{2\pi}\right)^D \sum_\sigma v_+^\dagger(k)\ \widetilde{H}(k)^{,\mu}\ T_\alpha\ v_+(k) \quad (3.20)$$

$$\langle +|\frac{\delta^2 V}{\delta \widetilde{A}^\alpha_\mu(p_1)\ \delta \widetilde{A}^\beta_\nu(p_2)}|+\rangle = (2\pi)^D\ \delta_{2\pi}(p_1+p_2) \int_0^1 dt_1\ dt_2 \cdot$$
$$\cdot \int_{BZ} \left(\frac{dk}{2\pi}\right)^D \sum_\sigma v_+^\dagger(k)\ \widetilde{H}(k - t_1p_1 - t_2p_2)^{,\mu\nu}\{\theta(t_1-t_2)T_\alpha T_\beta +$$
$$+\theta(t_2-t_1)T_\beta T_\alpha\}\ v_+(k) \quad (3.21)$$

The first order term (3.20) vanishes if the crystal symmetry is large enough and we shall assume this is the case.

4 Chiral anomalies

The computation of perturbative corrections to the ground states was considered in Sec.3. Here we wish to examine the response of these states to a gauge transformation.

Gauge transformations are implemented by the unitary operators, U_θ, defined by (3.9). Acting on the Hamiltonians (3.1) they give

$$U_\theta\ H_\pm(A)\ U_\theta^{-1} = H_\pm(A^\theta) \quad (4.1)$$

where A^θ is given by (3.5). Since the perturbative ground state is not degenerate we must have

$$U_\theta|A\pm\rangle\ = |A^\theta\pm\rangle\ e^{i\Phi_\pm(\theta,A)} \quad (4.2)$$

where $\Phi_\pm$ is real. The ground state energies, $E_\pm(A)$, must be invariant. The functionals $\Phi_\pm$ satisfy a composition rule that reflects the group property,

$$e^{i\theta_1}\ e^{i\theta_2} = e^{i\theta_{12}} \quad (4.3)$$

Applying the operators U_{θ_1} and U_{θ_2} successively to the states $|A\pm\rangle$, using (4.2), one finds

$$\Phi_\pm(\theta_{12}, A) = \Phi_\pm(\theta_1, A^{\theta_2}) + \Phi_\pm(\theta_2, A) \tag{4.4}$$

which must hold identically in θ_1, θ_2 and A when θ_{12} is expressed in terms of θ_1 and θ_2. For infinitesimal θ_1 and θ_2 (4.3) gives

$$\theta_{12} = \theta_1 + \theta_2 + \frac{i}{2}\,[\theta_1, \theta_2] + \ldots$$

or, using a basis of hermitian matrices to write $\theta = \theta^\alpha\, T_\alpha$,

$$\theta_{12}^\alpha = \theta_1^\alpha + \theta_2^\alpha + \frac{1}{2}\,\theta_1^\gamma\,\theta_2^\beta\,c_{\beta\gamma}{}^{\alpha} + \ldots \tag{4.5}$$

where $c_{\beta\gamma}{}^{\alpha}$ is a structure constant. This formula is accurate up to second order. Substituting the expansion,

$$\Phi(\theta, A) = \int dx\;\theta^\alpha(x)\;\Phi_\alpha(x|A) + \frac{1}{2}\int dx\;dx'\;\theta^\alpha(x)\;\theta^\beta(x')\;\Phi_{\alpha\beta}(x, x'|A) + \ldots$$

into the composition rule (4.4) one finds, in second order,

$$\frac{1}{2}\,\delta(x-x')\;c_{\alpha\beta}{}^{\gamma}\,\Phi_{\gamma\pm}(x|A) + \frac{1}{2}\Big(\Phi_{\alpha\beta\pm}(x, x'|A) + \Phi_{\beta\alpha\pm}(x', x|A)\Big) = -\nabla'_\mu\left(\delta\Phi_{\alpha\pm}(x|A)/\delta A^\beta_\mu(x')\right) \tag{4.6}$$

where ∇_μ denotes the covariant derivative ($A^\theta_\mu = A_\mu + \nabla_\mu\theta + \ldots$).

The ground state overlap $\langle A+|A-\rangle$ may not be gauge invariant. According to (4.2) it satisfies

$$\langle A^\theta + | A^\theta - \rangle \;=\; \langle A + | A - \rangle\; e^{ig(\theta, A)} \tag{4.7}$$

where $g(\theta, A) = \Phi_+(\theta, A) - \Phi_-(\theta, A)$ may not vanish. This is how the chiral anomaly is expressed in the overlap formulation. From the composition formulae (4.6) it follows, in particular, that the first order part of $g(\theta, A)$ must satisfy the equations

$$\nabla_\mu\,\frac{\delta\; g_\beta(x'|A)}{\delta\; A^\alpha_\mu(x)} - \nabla'_\mu\,\frac{\delta\; g_\alpha(x|A)}{\delta\; A^\beta_\mu(x')} = \delta(x-x')\;c_{\alpha\beta}{}^{\gamma}\;g_\gamma(x|A) \tag{4.8}$$

which will be recognized as the Wess–Zumino consistency conditions. The functional $g_\alpha(x|A)$ should therefore be interpreted as the so-called "consistent" anomaly [11].

To compute the angle Φ_+ it is sufficient to consider one component of the defining equation (4.2),

$$e^{i\Phi_+(\theta,A)} = \frac{\langle +|U_\theta|A+\rangle}{\langle +|A_+^\theta\rangle}$$

For infinitesimal θ this gives, using (3.9),

$$\begin{aligned}\Phi_+(\theta,A) &= \sum_n \frac{\langle +|\psi(n)^\dagger\ \theta(n)\ \psi(n)|A+\rangle}{\langle +|A+\rangle} + \\ &+\frac{1}{i}\int dx\ \theta^\alpha(x)\ \nabla_\mu\ \frac{\delta}{\delta A_\mu^\alpha(x)}\ \ell n\ \langle +|A+\rangle \qquad (4.9)\end{aligned}$$

Since $\langle +|A+\rangle = \alpha_+(A)$ is defined to be real and positive the second term here is pure imaginary. It makes no contribution to the real angle, Φ_+. Hence, the first order part of Φ_+ is given by

$$\begin{aligned}\Phi_+(\theta,A) &= \mathrm{Re}\sum_n \frac{\langle +|\psi(n)^\dagger\ \theta(n)\ \psi(n)|A+\rangle}{\langle +|A+\rangle} \\ &= \mathrm{Re}\sum_n \langle +|\psi(n)^\dagger\ \theta(n)\ \psi(n)\left(1-G_+(V-\Delta E_+)\right)^{-1}|+\rangle \qquad (4.10)\end{aligned}$$

on substituting the perturbation series (3.14). A similar expression gives $\Phi_-(\theta,A)$.

There is no reason to expect the quantity $g = \Phi_+ - \Phi_-$ to vanish in general. The Hamiltonians $\widetilde{H}_\pm(k)$ depend on the auxiliary parameter Λ through the combinations, $B(k) \pm |\Lambda|$, respectively. There is no symmetry operator that relates them. However, it must be kept in mind that Λ is not an ultraviolet regulator. In the natural units we are using, it is a small number. The physically significant features of the system are to be looked for in the infra-red sector,

$$|k| \ll |\Lambda| \ll 1$$

Amplitudes should be expanded in powers of k and the coefficients should then be evaluated at $\Lambda = 0$. Singularities here correspond to ultraviolet divergences and must be compensated in the usual way by counterterms. The chiral anomaly, if it is present, will appear as a discontinuity in the angle Φ at $\Lambda = 0$. To illustrate the mechanism we consider firstly the 2-dimensional case, where the anomaly should come in the second order part of the overlap $\langle A+|A-\rangle$ or, equivalently, in the first order part of $\Phi_+ - \Phi_-$. The 4-dimensional case will be considered below.

The first order contributions to $\Phi_\pm$ are easy to evaluate using the matrix elements (3.18),

$$\begin{aligned}
\Phi_+^{(1)}(\theta, A) &= \frac{1}{2} \sum_n \langle +|\psi(n)^\dagger\ \theta(n)\ \psi(n)\ G_+\ V^{(1)}|+\rangle + c.c. \\
&= -\frac{1}{2} \sum_n \int \left(\frac{dk_1}{2\pi}\right)^2 \left(\frac{dk_2}{2\pi}\right)^2 \langle +|\psi(n)^\dagger\ \theta(n)\ \psi(n)|1,\bar{2}+\rangle\ \ (\omega_+(1)+\omega_+(2))^{-1} \cdot \\
&\quad \cdot \int \left(\frac{dp}{2\pi}\right)^2 \langle 1,\bar{2}+|\ \frac{\delta V}{\delta \widetilde{A}_\mu^\alpha(p)}\ |+\rangle\ \widetilde{A}_\mu^\alpha(p) + c.c. \\
&= \frac{1}{2} \int \left(\frac{dk_1}{2\pi}\right)^2 \left(\frac{dk_2}{2\pi}\right)^2 v_+^\dagger(2)\ \tilde{\theta}(2-1)\ u_+(1)\,(\omega_+(1)+\omega_+(2))^{-1} \cdot \\
&\quad \cdot \int \left(\frac{dp}{2\pi}\right)^2 (2\pi)^2\ \delta_{2\pi}(k_2-k_1+p) \int_0^1 dt\ u_+^\dagger(1)\ \widetilde{H}(k_1-tp)^{,\mu}\ T_\alpha\ v_+(2)\ \widetilde{A}_\mu^\alpha(p) + c.c. \\
&= \int \left(\frac{dp}{2\pi}\right)^2 \tilde{\theta}_\alpha(-p)\ F_+^\mu(p)\ \widetilde{A}_\mu^\alpha(p) \qquad (4.11)
\end{aligned}$$

where $F_+^\mu(p) = F_+^\mu(-p)^*$ is given by the loop integral,

$$\begin{aligned}
F_+^\mu(p) &= \frac{1}{2} \int_0^1 dt \int_{BZ} \left(\frac{dk}{2\pi}\right)^2 \left(\omega_+\big(k+\frac{p}{2}\big) + \omega_+\big(k-\frac{p}{2}\big)\right)^{-1} \cdot \\
&\quad \cdot \mathrm{tr}\left(U_+\big(k+\frac{p}{2}\big)\ \widetilde{H}\left(k+\big(\frac{1}{2}-t\big)p\right)^{,\mu}\ V_+\big(k-\frac{p}{2}\big) + \right. \\
&\quad \left. + V_+\big(k+\frac{p}{2}\big)\ \widetilde{H}\left(k+\big(\frac{1}{2}-t\big)p\right)^{,\mu}\ U_+\big(k-\frac{p}{2}\big)\right) \qquad (4.12)
\end{aligned}$$

The Hamiltonians,

$$\widetilde{H}_\pm(k) = \gamma_5\ i\gamma^\nu\ C_\nu(k) + (B(k) \pm |\Lambda|)\,\gamma_5\ T_c$$

commute with the generators T_α. The matrices $U_\pm, V_\pm$ project onto positive and negative energy eigenstates of $\widetilde{H}_\pm$,

$$\begin{aligned}
U_\pm(k) &= \frac{\omega_\pm(k) + \widetilde{H}_\pm(k)}{2\omega_\pm(k)} \\
&= 1 - V_\pm(k)
\end{aligned}$$

where $\omega_\pm = \sqrt{C^2 + (B \pm |\Lambda|)^2}$. Since $U_+V_+ = 0$ it is clear that (4.12) vanishes at $p = 0$. The first derivative reduces to the simple form

$$F_\pm^{\mu,\nu}(0) = \frac{1}{16} \int_{BZ} \left(\frac{dk}{2\pi}\right)^2 \frac{1}{\omega_\pm^3}\ \mathrm{tr}\left([\widetilde{H}^{,\mu}, \widetilde{H}^{,\nu}]\widetilde{H}_\pm\right) \qquad (4.13)$$

The trace over 2×2 Dirac matrices gives, for each flavour,

$$\begin{aligned} \mathrm{tr}\left([\widetilde{H}^{,\mu}, \widetilde{H}^{,\nu}]\widetilde{H}_{\pm}\right) &= 4i\ \varepsilon^{\alpha\beta}\ C_{\alpha}{}^{,\mu}\ C_{\beta}{}^{,\nu}\ (B \pm |\Lambda|)T_c - 4i\ \varepsilon^{\alpha\beta}\ (C_{\alpha}{}^{,\mu}\ B^{,\nu} - C_{\alpha}{}^{,\nu}\ B^{,\mu})C_{\beta}\ T_c \\ &= 4i\ \varepsilon^{\mu\nu}\ T_c\Big\{(B \pm |\Lambda|)\ \det C - C_{\alpha}{}^{,\alpha}\ B^{,\beta}\ C_{\beta} + C_{\alpha}{}^{,\beta}\ B^{,\alpha}\ C_{\beta}\Big\} \qquad (4.14) \end{aligned}$$

A discontinuity at $\Lambda = 0$, i.e. $F_{+}^{\mu,\nu} \neq F_{-}^{\mu,\nu}$, can arise in (4.13) from infra–red singularities, points where $\omega_{\pm}(k) = 0$ at $\Lambda = 0$. One such point is the origin since $C_{\mu}(k) \sim k_{\mu}$ and $B(k) \sim k^2$ near $k = 0$. In general one must sum the contributions from all points where $C_{\mu} = B = 0$. The contribution of one such point is evaluated in the Appendix,

$$F_{+}^{\mu,\nu}(0) - F_{-}^{\mu,\nu}(0) = \frac{1}{4\pi}\ i\ \varepsilon^{\mu\nu}\ T_c\ \frac{\det C}{|\det C|} \qquad (4.15)$$

If $B(p)$ was set equal to zero everywhere then there would be a contribution from every zero of $C_{\mu}(k)$, the total for each flavour would then contain the factor

$$\sum_{zeroes} \frac{\det C}{|\det C|} = 0$$

The vanishing of this sum is a consequence of the toroidal topology of momentum space (Poincaré–Hopf theorem). If $B(p) \equiv 0$ then there is no anomaly. On the other hand if $B(p)$ vanishes only at $p = 0$, there can be an anomaly, viz.

$$\Phi_{+}(\theta, A) - \Phi_{-}(\theta, A) = -\frac{1}{8\pi} \int dx\ \varepsilon^{\mu\nu}\ \mathrm{tr}(T_c\ \theta(x)\ F_{\mu\nu}(x)) \qquad (4.16)$$

We repeat this calculation for the 4–dimensional case. Now we assume that $B(k)$ has been chosen to eliminate any unwanted states, its only zero being at $k = 0$. The anomaly should appear in the second order part of $\Phi_{+} - \Phi_{-}$. A straightforward application of the formulae developed in Sec.3 gives

$$\begin{aligned} \Phi_{+}^{(2)}(\theta, A) &= \mathrm{Re} \sum_{n} \langle +|\psi(n)^{\dagger}\ \theta(n)\ \psi(n) \left(G_{+}V^{(2)} + G_{+}V^{(1)}G_{+}V^{(1)}\right)|+\rangle \\ &= \mathrm{Re}\ \frac{1}{\Omega^2} \sum_{1,2} v_{+}^{\dagger}(2)\ \tilde{\theta}(2-1)\ u_{+}(1)\langle 1, \bar{2}+|\left(G_{+}V^{(2)} + G_{+}V^{(1)}G_{+}V^{(2)}\right)|+\rangle \\ &= \frac{1}{2}\ \mathrm{Re} \int \left(\frac{dp}{2\pi}\right)^4 \left(\frac{dq}{2\pi}\right)^4\ \tilde{A}_{\mu}^{\alpha}(p)\ \tilde{A}_{\nu}^{\beta}(q)\ \tilde{\theta}^{\gamma}(-p-q)\ R_{\alpha\beta\gamma}^{\mu\nu}(p, q) \qquad (4.17) \end{aligned}$$

where, in the limit $\Omega \to \infty$, the kernel $R^{\mu\nu}_{\alpha\beta\gamma}$ is given by the 1–loop integral,

$$R^{\mu\nu}_{\alpha\beta\gamma}(p,q) = \int_{BZ} \left(\frac{dk}{2\pi}\right)^4 \left(\omega_+(k) + \omega_+(k-p-q)\right)^{-1} \int_0^1 dt\, dt' \cdot$$
$$\cdot \Bigg[-\mathrm{tr}\Big(U_+(k)\ \widetilde{H}(k-tp-t'q)^{,\mu\nu}\ V_+(k-p-q)\{\theta(t-t')T_\alpha T_\beta + \theta(t'-t)T_\beta T_\alpha\}T_\gamma\Big) +$$
$$+ \frac{2}{\omega_+(k-p-q)+\omega_+(k-p)}\ \mathrm{tr}\Big(U_+(k)\ \widetilde{H}(k-tp)^{,\mu}\ U_+(k-p)\ \widetilde{H}(k-p-t'q)^{,\nu} \cdot$$
$$\cdot V_+(k-p-q)T_\alpha T_\beta T_\gamma\Big) -$$
$$- \frac{2}{\omega_+(k)+\omega_+(k-p)}\ \mathrm{tr}\Big(U_+(k)\ \widetilde{H}(k-tp)^{,\mu}\ V_+(k-p)\ \widetilde{H}(k-p-t'q)^{,\nu} \cdot$$
$$\cdot V_+(k-p-q)T_\alpha T_\beta T_\gamma\Big)\Bigg] \qquad (4.18)$$

The analogous formulae for Φ_- are obtained by replacing Λ with $-\Lambda$ wherever it occurs.

Instead of examining every term in the somewhat daunting expression (4.18) we shall assume that the discontinuity at $\Lambda = 0$ is confined to components that contain the antisymmetric tensor, $\varepsilon^{\kappa\lambda\mu\nu}$. These parts are relatively easy to pick out. Notice, firstly, that the trace involving $\widetilde{H}^{,\mu\nu}$ makes no contribution since it has too few γ matrices. The second and third terms in (4.18) make similar contributions involving the Dirac trace

$$\frac{1}{4}\ \mathrm{tr}\left(\widetilde{H}(k)\ \widetilde{H}(k-tp)^{,\mu}\ \widetilde{H}(k-p)\ \widetilde{H}(k-p-t'q)^{,\nu}\ \widetilde{H}(k-p-q)\right) =$$
$$= 4\ \varepsilon^{\kappa\lambda\sigma\tau}\Big[(B(k)+\Lambda)\ C_\kappa(k-tp)^{,\mu}\ C_\lambda(k-p)\ C_\sigma(k-p-t'q)^{,\nu}\ C_\tau(k-p-q) -$$
$$-C_\kappa(k)\ B(k-tp)^{,\mu}\ C_\lambda(k-p)\ C_\sigma(k-p-t'q)^{,\nu}\ C_\tau(k-p-q) +$$
$$+C_\kappa(k)\ C_\lambda(k-tp)^{,\mu}\ \left(B(k-p)+\Lambda\right)\ C_\sigma(k-p-t'q)^{,\nu}\ C_\tau(k-p-q) -$$
$$-C_\kappa(k)\ C_\lambda(k-tp)^{,\mu}\ C_\sigma(k-p)\ B(k-p-t'q)^{,\nu}\ C_\tau(k-p-q) +$$
$$+C_\kappa(k)\ C_\lambda(k-tp)^{,\mu}\ C_\sigma(k-p)\ C_\tau(k-p-t'q)^{,\nu}\ \left(B(k-p-q)+\Lambda\right)\Big] \qquad (4.19)$$

plus terms not containing ε. Now expand. Since $\tilde{A}(p), \tilde{A}(q)$ are concentrated around the origin it is feasible to treat p and q as small variables. Moreover, since we are looking for the discontinuity at $\Lambda = 0$ which is caused by the infra-red singularity at $k = 0$, we can assume that the small Λ behaviour is dominated by the neighbourhood of $k = 0$.

Expanding (4.19) in powers of p, q and k we need to retain only the terms of second order,

$$\begin{aligned}
&4\ \varepsilon^{\kappa\lambda\sigma\tau}\Big(\Lambda\ \delta^{\mu}_{\kappa}(k-p)_{\lambda}\ \delta^{\nu}_{\sigma}(k-p-q)_{\tau}+\\
&\quad +k_{\kappa}\ \delta^{\mu}_{\lambda}\ \Lambda\ \delta^{\nu}_{\sigma}\ \ (k-p-q)_{\tau}+\\
&\quad +k_{\kappa}\ \delta^{\mu}_{\lambda}\ \ (k-p)_{\sigma}\ \delta^{\nu}_{\tau}\ \Lambda\Big)\\
&= 4\ \Lambda\ \varepsilon^{\mu\nu\lambda\tau}\ p_{\lambda}\ q_{\tau}
\end{aligned}$$

The relevant part of (4.18) therefore reduces to

$$\begin{aligned}
R^{\mu\nu}_{\alpha\beta\gamma}(p,q) &\simeq -\frac{1}{2}\ \Lambda\ \varepsilon^{\mu\nu\lambda\tau}\ p_{\lambda}\ q_{\tau}\ \mathrm{tr}(T_cT_{\alpha}T_{\beta}T_{\gamma})\int\left(\frac{dk}{2\pi}\right)^4\frac{1}{\omega^5}\\
&= -\frac{1}{24\pi^2}\ \frac{\Lambda}{|\Lambda|}\ \varepsilon^{\mu\nu\lambda\tau}\ p_{\lambda}\ q_{\tau}\ \mathrm{tr}(T_cT_{\alpha}T_{\beta}T_{\gamma}) \qquad (4.20)
\end{aligned}$$

near $\Lambda = 0$. The anomaly is given by

$$\begin{aligned}
\Phi_{+}-\Phi_{-} &= -\frac{1}{24\pi^2}\int\left(\frac{dp}{2\pi}\right)^4\left(\frac{dq}{2\pi}\right)^4\ \tilde{A}^{\alpha}_{\mu}(p)\ \tilde{A}^{\beta}_{\nu}(q)\ \tilde{\theta}^{\gamma}(-p-q)\ \cdot\\
&\qquad \cdot\varepsilon^{\mu\nu\lambda\tau}\ p_{\lambda}\ q_{\tau}\ \mathrm{tr}(T_cT_{\alpha}T_{\beta}T_{\gamma})\\
&= \frac{1}{24\pi^2}\int d^4x\ \varepsilon^{\mu\nu\lambda\tau}\ \mathrm{tr}\left(T_c\ \partial_{\lambda}\ A_{\mu}(x)\ \partial_{\tau}\ A_{\nu}\ \theta(x)\right) \qquad (4.21)
\end{aligned}$$

to second order. Presumably the full non-Abelian structure is enforced by the consistency equations (4.8).

To construct anomaly-free models it is necessary to combine chiral multiplets in a suitable way. In effect this means choose a chirality matrix, T_c, that commutes with $\theta(x)$ and satisfies

$$\varepsilon_{\kappa\mu\lambda\nu}\ \mathrm{tr}(T_c\ \theta(x)\ \partial_{\kappa}\ A_{\mu}\ \partial_{\lambda}\ A_{\nu}) = 0$$

For example for one family of quarks and leptons in the standard $SU(2)_L \times U(1)_Y$ model choose $T^{lep}_c = diag(1,1,-1)$ and $T^{quark}_c = diag(1,1,-1,-1)$ for each colour.

5 The bilinear part of $\langle A+|A-\rangle$

In the foregoing section it was proven that the lattice effective action $\Gamma(A) = -ln\langle A+|A-\rangle$ exhibits the correct anomalous behaviour under local gauge transformations. In this section we would like to show the plausibility of (1.5), in any number of dimensions, up to the second order terms in A.

For slowly varying weak background gauge fields we shall approximate the lattice formulation of the last three sections with a continuum theory described by the set of equations (1.4) to (1.6). The starting point will be the continuum Hamiltonians

$$H_{\pm} = \int d^D x \; \psi^{\dagger}(x) \; \gamma_5 \Big(\gamma_\mu (\partial_\mu - i \; A_\mu) \pm |\Lambda| \Big) \; \psi(x) \tag{5.1}$$

where A_μ is the Lie algebra valued vector potential.

To evaluate $\Gamma(A) = -\ell n \; \langle A + | A - \rangle$ up to second order terms in A we use the perturbative solution (3.14). It is not hard to show that for finite values of $|\Lambda|$ we have the following non–vanishing contributions

$$\begin{aligned} \frac{\langle A + | A - \rangle}{\langle + | - \rangle} &= 1 - \frac{1}{2} \langle + | V G_+^2 V | + \rangle - \frac{1}{2} \langle - | V G_-^2 V | - \rangle + \\ &+ \frac{\langle + | V G_+ V G_+ | - \rangle}{\langle + | - \rangle} + \frac{\langle + | V G_+ G_- V | - \rangle}{\langle + | - \rangle} + \frac{\langle + | G_- V G_- V | - \rangle}{\langle + | - \rangle} + \ldots \end{aligned} \tag{5.2}$$

where $G_{\pm}$ are defined by (3.12) and

$$V = \int d^D x \; \psi^{\dagger}(x) \; i \; A\!\!\!/(x) \; \gamma_5 \; \psi(x) \tag{5.3}$$

Each term appearing in (5.2) can be expressed in terms of matrix elements of V in which up to 4–particle states can contribute. For example

$$\langle + | V G_+^2 V | + \rangle = \frac{1}{\Omega^2} \sum_{1,2} \frac{\langle + | V | 1\bar{2} + \rangle \langle 1\bar{2} + | V | + \rangle}{(\omega_1 + \omega_2)^2} \tag{5.4a}$$

and

$$\begin{aligned} \frac{\langle + | V G_+ V G_+ | - \rangle}{\langle + | - \rangle} &= \frac{1}{\Omega^4} \sum_{1,2,3,4} \frac{\langle + | V | 1\bar{2} + \rangle \langle 1\bar{2} + | V | 3\bar{4} + \rangle \langle 3\bar{4} + | - \rangle}{(\omega_1 + \omega_2)(\omega_3 + \omega_4) \langle + | - \rangle} \\ &+ \frac{1}{\Omega^6} \sum_{1 \ldots 6} \frac{\langle + | V | 1\bar{2} + \rangle \langle 1\bar{2} + | V | 345\bar{6} + \rangle \langle 345\bar{6} + | - \rangle}{4(\omega_1 + \omega_2)(\omega_3 + \omega_4 + \omega_5 + \omega_6) \langle + | - \rangle} \end{aligned} \tag{5.4b}$$

etc. The notation used here is identical to that of Sec.3. In particular the labels $1, 2, \ldots,$ stand for momentum, spin and flavour.

Expressing V and the states in terms of Fock space operators one can calculate all the necessary matrix elements. Here we give one of them as an example

$$\begin{aligned} \langle 1\bar{2}+|V|3\bar{4}+\rangle &= \Omega\ \delta_{24}\ u_+^\dagger(1)\ i\ \tilde{A\!\!\!/}(k_1-k_3)\ \gamma_5\ u_+(3) \\ &\quad -\Omega\ \delta_{12}\ v_+^\dagger(4)\ i\ \tilde{A\!\!\!/}(k_4-k_2)\ \gamma_5\ v_+(2) \end{aligned} \tag{5.5}$$

where

$$u_\pm(k,\lambda) = \frac{\omega(k)+\gamma_5(i\ k\!\!\!/ \pm |\Lambda|)}{\sqrt{2\omega(k)\ (\omega(k)\pm|\Lambda|)}}\ \chi(\lambda) \tag{5.6a}$$

$$v_\pm(k,\lambda) = \frac{\omega(k)-\gamma_5(i\ k\!\!\!/ \pm |\Lambda|)}{\sqrt{2\omega(k)\ (\omega(k)\mp|\Lambda|)}}\ \chi(\lambda) \tag{5.6b}$$

and $\omega(k)=\sqrt{k^2+\Lambda^2}$.

Using these equations we can put equations such as (5.4) into the form

$$\langle +|VG_+^2V|+\rangle = \frac{1}{\Omega^2}\sum_{k_1,k_2}\frac{\mathrm{tr}\Big(i\ \tilde{A\!\!\!/}(k_2-k_1)\gamma_5 U_+(k_1) i\ \tilde{A\!\!\!/}(k_1-k_2)\gamma_5\ V_+(k_2)\Big)}{\Big(\omega(k_1)+\omega(k_2)\Big)^2} \tag{5.7}$$

where

$$U_\pm(k) = \frac{\omega(k)+\gamma_5(i\ k\!\!\!/ \pm|\Lambda|)}{2\omega(k)} = 1 - V_\pm(k) \tag{5.8}$$

Every term in (5.2) can be rewritten in a form similar to (5.7). These expressions of course in general suffer from ultraviolet divergences. It would be more desirable to carry out these calculations on the lattice regularized version of the overlap. We shall come back to this problem in the future but, for now we give only the more brief and intuitive continuum version. For the time being let us assume that the continuum theory is regularized in some way and examine the $|\Lambda|\to\infty$ limit of expressions like (5.7). It is not hard to see that in this limit (5.7) vanishes. In fact the only non–vanishing contribution to (5.2), in the limit of $|\Lambda|\to\infty$, originate from the last three terms of this equation. It can be shown that as $|\Lambda|\to\infty$, the overlap given by (5.2) reduces to

$$\frac{\langle A+|A-\rangle}{\langle +|-\rangle} = 1 - \left(\frac{1}{8}+\frac{1}{4}+\frac{1}{8}\right)\int\frac{d^Dp}{(2\pi)^D}\ \tilde{A}_\mu{}^\alpha(p)\ \Pi_{\mu\nu}(p)\ \tilde{A}_\nu^\alpha(-p) \tag{5.9a}$$

where

$$\Pi_{\mu\nu} = \int\frac{d^Dk}{(2\pi)^D}\ \mathrm{tr}\left(\gamma_\mu\ \frac{1+\gamma_5}{2}\ \frac{k\!\!\!/ - p\!\!\!/}{(k-p)^2}\ \gamma_\nu\ \frac{1+\gamma_5}{2}\ \frac{k\!\!\!/}{k^2}\right) \tag{5.9b}$$

The numerical coefficients in the second term on the right–hand side of (5.9a) indicate the contribution of the respective terms in (5.2).

Equations (5.9a) and (5.9b) are in agreement with the bilinear part of the effective action of a Weyl fermion coupled to Yang–Mills fields in a D–dimensional Euclidean space. Namely if we start from

$$\begin{aligned} e^{-\Gamma(A)} &= \int (d\bar{\psi}\; d\psi)\; e^{-S} \\ S &= \int d^D x\; \bar{\psi}(x) \left(\partial\!\!\!/ - i\; A\!\!\!/\; \frac{1+\gamma_5}{2} \right)\; \psi(x) \end{aligned}$$

and calculate $\Gamma(A)$ to second order in A we obtain $\Gamma(A) = -\ell n\; \frac{\langle A+|A-\rangle}{\langle +|-\rangle}$, with $\langle A+|A-\rangle$ given by (5.9). This exercise indicates that as $|\Lambda| \to \infty$, the unphysical states not only decouple in the free theory as shown in [5] but they also do so in the quantum loops of the interacting theory, leaving behind only the contribution of massless physical chiral fermions. Together with the anomaly calculations this gives further support to the identification (1.6).

Another indication of the validity of the overlap approach is contained in a recent note of Narayanan and Neuberger [12] in which the chiral determinant on a 2–dimensional torus in the presence of non–trivial background Polyakov loop variables are examined.

6 Gravitational anomalies

In this section we show that the overlap formalism correctly reproduces the gravitational anomalies. We shall examine only continuous two–dimensional theories but we believe that the extension of our calculations to higher dimensions should be in principle straightforward, although more cumbersome. Also, being concerned with the continuum theory, as we are in this section, we shall consider the limit $|\Lambda| \to \infty$ of the overlap.

The Hamiltonians which, in the limit of $|\Lambda| \to \infty$, produce the appropriate effective action for the coupling of 2–dimensional chiral fermions to gravity are given by

$$H_{\pm} = \int d^2x\; \psi^\dagger(x)\; \sigma^3 \big(\sigma^a\; e_a{}^{\mu}(x)\; \nabla_\mu \pm |\Lambda| \big)\; \psi(x) \tag{6.1}$$

where $\sigma^3, \sigma^a,\ a = 1,2$ are the Pauli spin matrices and $e_a{}^{\mu}(x)$ are the components of a zwei–bein. The covariant derivative ∇_μ is defined by

$$\nabla_\mu \psi = \left(\partial_\mu - \frac{i}{2}\, \omega_\mu\, \sigma_3 - i\, A_\mu - \frac{1}{2}\, \partial_\mu\, \ell n\, e \right)\, \psi(x) \tag{6.2}$$

where $e = \det\, e_\mu{}^a(x)$ and A_μ is a $U(1)$–Maxwell field. The spin connection ω_μ is given by

$$\omega_\mu = -\frac{1}{2}\, e_\mu{}^a\, \frac{\varepsilon^{\alpha\beta}}{e}\, \partial_\alpha\, e_\beta{}^a \tag{6.3}$$

where $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha} = \varepsilon_{\alpha\beta}$ and $\varepsilon^{12} = 1$.

The 2–component spinor field $\psi(x)$ can undergo three independent local transformations, namely

i) local frame rotations: $$\psi'(x) = e^{\frac{i}{2}\sigma_3 \phi(x)}\, \psi(x) \tag{6.4a}$$

ii) $U(1)$–gauge transformations: $$\psi'(x) = e^{i\theta(x)}\, \psi(x) \tag{6.4b}$$

iii) general coordinate transformations: $$\psi'(x') = \left(\det\, \frac{\partial x^\mu}{\partial x'^\nu} \right)^{1/2}\, \psi(x) \tag{6.4c}$$

The transformation rule (6.4c) indicates that $\psi(x)$ is a scalar density of weight 1/2 under the diffeomorphism group.

It is known that the effective action of charged chiral fermions in a gravitational background responds anomalously to all three groups of the above transformations [13]. Our intention in this section is to recover this anomalous behaviour within the overlap formalism. To this end we shall assume that the external fields are weak and evaluate the angles $\Phi_\pm$ introduced in Sec.4 up to first order in the external fields and the parameters of transformations. Writing

$$e_a{}^{\mu}(x) = \delta_a{}^{\mu} + h_a{}^{\mu}(x) \tag{6.5}$$

we need to evaluate

$$\Phi_+(A, h; \varphi, \theta, \xi) = \frac{1}{\alpha_+}\, \mathrm{Re} \int d_x^2\, \langle +|\psi^+(x)\, \Sigma(x)\, \psi(x)|A, h+\rangle \tag{6.6}$$

where

$$\Sigma(x) = \frac{1}{2}\, \sigma_3\, \varphi(x) + \theta(x) + \xi^\lambda(x)\, P_\lambda(x) \tag{6.7a}$$

where the operators $P_\lambda(x)$ are the generators of the diffeomorphisms and are defined by

$$\psi^\dagger(x)\ P_\lambda(x)\ \psi(x) = \frac{i}{2}\ \left(\partial_\lambda\ \psi^\dagger(x)\ \psi(x) - \psi^\dagger(x)\ \partial_\lambda\ \psi(x)\right) \tag{6.7b}$$

Φ_- can be obtained from Φ_+ by changing $|\Lambda|$ to $-|\Lambda|$.

Using the same notation as in Sec.3, it is easy to see that

$$\begin{aligned} A_L &\equiv \frac{\delta}{\delta\varphi(x)}\ (\Phi_+ - \Phi_-)\Big|_{|\Lambda|\to\infty} \\ &= \mathrm{Re}\left[\frac{1}{\Omega^2}\sum_{k_1k_2}\frac{1}{\omega_1+\omega_2}\langle +|\psi^+(x)\ \frac{\sigma_3}{2}\ \psi(x)|k_1\bar{k}_2+\rangle\langle +k_1\bar{k}_2|V|+\rangle - (|\Lambda| \to -|\Lambda|)\right]_{\substack{\Omega\to\infty \\ |\Lambda|\to\infty}} \end{aligned} \tag{6.8a}$$

$$\begin{aligned} A_{Maxw.} &\equiv \frac{\delta}{\delta\theta(x)}\ (\Phi_+ - \Phi_-)\Big|_{|\Lambda|\to\infty} \\ &= \mathrm{Re}\left[\frac{1}{\Omega^2}\sum_{k_1k_2}\frac{1}{\omega_1+\omega_2}\langle +|\psi^+(x)\ \psi(x)|k_1\bar{k}_2+\rangle\langle +k_1\bar{k}_2|V|+\rangle - (|\Lambda| \to -|\Lambda|)\right]_{\substack{\Omega\to\infty \\ |\Lambda|\to\infty}} \end{aligned} \tag{6.8b}$$

$$\begin{aligned} T_\mu &\equiv \frac{\delta}{\delta\xi^\mu(x)}\ (\Phi_+ - \Phi_-)\Big|_{|\Lambda|\to\infty} \\ &= \mathrm{Re}\Big[\frac{1}{\Omega^2}\sum_{k_1k_2}\frac{1}{\omega_1+\omega_2}\langle +|\frac{i}{2}\left(\partial_\mu\ \psi^\dagger(x)\ \psi(x) - \psi^\dagger(x)\ \partial_\mu\ \psi(x)\right)|k_1\bar{k}_2+\rangle\langle +k_1\bar{k}_2|V|+\rangle - \\ &\qquad - (|\Lambda| \to -|\Lambda|)\Big]_{\substack{\Omega\to\infty \\ |\Lambda|\to\infty}} \end{aligned} \tag{6.8c}$$

where

$$V = \int d^2x\ \psi^\dagger(x)\ \sigma_3 \left\{\sigma^a\ h_a{}^\mu(x)\ \partial_\mu + \sigma^\mu\left(-\frac{i}{2}\ \omega_\mu\ \sigma_3 - i\ A_\mu + \frac{1}{2}\ \partial_\mu\ h\right)\right\}\psi(x) \tag{6.9}$$

and $h(x) = \delta^a_\mu\ h_a{}^\mu(x)$.

Using the notation developed in previous sections we can easily evaluate the matrix elements appearing in (6.8). For example

$$\begin{aligned} \langle k_1\bar{k}_2 + |V|+\rangle &= u_+^\dagger(k_1)\ \sigma_3\sigma^\mu\Big[i\ k_{2\lambda}\ \tilde{h}_\mu{}^\lambda(k_1-k_2) - \frac{i}{2}\ \sigma_3\ \tilde{\omega}_\mu(k_1-k_2) - \\ &\qquad - i\ \tilde{A}_\mu(k_1-k_2) + \frac{i}{2}\ (k_1-k_2)_\mu\ \tilde{h}(k_1-k_2)\Big]v_+(k_2) \end{aligned}$$

Substituting these matrix elements in (6.8) and letting $\Omega \to \infty$ after some straightforward algebra one obtains

$$
\begin{aligned}
A_L &= |\Lambda| \int \frac{d^2p}{(2\pi)^2} \, e^{ip\cdot x} \, G_{\mu\nu}(p) \, \tilde{h}_{\mu\nu}(p) & (6.10a)\\
A_{Maxw.} &= |\Lambda| \int \frac{d^2p}{(2\pi)^2} \, e^{ip\cdot x} \, G(p) \, i \, \varepsilon_{\mu\nu} \, p_\nu \, \tilde{A}_\mu(p) & (6.10b)\\
T_\mu &= |\Lambda| \int \frac{d^2p}{(2\pi)^2} \, e^{ip\cdot x} \, G_{\mu\lambda}(p) \, i \, \varepsilon_{\sigma\nu} \, p_\nu \, \tilde{h}_{\sigma\lambda}(p) & (6.10c)
\end{aligned}
$$

where

$$
\begin{aligned}
G_{\mu\nu}(p) &= \int \frac{d^2k}{(2\pi)^2} \frac{k_\mu \, k_\nu}{\left(\omega\left(k+\frac{p}{2}\right)+\omega\left(k-\frac{p}{2}\right)\right)\omega\left(k+\frac{p}{2}\right)\omega\left(k-\frac{p}{2}\right)} & (6.11a)\\
G(p) &= \int \frac{d^2k}{(2\pi)^2} \frac{1}{\left(\omega\left(k+\frac{p}{2}\right)+\omega\left(k-\frac{p}{2}\right)\right)\omega\left(k+\frac{p}{2}\right)\omega\left(k-\frac{p}{2}\right)} & (6.11b)
\end{aligned}
$$

and the $|\Lambda| \to \infty$ limit of (6.10) is understood. The kernel $G(p)$ is given by a convergent integral. Therefore in the limit of $|\Lambda| \to \infty$ it becomes $G(0) = \frac{1}{4\pi|\Lambda|}$. Inserting this in (6.10b) we obtain

$$
A_{Maxw.} = \frac{1}{4\pi} \, \varepsilon_{\mu\nu} \, \partial_\nu \, A_\mu(x) \tag{6.12}
$$

To evaluate the $|\Lambda| \to \infty$ limit of $G_{\mu\nu}(p)$, we expand the integrand in powers of $\frac{p}{|\Lambda|}$. The leading term will be a p–independent linearly divergent integral which should be handled by adopting a suitable subtraction scheme. The terms of order $\frac{1}{|\Lambda|}$, on the other hand, are finite and are given by

$$
G_{\mu\nu}(p) = \frac{1}{48\pi} \, (p_\mu \, p_\nu - p^2 \, \delta_{\mu\nu}) \, \frac{1}{|\Lambda|} + 0\left(\frac{1}{|\Lambda|^3}\right) \tag{6.13}
$$

Substituting this in (6.10a) and (6.10c) we obtain

$$
\begin{aligned}
A_L &= \frac{1}{48\pi} \, (\partial^2 \, \delta_{\mu\nu} - \partial_\mu \, \partial_\nu) h_{\mu\nu}(x) & (6.14a)\\
T_\mu &= \frac{1}{48\pi} \, (\partial^2 \delta_{\mu\lambda} - \partial_\mu \partial_\lambda) \varepsilon_{\sigma\nu} \, \partial_\nu \, h_{\sigma\lambda}(x) & (6.14b)
\end{aligned}
$$

These results which were contained in [14] agree with the linearized versions of the corresponding equations of [15].

7 Outlook

In our opinion one of the more urgent questions to be addressed in the overlap approach is to extend the calculations of Sec.5 to all orders in A in a properly regularized theory, i.e. on the lattice. This will increase our confidence in the validity of the approach. Of course, eventually, the dynamics of the boson fields should also be incorporated.

Ultimately the value of the scheme will depend on its usefulness in non–perturbative studies of standard model.

Appendix: The Λ–discontinuity

The discontinuity at $\Lambda = 0$ of the functional $\Phi(\theta, A)$ is computed for the 2–dimensional case. This concerns the first order term obtained in Sec.4

$$\Phi_\pm(\theta, A) = \int \left(\frac{dp}{2\pi}\right)^2 \tilde{\theta}_\alpha(-p)\; F^\mu_\pm(p)\; \tilde{A}^\alpha_\mu(p) \tag{A.1}$$

where $F^\mu_+(p)$ is given by (4.12). Of special concern is its first derivative evaluated at $p = 0$, given by (4.13) and (4.14),

$$F^{\mu,\nu}_\pm(0) = \frac{i}{4}\, \varepsilon^{\mu\nu} \int_{BZ} \left(\frac{dk}{2\pi}\right)^2 \frac{1}{\omega^3_\pm}\, \langle T_c\rangle \Big\{(B \pm |\Lambda|) \det C + \Big(C_\alpha{}^{,\beta}\, B^{,\alpha} - C_\alpha{}^{,\alpha}\, B^{,\beta}\Big)\, C_\beta\Big\} \tag{A.2}$$

where $\langle T_c\rangle$ is defined by the flavour trace,

$$\begin{aligned} \mathrm{tr}(T_c\, \theta\, A_\mu) &= \mathrm{tr}(T_c\, T_\beta\, T_\alpha)\, \theta^\beta\, A^\alpha_\mu \\ &= \langle T_c\rangle\, \theta_\alpha\, A^\alpha_\mu \end{aligned} \tag{A.3}$$

In the integral B and C_μ are functions of k and $\det C$ is the determinant of $\partial C_\mu/\partial k_\nu$. The energies are given by

$$\omega_\pm = \sqrt{g^{\mu\nu}\, C_\mu\, C_\nu + (B \pm |\Lambda|)^2} \tag{A.4}$$

where $g_{\mu\nu}$ is the lattice metric normalized such that $\det g = 1$. The integral can develop infra–red singularities for $\Lambda = 0$ at points where $\omega_\pm$ vanishes, i.e. wherever $B = C_\mu = 0$. These singularities contribute to the discontinuity, $F_+ - F_-$, at $\Lambda = 0$.

The purpose of this appendix is to estimate the contribution of one such singularity. Suppose that C_μ and B have simple zeroes at $k = \hat{k}$,

$$\begin{aligned} C_\mu(k) &= C_\mu{}^{,\nu}(\hat{k})\, (k - \hat{k})_\nu + \ldots \equiv p_\mu + \ldots \\ B(k) &= B^{,\nu}(\hat{k})\, (k - \hat{k})_\nu + \ldots \equiv b^\mu\, p_\mu + \ldots \end{aligned} \tag{A.5}$$

To estimate the small Λ behaviour we can use the method of steepest descents. Write

$$\frac{1}{\omega^3} = \frac{1}{\Gamma(3/2)} \int_0^\infty d\alpha\; \alpha^{1/2}\, e^{-\alpha\omega^2}$$

and express (A.2) as a Laplace transform

$$F_{\pm}^{\mu,\nu}(0) = i\;\varepsilon^{\mu\nu}\;\langle T_c\rangle \int_0^\infty d\alpha\; e^{-\alpha\Lambda^2}\; K_{\pm}(\alpha,\Lambda) \tag{A.6}$$

where

$$K_{\pm}(\alpha,\Lambda) = \frac{\alpha^{1/2}}{4\Gamma(3/2)} \int \left(\frac{dk}{2\pi}\right)^2 \; e^{-\alpha(\omega_{\pm}^2-\Lambda^2)} \left\{(B \pm |\Lambda|)\det C + (C_\alpha{}^{,\beta}\; B^{,\alpha} - C_\alpha{}^{,\alpha}\; B^{,\beta})C_\beta\right\} \tag{A.7}$$

The small Λ behaviour of (A.6) is determined by the large α behaviour of (A.7) and this is dominated by contributions from the neighbourhood of $\hat{k}$, i.e. small values of the momentum p_μ defined in (A.5). Change the integration variable from k to p and expand in powers of p so that, for large α,

$$\begin{aligned} K_{\pm}(\alpha,\Lambda) \;\; \simeq \;\; & \frac{\alpha^{1/2}}{4\Gamma(3/2)}\;\frac{\det C}{|\det C|} \int \left(\frac{dp}{2\pi}\right)^2 \exp\left[-\alpha\left(\pm 2|\Lambda| b^\mu p_\mu + (g^{\mu\nu} + b^\mu b^\nu)p_\mu p_\nu + \ldots\right)\right] \cdot \\ & \cdot \left\{\pm|\Lambda| + \left(b^\beta + \frac{C_\alpha{}^{,\beta}\; B^{,\alpha} - C_\alpha{}^{,\alpha}\; B^{,\beta}}{\det C}\right) p_\beta + \ldots\right\} \end{aligned} \tag{A.8}$$

The integration can be extended to the entire plane without affecting the asymptotic development in inverse powers of α. Use the Gaussian formula

$$\int \left(\frac{dp}{2\pi}\right)^2 \; e^{-\alpha M^{\mu\nu}p_\mu p_\nu + \xi^\mu p_\mu} = \frac{1}{4\pi\alpha}\;|\det M|^{-1/2} \exp\left[\frac{1}{4\alpha}\xi^\mu M^{-1}_{\mu\nu}\xi^\nu\right] \tag{A.9}$$

Here we have

$$\begin{aligned} M^{\mu\nu} &= g^{\mu\nu} + b^\mu b^\nu \\ \xi^\mu &= \mp 2\alpha|\Lambda| b^\mu + \eta^\mu \end{aligned}$$

where η^μ is small. Since

$$\det M = 1 + b^2 \quad \text{and} \quad M^{-1}_{\mu\nu} = g_{\mu\nu} - (1+b^2)^{-1}\; b_\mu b_\nu$$

the right–hand side of (A.9) is given by

$$\frac{1}{4\pi\alpha}\;(1+b^2)^{-1/2} \exp\left(\frac{\alpha\Lambda^2 b^2}{1+b^2}\right) \cdot \left(1 \mp \frac{|\Lambda| b_\mu \eta^\mu}{1+b^2} + 0(\eta^2)\right)$$

Hence, (A.8) reduces to

$$K_{\pm}(\alpha,\Lambda) \simeq \pm|\Lambda| \frac{\det C}{|\det C|} \frac{\alpha^{-1/2}(1+b^2)^{-1/2}}{16\pi\,\Gamma(3/2)} \exp\left(\frac{\alpha\Lambda^2 b^2}{1+b^2}\right) \cdot$$

$$\cdot \left[1 - \left(b^\beta + \frac{C_\alpha{}^{,\beta} B^{,\alpha} - C_\alpha{}^{,\alpha} B^{,\beta}}{\det C}\right) \frac{b_\beta}{1+b^2} + 0\left(\Lambda, \frac{1}{\alpha\Lambda}\right)\right] \quad \text{(A.10)}$$

The neglected terms are unimportant because, in effect, $\alpha \sim \Lambda^{-2}$ for $\Lambda \to 0$. The second term in the square brackets vanishes because, on using the definitions (4.5),

$$\begin{aligned}(C_\alpha{}^{,\beta} B^{,\alpha} - C_\alpha{}^{,\alpha} B^{,\beta}) b_\beta &= (C_\alpha{}^{,\beta} C_\gamma{}^{,\alpha} - C_\alpha{}^{,\alpha} C_\gamma{}^{,\beta}) b^\gamma b_\beta \\ &= -b^2 \det C\end{aligned}$$

Finally, substituting (A.10) into (A.6) one obtains

$$F_{\pm}^{\mu,\nu}(0) = \pm\, i\, \varepsilon^{\mu\nu} \langle T_c \rangle \frac{1}{8\pi} \frac{\det C}{|\det C|} + 0(\Lambda)$$

The discontinuity at $\Lambda = 0$ therefore receives the contribution

$$F_{+}^{\mu,\nu}(0) - F_{-}^{\mu,\nu}(0) = i\, \varepsilon^{\mu\nu} \langle T_c \rangle \frac{1}{4\pi} \frac{\det C}{|\det C|} \quad \text{(A.11)}$$

from every simple zero on the torus. Note that this result does not depend on $B^{,\mu}(\hat{k})$. The order of the zero in B is not important.

References

[1] R. Narayanan and H. Neuberger, "A construction of lattice chiral gauge theory", IASSNS–HEP–94/99, RU–94–93, hep–th/9411108.

[2] D.B. Kaplan, Phys. Lett. B288 (1992) 342.

[3] C.G. Callan Jr. and J.A. Harvey, Nucl. Phys. B250 (1985) 427;
S.A. Frolov and A.A. Slavnov, Phys. Lett. B309 (1993) 344;
Kaplan's work gave rise to an upsurge of interest in the problem of chiral lattice fermions, see for example
H. Aoki, S. Ito, J. Nishimura and M. Oshikawa, Mod. Phys. Lett. A9 (1994) 1755;
S. Aoki and H. Hirose, Phys. Rev. D49 (1994) 2604;
S. Aoki and Y. Kikukawa, Mod. Phys. Lett. A8 (1993) 3517;
S. Chandrasekharan, Phys. Rev. D49 (1994) 1980;
M. Creutz and I. Horvath, Nucl. Phys. B(Proc. Suppl.)34 (1994) 583;
J. Distler and S–Y. Rey,"3 into 2 doesn't go", PUPT–1386, hep–lat/9305026;
V. Furman and Y. Shamir, " Axial symmetries in lattice QCD with Kaplan fermions", WIS–94–19–PH, hep–lat/9405004;
M.F.L. Golterman, K. Jansen and D.B. Kaplan, Phys. Lett. B301 (1993) 219;
M.F.L. Golterman, K. Jansen, D.N. Petcher and J. Vink, Phys. Rev. D49 (1944) 1606;
M.F.L. Golterman and Y. Shamir, "Domain wall fermions in a waveguide: the phase diagram at large Yukawa coupling", WASH–U–HEP–94–61, hep–lat/9409013;
A. Hulsebos, C.P. Korthals–Altes and S. Nicolis, "Gauge theories with a layered phase", CPT–94–P–3036, hep–th/9406003;
K. Jansen, Phys. Lett. B288 (1992) 348; and"Domain wall fermions and chiral gauge theories", Desy preprint DESY–94–188, hep–lat/9410108
K. Jansen and M. Schmaltz, Phys. Lett. B296 (1992) 374;
D.B. Kaplan, Nucl. Phys. B(Proc. Suppl.)30 (1993) 597;

T. Kawano and Y. Kikukawa, "On the large mass limit of continuum theories in Kaplan formulation", KUNS–1239, hep–th/9402141;

R. Narayanan and H. Neuberger, Phys. Lett. B302 (1993) 62; Phys. Rev. Lett. 71 (1993) 3251; Nucl. Phys. B(Proc. Suppl.)34 (1994) 587

C.P. Korthals–Altes, S. Nicolis and J. Prades, Phys. Lett. B316 (1993) 339;

R. Narayanan, Nucl. Phys. B(Proc. Suppl.)34 (1994) 95;

Y. Shamir, Nucl. Phys. B406 (1993) 90; Nucl. Phys. B417 (1993) 167; Phys. Lett. B305 (1992) 357;

Z. Yand, Phys. Lett. B296 (1992) 151

[4] R. Narayanan and H. Neuberger, Nucl. Phys. B412 (1994) 574

[5] S. Randjbar–Daemi and J. Strathdee, "On the overlap formulation of chiral gauge theory", ICTP preprint IC/94/396, hep–th/9412165

[6] S. Aoki and R. Levien, "Kaplan–Narayanan–Neuberger lattice fermions pass a perturbative test", preprint UTHEP–289, hep–th/9411137

[7] J.W. Milnor, Topology from Differential Viewpoint (The University press of Virginia, Charlottesville, 1965)

[8] L.H. Karsten, Phys. Lett. 104B (1981) 315;

L.H. Karsten and J. Smit, Nucl. Phys. B183 (1981) 103;

H.B. Nielsen and M. Ninomya, Nucl. Phys. B185 (1981) 20; Nucl. Phys. B193 (1981) 173; Phys. Lett. 105B (1981) 219;

J. Smit, Acta Phys. Pol. B17 (1986) 531

[9] These representations are constructed in

S. Randjbar–Daemi and J. Strathdee, "A four fermion lattice model", ICTP preprint IC/94/360

[10] See for example the Roma approach to lattice chiral gauge theories in: A. Borrelli, L. Maiani, G.C. Rossi, R. Sisto and M. Testa, Nucl. Phys. B333 (1990) 335;

L. Maiani, G.C. Rossi and M. Testa, Phys. Lett. 292B (1992) 397;
G.C. Rossi, A. Sarno and R. Sisto, Nucl. Phys. B398 (1993) 101;
and the recent paper
A.A. Slavnov,"Generalized Pauli–Villars regularization for undoubled lattice fermions", preprint SMI–9–94, hep–th/9412233;
Reviews of lattice chiral fermions can be found in the proceedings of annual lattice gauge theory meetings published as Nucl. Phys. B(Proc. Suppl.). See for example the issue Nucl. Phys. B(Proc. Suppl.)29B,C (1992) and the review article
D.N. Petcher, Nucl. Phys. B(Proc. Suppl.)30 (1993) 50

[11] W. Bardeen and B. Zumino, Nucl. Phys. B244 (1984) 421

[12] R. Narayanan and H. Neuberger, "Two–dimensional twisted chiral fermions on the lattice", IASSN–HEP and RU preprint RU–94–99, hep–lat/9412104

[13] L. Alvarez–Gaumé and E. Witten, Nucl. Phys. B234 (1983) 269

[14] S. Randjbar–Daemi and J. Strathdee, "Gravitational Lorentz anomaly from the overlap formula in 2–dimensions", ICTP preprint IC/94/401, hep–th/9501012

[15] H. Leutwyler, Phys. Lett. 153B (1985) 65