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Acceleration Theorems

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Acceleration Theorems

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Abstract

Electromagnetic fields can be separated into near and far components. Near fields are extensions of static fields. They do not radiate, and they fall off more rapidly from a source than far fields. Near fields can accelerate particles, but the ratio of acceleration to source fields at a distance R, is always less than R/λ or 1, whichever is smaller. Far fields can be represented as sums of plane parallel, transversely polarized waves that travel at the velocity of light. A single such wave in a vacuum cannot give continuous acceleration, and it is shown that no sum of such waves can give net first order acceleration. This theorem is proven in three different ways; each method showing a different aspect of the situation.

INTRODUCTION

In the quest to build higher energy accelerators that are both smaller and cheaper, we search for ways to increase the acceleration gradient. Acceleration is conventionally obtained in structures with dimensions comparable to the wavelength (cavities or linacs). Breakdown on the surfaces of the nearby structure limits the gradient attainable. Acceleration is possible far from any structure in a gas (Inverse Cherenkov acceleration) but the gas can breakdown and Coulomb scattering blows up emittance. Acceleration is possible in vacuum, far from any structure if a fixed or wiggling magnetic field is present (Inverse FEL), but synchrotron radiation limits the energy at which this can be used. Acceleration is also possible in a plasma, but both plasma instabilities and Coulomb scattering blow up emittance, limiting its applications.

If it were possible to get good acceleration, far from the source, in a vacuum, without magnetic fields, it would be very desirable. We note that the fields at a focus can be far higher than those at the aperture of the lens or mirror that produces it, and we wonder if such fields at a focus could not be used to accelerate. Several papers have been written that claim such acceleration. Unfortunately. we conclude that these claims cannot be correct.

Fields can be divided into "near fields" whose strength at a distance are never greater than those at their source, and far fields whose fields can be focussed to values far higher than those at their source. Near fields can accelerate, but it has been shown[1] that far fields, in vacuum, without magnetic bending, cannot accelerate. In this paper we expand the presentation of this theorem, and give three proofs of it.

NEAR AND FAR FIELDS

Fields generated by a moving charge

Electromagnetic fields can be separated into two components: near fields and far fields. There are several ways of identifying these two different kinds. The simplest is to examine the fields generated by a moving charge. From almost any textbook[2] we find that from a charge e, moving with a velocity $\vec{\beta}\ c$, at a time t, the electric field vector $\vec{\mathcal{E}}(\vec{x},t')$ at a location $\vec{x}$ relative to the charge, at a time $t' = t + \frac{R}{c}$, is given by:

$$\vec{\mathcal{E}}(\vec{x},t') = \frac{e\left(\frac{(\vec{n}-\vec{\beta})}{\gamma^2(1-\vec{\beta}.\vec{n})^3}\right)}{R^2} + \frac{\frac{e}{c}\left(\frac{\vec{n}\times(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}}{(1-\vec{\beta}.\vec{n})^3}\right)}{R}$$

where $\vec{n}$ is the unit vector in direction $\vec{x}$, R is the absolute value of $\vec{x}$, γ is the electron energy divided by its mass, $\dot{\vec{\beta}}$ is the time derivative of the vector $\vec{\beta}$, and c is the velocity of light.

The expressions may be more familiar if we take the non-relativistic limit $\beta \ll 1$:

$$\vec{\mathcal{E}}(\vec{x},t') = \frac{e}{R^2}(\vec{n}) + \frac{e}{cR}(\vec{n}\times\vec{n}\times\dot{\vec{\beta}})$$

In either case, it is seen that the first term is inversely proportional to R^2, ie it falls like the static Coulomb field of the charge. This is the static or near field component.

The second term falls only with the inverse of R. This is the dependence of a radiation field, whose energy falls with the area over which it is spread. ie whose energy falls as R^{-2} and thus whose fields fall as R^{-1}. This is the far field.

Electromagnetic Waves

Another way of seeing the difference between near and far fields is by looking for solutions to Maxwell's equations. From these, in a vacuum, we can easily obtain the wave equation:

$$\Delta^2\vec{\mathcal{E}} = -\frac{\omega^2}{c^2}\vec{\mathcal{E}}$$

All solutions to this equation can be expressed as sums of waves of the form

$$\vec{\mathcal{E}}(\vec{r},t) = \vec{\mathcal{E}}_o\ e^{i[(\vec{k}.\vec{r})-\omega t]}$$

where

$$\vec{k}^2 = \frac{\omega^2}{c^2} = \left(\frac{2\pi}{\lambda}\right)^2$$

Far Fields

If the vector $\vec{k}$ is real then these equations refer to plane parallel waves travelling in the direction $\vec{k}$ at the velocity of light c. They are the source of radiation and are thus far fields. If we apply the Maxwell's equation

$$\nabla \cdot \vec{\mathcal{E}} = 0$$

then we obtain

$$\vec{k} \cdot \vec{\mathcal{E}} = 0$$

ie the electric fields are perpendicular to the direction of propagation: they are transversely polarized.

Since such far fields travel at the velocity of light (we are only considering waves in a vacuum), they can only remain in phase with a relativistic particle if the wave and particle are travelling in the same direction. If they are not travelling in the same direction then any interaction between them will be periodic and will add to nothing. But if they are travelling in the same direction, even though they remain in phase, the electric fields are perpendicular to the direction of propagation, and there is again no acceleration.

Near Fields

In the above we have taken the vector $\vec{k}$ to be real. But if we allow the vector $\vec{k}$ to be complex then the meaning of the equations is more complicated. Let

$$\vec{k} = \vec{k}_r + i\vec{g}$$

where $\vec{k}_r$ and $\vec{g}$ are real vectors. Then the fields are given by

$$\vec{\mathcal{E}}(\vec{r}, t) = \vec{\mathcal{E}}_o \; e^{-(\vec{g} \cdot \vec{r})} \; e^{i[(\vec{k}_r . \vec{r}) - \omega t]}$$

and propagate in the direction $\vec{k}_r$ but fall exponentially in direction $\vec{g}$. They are known as evanescent waves. Such waves are surface phenomena and cannot propagate out in 3 dimensions. They are not conventional radiation fields, and fall more rapidly from their source than a true radiation field. They are near fields.

The velocity of propagation of such a wave will be given by

$$v = \frac{\omega}{|\vec{k}|}$$

but from

$$\vec{k}^2 = \frac{\omega^2}{c^2}$$

we obtain

$$v_{\vec{k}}^2 = \frac{c^2}{1 + (\frac{gc}{\omega})^2} < c^2$$

For this reason such fields are also known as "slow" fields. We should note however that the velocity of such waves in a directions at an angle θ to the vector $\vec{k}$ will have a velocity equal to $v_{\vec{k}}/cos(\theta)$ which can be matched to that of any particle that we wish to accelerate.

In order to find the polarization of such waves, we again apply the Maxwell's equation

$$\nabla \cdot \vec{\mathcal{E}} = 0$$

For a complex wave we obtain

$$i\vec{k}_r \cdot \vec{\mathcal{E}} = \vec{g} \cdot \vec{\mathcal{E}}$$

ie the electric fields $\vec{\mathcal{E}}$ are not necessarily perpendicular to the direction of propagation $\vec{k}_r$. This fact, together with the slow velocity of such near fields allow them to remain in phase with a particle, and to accelerate that particle if it is charged.

Ratio of Surface to Acceleration Fields

Any fields in a vacuum can be expressed as sums of waves travelling in various directions. Both real (far) and complex (near) waves must be included. In an accelerating structure, for instance, there are clearly both kinds of waves. There must be near waves or there would be no acceleration. But there must be other waves also since the near waves cannot usually be found that would satisfy the boundary conditions of the cavity. Simple modes in rectangular of cylindrical waveguides are composed of pure far fields, and they do not accelerate. When a periodicity, such as periodic irises are introduced then near fields are added and acceleration is achieved.

If we wish to estimate the surface fields on the structure walls that are needed to achieve a certain acceleration, we must take into account both near and far fields. But a lower limit on the surface fields can be obtained from the near fields alone. The addition of "far" fields will only increase the surface fields, but will not add to the acceleration.

Consider, for instance, the case of acceleration over a surface. The surface could be a dielectric or grating. In either case evanescent (near) fields can be established near the surface, falling exponentially from that surface. For this case we will take the direction $\vec{g}$, in which the fields are falling exponentially, to be perpendicular to the surface, and call this direction y. The direction of propagation we take to be within the surface, in directions x and z. The particle to be accelerated we take to be moving at velocity c in the z direction at a height y above the surface.

In this case, for the accelerating near field, we can write:

$$\mathcal{E}_z(y) = \mathcal{E}_y(0) \frac{g_y}{i\, k_z} e^{-(g_y\, y)}$$

with the periodic time dependence implied. If this field is to accelerate a relativistic particle travelling in the z direction, then its velocity of propagation in z must equal c; ie

$$k_z = \frac{2\pi}{\lambda}$$

so for small g_y when $\mathcal{E}_z << \mathcal{E}_y$ then:

$$\frac{Acc\ Field}{Surface\ Field} \leq \frac{\mathcal{E}_z(y)}{\mathcal{E}_y(0)} = -i\ \frac{\lambda}{2\pi}\ (g_y\ e^{-(g_y\ y)})$$

If we chose g_y to maximize acceleration then

$$\frac{d}{dg_y}(g_y\ e^{-g_y y}) = 0$$

$$g_y = 1/y$$

and provided now that $y << \lambda$

$$\frac{Acc\ Field}{Surface\ Field} \leq \frac{\mathcal{E}_z(y)}{\mathcal{E}_y(0)} = -i\ \frac{1}{2\pi}\ (\frac{\lambda}{y})e^{-1}$$

This inverse dependency of the ratio of accelerating to surface fields, with the distance to the source divided by the wavelength, is very general. The constant changes with the geometry, but the dependency remains the same. In the cylindrical case, as we will see below, the 2π is replaced by a single π, but the ratio of axial acceleration to the surface fields at a radius r is proportional to λ/r.

Cylindrical Symmetry

If we integrate the general wave solutions over the azimuthal angle about a given direction, then we obtain the cylindrically symmetric solutions:

$$\mathcal{E}_z(z, r, t) = \mathcal{E}_o J_o(k_r r) e^{i(k_z z - \omega t)}$$

$$\mathcal{E}_r(z, r, t) = -i\frac{k_z}{k_r}\mathcal{E}_o J_1(k_r r) e^{i(k_z z - \omega t)}$$

where J_o and J_1 are Bessel functions, and

$$k_r^2 + k_z^2 = (\frac{2\pi}{\lambda})^2$$

If k_r is real and finite then these equations represent far fields. They are the waves that can propagate in a smooth cylindrical waveguide, and they are the waves in Cherenkov radiation. We note that $k_z < (2\pi/\lambda)$, and thus the wave velocity in the z direction is greater than that of light. Thus such waves can not continuously accelerate any real charge.

But if k_r is zero or imaginary, then the equations represent near fields. Now $k_z \geq (2\pi/\lambda)$, the wave velocity is less than or equal to the velocity of light, and acceleration is possible.

In the particular case of $k_r \to 0$ (ie the solution that will accelerate a relativistic particle), then:

$$\mathcal{E}_z(z, r, t) = \mathcal{E}_o e^{i(k_z z - \omega t)} \quad \text{(independent of r)}$$

$$\mathcal{E}_r(z,r,t) \to -i\frac{k_z}{k_r}(\frac{k_r r}{2})\mathcal{E}_o e^{i(k_z z-\omega t)} = -ik_z\frac{r}{2}\mathcal{E}_o e^{i(k_z z-\omega t)}$$

$$\frac{Acc\ Field}{Surface\ Field} \leq \frac{\mathcal{E}_z(o)}{\mathcal{E}_r(r)} = -i\,\frac{1}{\pi}\,(\frac{\lambda}{r})$$

which is the same as the ratio in the plane case above, but for the factor 2, and the substitution of r for y.

Acceleration Theorem

We have concluded above that a single far field can not produce continuous acceleration, but we can make a stronger statement:

No combination of far fields, in an otherwise field free vacuum, can produce first order net acceleration.

As a consequence of this theorem, acceleration under these restricted circumstances can only come from near fields, and then

In an otherwise field free vacuum, the ratio of the first order average accelerating gradient, to the maximum fields at a distance R, is always less than a factor of the order of λ/R where λ is the wavelength of the accelerating field.

Let us consider the qualifiers in order:

"otherwise field free"

No static or dynamic fields, other than those of the accelerating fields, may be present. With a static field there can be inverse synchrotron radiation; with a wiggler field there can be inverse free electron laser acceleration; both can give net first order acceleration of a particle.

"in a vacuum"

There can be no other charges or electrically active material present. In the presence of a plasma, there can be accelerating fields due to waves in that plasma. In the presence of other charges, there can be collective acceleration. And in a dielectric medium inverse Cherenkov acceleration can occur.

"first order"

We are only considering acceleration in which the change in energy is linear with the amplitude of the accelerating fields. There can be second order effects due to Compton scattering. In classical terms, this requires the re-radiation of energy, and the acceleration is then proportional to square of the accelerating fields.

"net acceleration"

The acceleration we are discussing is the net change of energy from and to points that are free of the accelerating field. Far fields can and will produce local changes in energy, but these will be reversed before the particle has exited the field region.

It is often objected that the field could be "cut off" after such local acceleration, and thus the acceleration produced by the far waves preserved. What the theorem says, is that no suitable cutoff can be achieved by remote focussing, holograms or phase manipulations; such attempts will always introduce deceleration that just cancels any acceleration previously given. The only way to "cut off" the radiation in such a way as to keep the acceleration, is to introduce a physical obstruction such as a foil, plasma or iris. The foil or plasma clearly violate our conditions. In the case of the iris, then the net acceleration obtained, relative to the fields on the iris, will now be found to be inversely proportional to R/λ, where R is now the radius of the iris. The acceleration is thus a near field effect of the iris.

PROOFS

Proof 1: Sums of Waves

All far fields can be expressed as sums of plane, transversely polarized waves travelling at the velocity of light. The continuous acceleration of a relativistic particle by any one such wave, and thus the net acceleration from $-\infty$ *to* $+\infty$, is zero. Since the first order acceleration from the sum of such waves equals the sum of accelerations from the individual waves, so the net acceleration, from $-\infty$ *to* $+\infty$ for any system of far fields must be zero. Since fields separated by field free regions, are independent, so if the acceleration, from $-\infty$ *to* $+\infty$ is zero, then the acceleration from and to any field free regions, must also be zero. Q.E.D.

This proof applies only to the acceleration of relativistic particles. The later proofs will be seen to be more general. Let us now expand on each of the components of the proof:

A single far field wave

In a vacuum, at distances large compared with the wavelength of the radiation, all fields can be represented as sums (taken over frequency, phase, and direction) of waves that are

a) plane
b) transversely polarized
c) propagating at the velocity of light

The net acceleration of a relativistic particle (taken from $-\infty$ to $+\infty$) by any one such wave is zero. If the particle and wave are travelling in the same direction then there is no acceleration because the electric field is perpendicular to the direction of motion. But if they are travelling in different directions then the wave and particle will not remain in phase and the acceleration will be periodic. Such acceleration when integrated from $-\infty$ to $+\infty$ will be zero.

Sums of waves

The first order acceleration of a relativistic particle by the sum of any number of fields is equal to the sum of accelerations from each of the component fields taken individually. This follows from the fact that the acceleration from one component will neither change the particle trajectory (which, for relativistic particles, is always essentially straight), nor its velocity (which is always c). Thus the acceleration from any one component will not affect that of any other component.

Since we are restricting the discussion to first order (acceleration linear in the electric field) the contributions from all components are linear in the component fields.

If the contributions are both independent and linear, then the consequence of all the contributions at the same time must be the same as the sum of the each contributions when taken alone.

From and to field free regions

If acceleration is zero from $-\infty$ to $+\infty$, then it is also zero from any field free region to any other field free region. The fields beyond any field free region are decoupled from the fields before it and can thus be arbitrarily chosen to be zero without affecting the generality of the discussion. We can thus chose the fields outside of two field free regions to be zero and the integral between those regions is equal to that from $-\infty$ to $+\infty$.

Proof 2: Quantum Mechanics

Far fields may be represented as a sum of on mass shell (ie zero mass) photons. Linear acceleration in a field free region can only be represented by the successive absorption of individual such photons with no simultaneous emission. The interaction of any such a single photon with a charged particle with finite mass is forbidden by energy and momentum conservation. Thus there can be no such acceleration by far fields. Q.E.D.

This proof is both more general than the first, in that it covers particles of any initial velocity, but it is restricted to the acceleration of particles with finite mass (not that this is much of a restriction), and is even easier than the first.

From the quantum point of view, far fields, in vacuum, are made up of massless (on mass shell) photons. Near fields are from photons that are off their mass shell and have mass. Such photons can, as a consequence, only exist over short distances. We will again consider the components of the proof in more detail:

Energy and momentum conservation

First order (acceleration proportional to the field amplitude) acceleration by an electromagnetic field can be represented as the successive, but separate, interactions of photons with the particle. Far from the source of the fields, the corresponding photons will be massless. But the interaction of a massless photon with a charged particle, without emission of a second photon, can not conserve energy and momentum, and thus there can be no such interaction and no such acceleration.

The proof of this is simplest if we are in the rest frame of the initial charge whose energy is then equal to its mass m and whose momentum is zero. After the interaction its energy will be $m + E_\gamma$ and momentum p_γ. If the particle is to maintain its same mass then

$$m^2 = (m + E_\gamma)^2 - (p_\gamma)^2$$

and

$$p_\gamma^2 = E_\gamma^2 + 2mE_\gamma$$

Thus we find that for any finite photon energy E_γ and finite mass of particle to accelerate m, then $p_\gamma > E_\gamma$. But for an on mass shell photon $m_\gamma = 0$ and $p_\gamma = E_\gamma$, so the above process cannot take place.

Multiphoton effects

The interaction cannot be represented by the absorption of two photons, since this is clearly a second order effect. Nor can the interaction cannot be represented by the interaction and emission of a photon (which could conserve energy and momentum), since this also is a second order process. The process is known as Compton scattering and is proportional to the flux of initial photons. It is thus proportional to the energy flow, or accelerating field squared. It is not proportional to the accelerating field as required.

A first order acceleration could take place by the absorption of two photons if one of the photons represents the interaction with a fixed or wiggler field, but the presence of such fields is also excluded in the formulation of the theorem.

Near fields

A first order acceleration can take place if the momentum and energy of the photon are not equal, ie if it is off its mass shell. The allowed energy nonconservation ΔE over a distance R is given by

$$\Delta E \; R \approx \hbar c$$

and the on mass shell energy E

$$E = h\nu$$

so for a change of energy comparable with the on mass shell energy:

$$R \approx \frac{\hbar c}{h\nu} = \frac{\lambda}{2\pi}$$

which is the same condition we had above for comparable source and acceleration gradients.

Proof 3: Energy Conservation

Any first order acceleration of a charge by a field must be accompanied by a corresponding decrease in that field's energy that is proportional to the field's amplitude. In vacuum, this can only occur by the destructive interference of the accelerating field with a field radiated by the test charge. In a field free vacuum far from any material a moving charge does not lose energy to radiation, and thus cannot decrease the accelerating field. Thus it cannot be accelerated. Q.E.D

This proof is valid for a test charge of any velocity or mass. It is thus the most general of our proofs. It is best understood by examining those cases that we know do accelerate. In every case, in the absence of an accelerating field, there is the emission of some form of radiation caused by something other than the vacuum. The accelerating field, when applied, interferes with the radiated field in such a way as to decrease the energy in the field. The accelerating mechanism is then often referred to as the "inverse" of the radiation mechanism. For instance:

Cause	Radiation	Acceleration
Irises	Wake Fields	Linac
Gas	Cherenkov	Inverse Cherenkov
Magnetic Field	Synchrotron	Inverse Synchrotron
Wiggler Fields	Free Electron Laser	Inverse Free Electron Laser

Acceleration in a vacuum, far from all sources, would be the inverse of nothing that exists. It cannot occur.

Near to a structure particles do lose energy to wake fields, and acceleration does occur. But the energy given to such wake fields falls, at longer distances, as $1/R^2$, and thus their field amplitudes fall as R^{-1}. At large distances they become negligible, and no acceleration is possible.

Inverse Cherenkov, Synchrotron, or FEL

When there is a source of radiation from the particle, such as Cherenkov, synchrotron, or wiggler radiation, then we can take its amplitude to be $\mathcal{E}_r$ and the amplitude of the accelerating field to be $\mathcal{E}_a$. The accelerating field energy without the particle to be accelerated will be proportional to $(\mathcal{E}_a)^2$. The final energy, if the particle is present, will be proportional to $(\mathcal{E}_a - \mathcal{E}_r)^2$. Thus the change of field energy ΔE will be:

$$\Delta E \propto (\mathcal{E}_a - \mathcal{E}_r)^2 - (\mathcal{E}_a)^2 = -2\mathcal{E}_a\mathcal{E}_r + \mathcal{E}_r^2$$

Since the fields radiated will in general be far smaller than those used to accelerate:

$$\Delta E \approx\propto -2\mathcal{E}_a\mathcal{E}_r$$

The energy gain, ie acceleration, of the particle must be equal and opposite to this. And we see that it is indeed proportional to the accelerating field as required for first order acceleration.

Compton Scattering

In contrast, one might consider radiation pressure, ie Compton scattering. This can occur in a vacuum far from all material. Classically, we consider an accelerating field which causes the particle to wiggle. This wiggling then radiates, and this radiation can interfere destructively with the initial accelerating field. But the radiated field is proportional to the applied accelerating field that generated it. ie

$$\mathcal{E}_r \propto \mathcal{E}_a$$

so

$$\Delta E \approx\propto 2\mathcal{E}_a\mathcal{E}_r \propto (\mathcal{E}_a)^2$$

We see again that this is second order in the accelerating field and not a violation of the theorem.

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