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TOPOLOGY CHANGE AND QUANTUM PHYSICS



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The role of topology in elementary quantum physics is discussed in detail. It is argued that attributes of classical spatial topology emerge from properties of state vectors with suitably smooth time evolution. Equivalently, they emerge from considerations on the domain of the quantum Hamiltonian, this domain being often specified by boundary conditions in elementary quantum physics. Several examples are presented where classical topology is changed by smoothly altering the boundary conditions. When the parameters labelling the latter are treated as quantum variables, quantum states need not give a well-defined classical topology, instead they can give a quantum superposition of such topologies. An existing argument of Sorkin based on the spin-statistics connection and indicating the necessity of topology change in quantum gravity is recalled. It is suggested therefrom and our results here that Einstein gravity and its minor variants are effective theories of a deeper description with additional novel degrees of freedom. Other reasons for suspecting such a microstructure are also summarized.

1 Introduction

There are indications from theoretical considerations that spatial topology in quantum gravity can not be a time-invariant attribute, and that its transmutations must be permitted in any eventual theory.

The best evidence for the necessity of topology change comes from the examination of the spin-statistics connection for the so-called geons [1, 2, 3]. Geons are solitonic excitations caused by twists in spatial topology. In the absence of topology change, a geon can neither annihilate nor be pair produced with a partner geon, so that no geon has an associated antigeon.

Now spin-statistics theorems generally emerge in theories admitting creation-annihilation processes [4, 5]. It can therefore be expected to fail for geons in gravity theories with no topology change. Calculations on geon quantization in fact confirm this expectation [2, 4, 5].

The absence of a universal spin-statistics connection in these gravity theories is much like its absence for a conventional nonrelativistic quantum particle which too cannot be pair produced or annihilated. Such a particle can obey any sort of statistics including parastatistics regardless of its intrinsic spin. But the standard spin-statistics connection can be enforced in nonrelativistic dynamics also by introducing suitable creation-annihilation processes [4].

There is now a general opinion that the spin-statistics theorem should extend to gravity as well. Just as this theorem emerges from even nonrelativistic physics once it admits pair production and annihilation [4], quantum gravity too can be expected to become compatible with this theorem after it allows suitable topology change [6]. In this manner, the desire for the usual spin-statistics connection leads us to look for a quantum gravity with transmuting topology.

Canonical quantum gravity in its elementary form is predicated on the hypothesis that spacetime topology is of the form $\Sigma \times \mathbf{R}$ (with \mathbf{R} accounting for time) and has an eternal spatial topology. This fact has led to numerous suggestions that conventional canonical gravity is inadequate if not wrong, and must be circumvented by radical revisions of spacetime concepts [7], or by improved approaches based either on functional integrals and cobordism or on alternative quantisation methods.

Ideas on topology change were first articulated in quantum gravity, and more specifically in attempts at semiclassical quantisation of classical gravity. Also it is an attribute intimately linked to gravity in the physicist's mind. These connections and the apparently revolutionary nature of topology change as an idea have led to extravagant speculations about twinkling topology in quantum gravity and their impact on fundamental conceptions in physics.

In this paper, we wish to point out that models of quantum particles exist which admit topology change or contain states with no well-defined classical topology. *This is so even*

though gravity does not have a central role in our ideas [some of which were previously reported in refs. [8] and [9]] and is significant only to the extent that metric is important for a matter Hamiltonian. These models use only known physical principles and have no revolutionary content, and at least suggest that topology change in quantum gravity too may be achieved with a modest physical input and no drastic alteration of basic laws.

We explain our views [8, 9] regarding the role of spatial topology in elementary quantum physics in Section 2, and in particular emphasize that domains of observables [10] and smoothness of time evolution have much to say on this matter. Section 3 develops these observations for a particle which moves on the union of two intervals. The domain of its Hamiltonian is characterized by an element u of $U(2)$ under the simplifying demand that momentum too be (essentially) self-adjoint on that domain. It is then the case that continuity properties of probability densities are compatible with continuous functions on two circles ($S^1 \cup S^1$) for certain u 's and with a single circle for certain other u 's. The configuration space of the particle is thus $S^1 \cup S^1$ and S^1 for these two u , whereas it is just two intervals for the remaining u . We then argue that topology change can be achieved by looking upon u as an external parameter and continuously changing it from one value to another.

It is not quite satisfactory to have to regard u as an external parameter and not subject it to quantum rules. In Section 4, we therefore promote it to an operator, introduce its conjugate variables and modify the Hamiltonian as well to account for its dynamics. The result is a closed quantum system. It has no state with a sharply defined u . We cannot therefore associate one or two circles with the quantum particle and quantum spatial topology has to be regarded as a superposition of classical spatial topologies. Depending on our choice of the Hamiltonian, it is possible to prepare states where topology is peaked at one or two S^1 's for a long time, or arrange matters so that there is transmutation from one of these states to the other.

Section 5 generalizes these considerations to higher dimensions and establishes that similar effects can be achieved in all dimensions by manipulating boundary conditions and their dynamics [11].

It is the contention of this paper that topology change can be achieved already in elementary quantum physics. We realise this phenomenon by promoting parameters entering boundary conditions to control parameters or degrees of freedom, in such a manner that the states of the latter affect spatial topology. There is a close relation of these ideas to what happens in the axion solution to the strong CP-problem [12], as we explain in the concluding Section 6. All this suggests that topology change is facilitated by the addition of degrees of freedom.

We have remarked on the desirability of topology change from the point of view of the spin-statistics theorem. There is perhaps a hint here that *quantum gravity with topology change has novel degrees of freedom and not just those of Einstein gravity or its minor variants.* We summarize further evidence for this point of view, taken also from geon physics, in Section 6. It is a matter for regret that all such ideas on quantum gravity

must for now remain beyond experimental control.

2 Topology in Quantum Physics

Quantum systems with classical attributes are generally characterized by infinite dimensional Hilbert spaces. This is the case already for elementary systems such as that of a particle on a manifold.

For a system like this, it is never the case that we can realistically observe all self-adjoint operators with equal ease. For example, the eigenstates of many of these operators have infinite mean values for energy, its square or angular momentum. In conventional quantum physics, they are tacitly discarded as observables for the simple reason that their measurement is tantamount to the preparation of the above unphysical eigenstates.

We thus see that a self-adjoint operator can be an observable only if it has additional attributes. Those which circumvent the problem of unphysical eigenstates can be formulated using considerations on domains [10]. A simple formulation of these attributes can be achieved if it is agreed that time evolution, and hence the Hamiltonian generating it, have very special roles in physics. It goes as follows.

Recall that the Hamiltonian H generally is an unbounded operator and cannot be applied on all vectors of the quantum physical Hilbert space \mathcal{H} . Rather it can be applied only on vectors of its domain $D(H)$ [10]. The latter is dense in \mathcal{H} , but is not all of \mathcal{H} , and is often specified by boundary conditions in simple quantum systems. [We will see examples of domains in subsequent sections.] The attribute in question of any observable \mathcal{O} , having only discrete spectrum, is that *it is a self-adjoint operator having all its eigenvectors in $D(H)$.*

If \mathcal{O} has (also) continuous spectrum, the definition of the conditions under which it is observable requires a little elaboration. Let us notice first that we cannot prepare a quantum state by observing a point of its continuous spectrum with no experimental uncertainty at all, so that the above attribute cannot be used now.

The conditions under which such an \mathcal{O} is an observable is facilitated by first considering a physical state ψ that has already been prepared by means of a complete set of measurements of observables having only discrete spectra. The previous definition of observables then implies that $\psi \in D(H)$. Suppose now that a measurement of \mathcal{O} is performed on ψ and that we observe the value x for \mathcal{O} (x belonging to the spectrum $S(\mathcal{O})$ of \mathcal{O}) with an experimental uncertainty ϵ . We can think of associating the set $\Omega_{x,\epsilon} = [x - \epsilon, x + \epsilon] \cap S(\mathcal{O})$ with this result. Then we know from the general principles of quantum theory that an instant after the measurement, the state of the system will be given by the new vector $\chi_{x,\epsilon}(\mathcal{O})\psi$, where $\chi_{x,\epsilon}(y)$ is the characteristic function of the interval $[x - \epsilon, x + \epsilon]$. (For the definition of bounded functions of self-adjoint operators, see [13]). Now, before going further, we realize at once that even this is an idealization not corresponding to reality,

for the boundaries of $\Omega_{x,\epsilon}$ cannot be specified with absolute precision. We thus replace the characteristic function $\chi_{x,\epsilon}(y)$ with some smooth, real function $f_{x,\epsilon}^\infty$, of fast decrease, approximately supported in $\Omega_{x,\epsilon}$. Summing up, our criterion for \mathcal{O} to be an observable is that, for any such $f_{x,\epsilon}^\infty$, the operator $f_{x,\epsilon}^\infty(\mathcal{O})$ should leave $D(H)$ invariant. This definition guarantees that also after the measurement the state of the system will be in the domain of the Hamiltonian.

We will hereafter accept these properties as fundamental attributes of any observable. The domain $D(H)$ in our scheme thus has a central role in quantum physics.

This domain strongly reflects the properties of the classical configuration space Q : In conventional quantum physics we generally insist for example that the density function $\psi^*\chi$ for any pair of vectors $\psi, \chi \in D(H)$ is a continuous function on Q , $\psi^*\chi \in C(Q)$. Conversely, Q can be recovered as a topological space from the C^* -algebra generated by these density functions by using the Gel'fand-Naimark theorem [14]. In view of this fundamental fact, *we will hereafter assume that in a quantum system, the classical configuration space and its topological attributes are to be inferred from the density functions associated with $D(H)$ in the manner indicated above.*

More formally, our assumption is that in quantum theory, we have an operator-valued hermitean form $\psi^*\chi$ with standard properties [15] for $\psi, \chi \in D(H)$, the scalar product (ψ, χ) being $\text{Tr } \psi^*\chi$. [This form in particular is to be positive in the sense that $\psi^*\psi$ is a non-negative operator which vanishes iff $\psi = 0$.] It is the above density function in conventional quantum theory. There it generates an abelian normed $*$ -algebra \mathcal{A}' , the norm and $*$ being operator norm and hermitean conjugation, the latter inverting $\psi^*\chi$ to $\chi^*\psi$: $(\psi^*\chi)^* = \chi^*\psi$. [It may not be abelian in unconventional quantum theories such as those on topological lattices [16, 8]]. According to our scheme, the reconstruction of Q from \mathcal{A}' is achieved by first completing \mathcal{A}' to a C^* -algebra \mathcal{A} , and then using the Gel'fand-Naimark theorem. Once Q has been found, we can of course identify \mathcal{A} with $C(Q)$.

The recovery of Q as a manifold requires also a C^∞ -structure on Q . This can be specified in algebraic language by giving a suitable subalgebra $\mathcal{A}^{(\infty)}$ of \mathcal{A} [17], the C^∞ -structure of Q being that one for which $\mathcal{A}^{(\infty)}$ consists of C^∞ functions.

There is a natural way to specify $\mathcal{A}^{(\infty)}$ too in our scheme: Let $D^\infty(H)$ be the subspace of $D(H)$ which is transformed into itself by arbitrary powers of H : $H^N D^\infty(H) \subset D^\infty(H)$, $N = 1, 2, \dots$. Then $\mathcal{A}^{(\infty)}$ is generated by $\psi^*\chi$ when ψ and χ run over $D^\infty(H)$.

There is a clear physical meaning to $D^\infty(H)$ in terms of ultraviolet cut-offs and smooth time evolution as we shall now show.

Let us assume for simplicity that the spectrum $\{E_n\}$ of H is entirely discrete, and let $H|E_n\rangle = E_n|E_n\rangle$, $\langle E_m|E_n\rangle = \delta_{mn}$. It is then evident that $|E_n\rangle \in D^\infty(H)$. Also any state vector of the form $\sum_n c_n|E_n\rangle \in D(H)$ where $\sum_n |c_n E_n^N|^2 < \infty$ for $N = 0, 1, 2, \dots$ belongs to $D^\infty(H)$. This condition is met if $|c_n|$ goes to zero exponentially fast as $n \rightarrow \infty$.

High energy components of state vectors in $D^\infty(H)$ are thus heavily suppressed.

A consequence is that time evolution of vectors in $D^\infty(H)$ is smooth in time, arbitrary time derivatives $\sum_n c_n (-iE_n)^N e^{-iE_n t} |E_n\rangle$ of $\sum_n c_n e^{-iE_n t} |E_n\rangle$ remaining in $D^\infty(H)$ if $\sum_n c_n |E_n\rangle \in D^\infty(H)$.

In contrast, time evolution of vectors $\sum d_n |E_n\rangle \in D(H)$ is generically only once-differentiable in time. Thus using the fact that $HD(H) \in \mathcal{H}$, we can see that $\frac{d}{dt} \sum d_n e^{-iE_n t} |E_n\rangle = \sum d_n (-iE_n) e^{-iE_n t} |E_n\rangle \in \mathcal{H}$, but can not prove further differentiability of this vector.

We find it striking that topological properties of the classical configuration space Q depend in this matter on the degree of temporal smoothness. Vectors in $D(H)$ which are C^1 in time determine Q as a topological space whereas vectors in $D(H)$ which are C^∞ in time determine Q as a manifold.

3 A Simple Model

We will be considering particle dynamics from here onwards until Section 6. The configuration space of a particle being ordinary space, we are thus imagining a physicist probing spatial topology using a particle.

Let us consider a particle with no internal degrees of freedom living on the union Q' of two intervals which are numbered as 1 and 2:

$$Q' = [0, 2\pi] \cup [0, 2\pi] \equiv Q'_1 \cup Q'_2. \quad (3.1)$$

It is convenient to write its wave function ψ as (ψ_1, ψ_2) , where each ψ_i is a function on $[0, 2\pi]$ and $\psi_i^* \psi_i$ is the probability density on Q'_i . The scalar product between ψ and another wave function $\chi = (\chi_1, \chi_2)$ is

$$(\psi, \chi) = \int_0^{2\pi} dx \sum_i (\psi_i^* \chi_i)(x). \quad (3.2)$$

It is interesting that we can also think of this particle as moving on $[0, 2\pi]$ and having an internal degree of freedom associated with the index i .

After a convenient choice of units, we define the Hamiltonian formally by

$$(H\psi)_i(x) = -\frac{d^2 \psi_i}{dx^2}(x) \quad (3.3)$$

[where ψ_i is assumed to be suitably differentiable in the interval $[0, 2\pi]$]. This definition is only formal as we must also specify its domain $D(H)$ [10]. The latter involves the statement of the boundary conditions (BC's) at $x = 0$ and $x = 2\pi$.

Arbitrary BC's are not suitable to specify a domain: A symmetric operator \mathcal{O} with domain $D(\mathcal{O})$ will not be self-adjoint unless the following criterion is also fulfilled:

$$\mathcal{B}_{\mathcal{O}}(\psi, \chi) \equiv (\psi, \mathcal{O}\chi) - (\mathcal{O}\psi, \chi) = 0 \text{ for all } \chi \in D(\mathcal{O}) \Leftrightarrow \psi \in D(\mathcal{O}). \quad (3.4)$$

For the differential operator H , the form $\mathcal{B}_H(\cdot, \cdot)$ is given by

$$\mathcal{B}_H(\psi, \chi) = \sum_{i=1}^2 \left[-\psi_i^*(x) \frac{d\chi_i(x)}{dx} + \frac{d\psi_i^*(x)}{dx} \chi_i(x) \right]_0^{2\pi}. \quad (3.5)$$

It is not difficult to show that there is a $U(4)$ worth of $D(H)$ here compatible with (3.4).

We would like to restrict this enormous choice for $D(H)$, our intention not being to study all possible domains for $D(H)$. So let us assume that the momentum P too is an observable and find possible $D(H)$ accordingly. That is to say, let us also insist that eigenstates of P [and wave packets constructed as previously from its generalized eigenvectors [18], if any] belong to $D(H)$. We will achieve this end by finding a domain $D(P)$ for P and verifying that eigenvectors of this P are all in $D(H)$ when its BC's are properly chosen.

The momentum P is defined by

$$(P\psi)_i(x) = -i \frac{d\psi_i(x)}{dx}. \quad (3.6)$$

Hence

$$\mathcal{B}_P(\psi, \chi) = -i \sum_{i=1}^2 [\psi_i^*(x) \chi_i(x)]_0^{2\pi}. \quad (3.7)$$

Let $u \in U(2)$ [regarded as 2×2 unitary matrices] and set

$$D_u(P) = \{\psi : \psi_i(2\pi) = u_{ij} \psi_j(0)\}. \quad (3.8)$$

In addition, ψ must of course be differentiable.

It is now easy to verify that P is self-adjoint for the domain $D_u(P)$ using the criterion (3.4). The eigenstates and spectrum of P are obtained by solving

$$P\psi = p\psi, \quad \psi \in D_u(P), \quad p \in \mathbf{R}. \quad (3.9)$$

There is only discrete spectrum. The solutions are given by exponentials, they are C^∞ and are obviously in the domain

$$D_u(H) = \left\{ \psi \in C^2(Q') : \psi_i(2\pi) = u_{ij} \psi_j(0), \quad \frac{d\psi_i}{dx}(2\pi) = u_{ij} \frac{d\psi_j}{dx}(0) \right\} \quad (3.10)$$

for H . Furthermore H is (essentially) self-adjoint on this domain as shown using (3.4). We therefore restrict attention to $D_u(H)$.

There are two choices of u which are of particular interest:

$$a) \quad u_a = \begin{pmatrix} 0 & e^{i\theta_{12}} \\ e^{i\theta_{21}} & 0 \end{pmatrix}, \quad (3.11)$$

$$b) \quad u_b = \begin{pmatrix} e^{i\theta_{11}} & 0 \\ 0 & e^{i\theta_{22}} \end{pmatrix}. \quad (3.12)$$

In case a , the density functions $\psi_i^* \chi_i$ fulfill

$$\begin{aligned} (\psi_1^* \chi_1)(2\pi) &= (\psi_2^* \chi_2)(0), \\ (\psi_2^* \chi_2)(2\pi) &= (\psi_1^* \chi_1)(0). \end{aligned} \quad (3.13)$$

Figure 1 displays (3.13), these densities being the same at the points connected by broken lines.

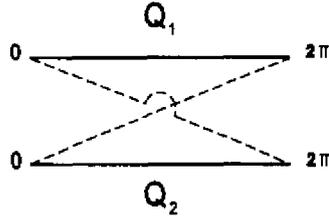


Figure 1: In case a), the density functions are the same at the points joined by broken lines in this Figure.

In case b), they fulfill, instead,

$$\begin{aligned} (\psi_1^* \chi_1)(2\pi) &= (\psi_1^* \chi_1)(0), \\ (\psi_2^* \chi_2)(2\pi) &= (\psi_2^* \chi_2)(0) \end{aligned} \quad (3.14)$$

which fact is shown in a similar way in Figure 2.

Continuity properties of $\psi_i^* \chi_i$ imply that we can identify the points joined by dots to get the classical configuration space Q . It is not Q' , but rather a circle S^1 in case a and the union $S^1 \cup S^1$ of two circles in case b .

The requirement $H^N D^\infty(H) \subset D^\infty(H)$ means just that arbitrary derivatives of $\psi_i^* \chi_i$ are continuous at the points joined by broken lines, that is on S^1 and $S^1 \cup S^1$ for the two cases. We can prove this easily using (3.10). Thus on regarding S^1 and $S^1 \cup S^1$ as the usual manifolds, $D^\infty(H)$ becomes $C^\infty(Q)$. In this way we also recover Q as manifolds.

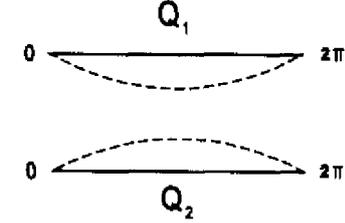


Figure 2: In case b), the density functions are the same at the points joined by broken lines in this Figure.

When u has neither of the values (3.11) and (3.12), then Q becomes the union of two intervals. The latter happens for example for

$$u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \quad (3.15)$$

In all such cases, Q can be regarded as a manifold with boundaries as shown by the argument above.

Summarizing, we see that the character of the underlying classical manifold depends on the domain $D_u(H)$ of the quantum Hamiltonian and can change when u is changed.

It is possible to reduce the u in the BC to $\mathbf{1}$ by introducing a connection. Thus since $U(2)$ is connected, we can find a $V(x) \in U(2)$ such that

$$V(0) = \mathbf{1}, \quad V(2\pi) = u^{-1}. \quad (3.16)$$

Using this V we can unitarily transform H to the new Hamiltonian

$$\begin{aligned} H' &= V H V^{-1}, \\ (H' \psi)_i(x) &= - \left[\frac{d}{dx} + A(x) \right]_{ij}^2 \psi_j(x), \\ A(x) &= V(x) \frac{d}{dx} V^{-1}(x). \end{aligned} \quad (3.17)$$

With suitable physical interpretation, the system defined by H' and the domain

$$D_1(H') = V D_u(H) \equiv \{ \phi : \phi = V \psi, \psi \in D_u(H) \} \quad (3.18)$$

is evidently equivalent to the system with Hamiltonian H and domain $D_u(H)$. Note in this connection that density functions on Q'_i are $\psi_i^* \chi_i$, and not $(V \psi)_i^* (V \chi)_i$.

4 Dynamics for Boundary Conditions

We saw in the previous section that topology change can be achieved in quantum physics by treating the parameters in the BC's as suitable external parameters which can be varied. Here we point out that there is no need for these parameters to be "external" as they too can be treated as quantum variables.

Dynamics for u which determines BC's is best introduced in the connection picture where the domain of H' is associated with $u = 1$. We assume this representation hereafter.

Quantisation of u is achieved as follows. Let $T(\alpha)$ be the antihermitean generators of the Lie algebra of $U(2)$ (the latter being regarded as the group of 2×2 unitary matrices) and normalized according to $\text{Tr } T(\alpha)T(\beta) = -N\delta_{\alpha\beta}$, N being a constant. Let \hat{u} be the matrix of quantum operators representing the classical u . It fulfills

$$\hat{u}_{ij}\hat{u}_{ik}^\dagger = \mathbf{1}\delta_{jk}, \quad [\hat{u}_{ij}, \hat{u}_{kk}] = 0, \quad (4.1)$$

\hat{u}_{ik}^\dagger being the adjoint of \hat{u}_{ik} . The operators conjugate to \hat{u} will be denoted by L_α . If

$$[T_\alpha, T_\beta] = c_{\alpha\beta}^\gamma T_\gamma,$$

$$c_{\alpha\beta}^\gamma = \text{structure constants of } U(2) \in \mathbf{R}, \quad (4.2)$$

L_α has the commutators

$$\begin{aligned} [L_\alpha, \hat{u}] &= -T(\alpha)\hat{u}, \\ [L_\alpha, L_\beta] &= c_{\alpha\beta}^\gamma L_\gamma, \\ [T(\alpha)\hat{u}]_{ij} &\equiv T(\alpha)_{ik}\hat{u}_{kj}. \end{aligned} \quad (4.3)$$

If \hat{V} is the quantum operator for V , $[L_\alpha, \hat{V}]$ is determined by (3.17) and (4.3), V being a function of u .

The Hamiltonian for the combined particle- u system can be taken to be, for example,

$$\begin{aligned} \hat{H} &= \hat{H}' + \frac{1}{2I} \sum_\alpha L_\alpha^2 \\ \hat{H}' &= - \left(\frac{d}{dx} + \hat{A}(x) \right)^2, \quad \hat{A}(x) \equiv \hat{V}(x) \frac{d}{dx} \hat{V}^{-1}(x), \end{aligned} \quad (4.4)$$

I being the moment of inertia.

Quantised BC's with a particular dynamics are described by (4.1), (4.3) and (4.4).

The general wave function in the domain of \hat{H} is a superposition of state vectors $\phi \otimes_{\mathbf{C}} |u\rangle$ where $\phi \in D_1(H')$ and $|u\rangle$ is a generalized eigenstate of \hat{u} :

$$\hat{u}_{ij}|u\rangle = u_{ij}|u\rangle, \quad \langle u'|u\rangle = \delta(u'^{-1}u). \quad (4.5)$$

The δ -function here is defined by

$$\int du f(u) \delta(u'^{-1}u) = f(u'), \quad (4.6)$$

du being the (conveniently normalized) Haar measure on $U(2)$. Also

$$\hat{A}(x)|u\rangle = A(x)|u\rangle. \quad (4.7)$$

It follows that the classical topology of one and two circles is recovered on the states $\sum C_\lambda \phi^{(\lambda)} \otimes_{\mathbf{C}} |u_a\rangle$ and $\sum D_\lambda \phi^{(\lambda)} \otimes_{\mathbf{C}} |u_b\rangle$, $[C_\lambda, D_\lambda \in \mathbf{C}, \phi^{(\lambda)} \in D_1(H')]$ with the two fixed values u_a , and u_b of (3.11) and (3.12) for u .

But these are clearly idealized unphysical vectors with infinite norm. The best we can do with normalizable vectors to localize topology around one or two circles is to work with wave packets

$$\begin{aligned} \int du f(u) \phi \otimes_{\mathbf{C}} |u\rangle, \\ \int du |f(u)|^2 < \infty \end{aligned} \quad (4.8)$$

where f is sharply peaked at the u for the desired topology. The classical topology recovered from these states will only approximately be one or two circles, the quantum topology also containing admixtures from neighbouring topologies of two intervals.

A localised state vector of the form (4.8) is not as a rule an eigenstate of a Hamiltonian like \hat{H} . Rather it will spread in course of time so that classical topology is likely to disintegrate mostly into that of two intervals. We can of course localise it around one or two S^1 's for a very long time by choosing I to be large, the classical limit for topology being achieved by letting $I \rightarrow \infty$. By adding suitable potential terms, we can also no doubt arrange matters so that a wave packet concentrated around $u = u_a$ moves in time to one concentrated around $u = u_b$. This process would be thought of as topology change by a classical physicist.

5 Generalizations

Considerations of the last sections admit generalizations to higher dimensions which we now indicate.

In analogy with the previous two-interval model, we now start our discussion with

$$C' = C_1 \cup C_2. \quad (5.1)$$

Here C_i are two cylinders. We assume for convenience that they are identical and can thus be identified with a common cylinder C :

$$C = \{(x_1, x_2) : x_1 \in [0, 2\pi], x_2 \in [0, 2\pi]\}. \quad (5.2)$$

Here $(0, x_2)$ and $(2\pi, x_2)$ are to be identified.

Let us consider a particle on Q' with spin associated with a two-valued index. The wave function ψ can then be written as (ψ_1, ψ_2) where each ψ_i has two components: $\psi_i = (\psi_i^{(1)}, \psi_i^{(2)})$. Here $\psi_i^{(a)}$ is a \mathbb{C} -valued function on C_i and $\psi_i^\dagger \psi_i$ is its probability density. As for the scalar product and Hamiltonian, we choose them to be

$$\begin{aligned} (\psi, \chi) &= \int_C dx_1 dx_2 \sum_i \psi_i^\dagger \chi_i(x) \equiv \int_C dx_1 dx_2 \psi^\dagger \chi(x), \\ (H\psi)_i^{(a)}(x) &= -\sum_{j=1}^2 \frac{\partial^2 \psi_j^{(a)}}{\partial x_j^2}(x), \quad x = x_1, x_2. \end{aligned} \quad (5.3)$$

Let

$$\begin{aligned} \mathcal{D} &= -i \sum_{i=1}^2 \alpha_i \partial_i, \\ \alpha_1 &= \tau_1, \quad \alpha_2 = \tau_3, \quad \tau_i = \text{Pauli matrices} \end{aligned} \quad (5.4)$$

be the Dirac operator. As $\mathcal{D}^2 = H$, we will use \mathcal{D} as the substitute for the momentum operator of Section 3 to simplify considerations.

Now, we have

$$\begin{aligned} \mathcal{B}_{\mathcal{D}}(\psi, \chi) &= (\psi, \mathcal{D}\chi) - (\mathcal{D}^\dagger \psi, \chi) = \\ &= i \int_{x_2=0} dx_1 \psi^\dagger \alpha_2 \chi - i \int_{x_2=2\pi} dx_1 \psi^\dagger \alpha_2 \chi. \end{aligned} \quad (5.5)$$

We will establish the feasibility of topology change even in this two-dimensional example, limiting ourselves for simplicity to a particular class of BC's for which \mathcal{D} is (essentially) self-adjoint. They are labelled by an element of $U(2)$ [regarded as 2×2 unitary matrices] and are given by

$$\begin{aligned} D_u(\mathcal{D}) &= \{\psi : \psi_i^{(k)} \in C^1(Q')\}; \\ \psi_i^{(k)}(x_1, 2\pi) &= u_{ij} \psi_j^{(k)}(x_1, 0). \end{aligned} \quad (5.6)$$

As for H , we can make it (essentially) self-adjoint on a dense subset $D_u(H)$ of $D_u(\mathcal{D})$ obtained by imposing also twice-differentiability and additional conditions:

$$D_u(H) = \{\psi : \psi_i^{(k)} \in C^2(Q')\};$$

$$\psi_i^{(k)}(x_1, 2\pi) = u_{ij} \psi_j^{(k)}(x_1, 0),$$

$$\partial_{x_2} \psi_i^{(k)}(x_1, 2\pi) = u_{ij} \partial_{x_2} \psi_j^{(k)}(x_1, 0) ;$$

$$\partial_{x_2} \equiv \frac{\partial}{\partial x_2}. \quad (5.7)$$

It is easily seen that for the choice $u = u_a$ [eq. (3.11)], the density functions remain continuous if we identify the points of the boundaries of C_1 and C_2 as shown in Figure 3. The result is a single torus. Alternatively we can also choose $u = u_b$ [eq. (3.12)]. With this latter u , Q becomes the union of two tori. For other choices of u in eq. (5.6), Q becomes the union of two cylinders.

More general topologies for Q can be obtained by allowing the matrix u in eq. (5.7) to depend on the coordinate x_1 . The choice

$$u(x_1) = \begin{cases} 1 & \text{for } x_1 \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi] \\ \tau_1 & \text{for } x_1 \in [\frac{\pi}{2}, \frac{3\pi}{2}] \end{cases} \quad (5.8)$$

for example makes Q a genus two surface as shown in Figure 3.

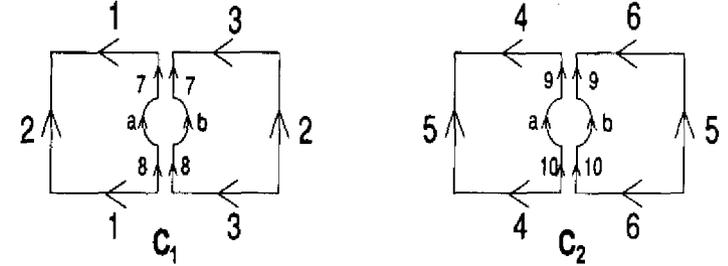


Figure 3: Opposite sides on the periphery with the same labels are to be identified in the direction of the arrows. With the cuts in the middle, C_1 and C_2 are thus cylinders. On identifying the lips of the cuts in the direction of the arrows in each cylinder separately, we get two tori. If instead we identify the lips of the cuts with the same labels in the direction of the arrows, we get a genus two manifold. If we do this identification for all except a and b , C_1 and C_2 become tori with holes.

There is the following simple way to see that (5.8) gives a genus two surface. With just the first line of (5.8), the effective configuration space is the union of two tori, each with a hole. These holes get identified with the second line of (5.8) to give us the genus two surface. This process is also shown in Figure 3.

We could in fact have begun with a pair of tori with holes and then used the second line of (5.8) as the BC. In this approach, we see clearly that what is involved is taking

connected sums [1-3] of tori. It also becomes obvious that we can in this way pass from manifolds with genera g_1 and g_2 to get a manifold with genus $g_1 + g_2$.

It is evident that topology change can be achieved here by changing u . For example a genus g manifold can be split into manifolds with genera g_1 and g_2 , g being $g_1 + g_2$. It should also be clear that much of the work in the previous sections can be adapted to such models.

There seems also to be no barrier to higher dimensional generalisations.

It is interesting that the process whereby surfaces are joined together in our approach to obtain a closed (compact and boundaryless) manifold is *local*, in that the identified set contains at most one point from each cut. But this is enough both for connected sums [1-3], and, in three dimensions, for surgery on links [19]. As any orientable, closed three-manifold can be changed to any other such manifold by surgery on links [19], we speculate that corresponding topology changes can also be achieved by our methods. There seems also to be no problem in taking connected sums [1-3] of these manifolds with \mathbf{R}^2 or \mathbf{R}^3 and extending these considerations to asymptotically flat spatial slices.

6 Final Remarks

We have seen in this article that topology change can be achieved in quantum physics by judicious introduction of new degrees of freedom. They control the BC's of operators associated with the classical configuration space. When they change, the classical configuration space too is changed and suffers topological transmutations.

There is striking resemblance of this mechanism for topology change and the axion approach to the strong CP problem in QCD [12]. The latter is caused by the fact that its Hamiltonian admits a one-parameter family of boundary conditions on its wave functions, labelled by an element $e^{i\theta}$ of $U(1)$. It has thus a one-parameter family of domains. The possibility of these BC's is reflected by the $\theta Tr(F \wedge F)$ term in the action.

If the BC's are now made dynamical, θ becomes the axion field a and the above term in the action becomes $a Tr(F \wedge F)$.

In this model with the additional axion degree of freedom, we find the θ -vacuum when a is frozen to the value θ . But a will in general fluctuate between different values so that the effective QCD θ depends on the state of a . These features resemble those found in the previous sections.

In any case, we see that topology change is facilitated by introducing new degrees of freedom. Let us now also recall the existence of indications that topology change should be permitted in quantum gravity to enforce the standard spin-statistics connection. The reasons leading to this opinion were summarized in the very first section.

The above two aspects relating to topology change suggest that *any eventual theory*

of quantum gravity will contain degrees of freedom going beyond those in Einstein gravity or its minor variants.

There is another line of thought indicating that Einstein's model for gravity is somehow only an effective theory of another underlying theory with more degrees of freedom. It has got repeatedly emphasised during conversations with Rafael Sorkin over the years and runs as follows. In molecular physics in the Born-Oppenheimer approximation, or in the collective model for nuclei, molecules and nuclei are generally described as rigid bodies with discrete symmetry groups $G \subset SU(2)$. These groups are also the fundamental groups $\pi_1(Q)$ of their configuration spaces Q , which on ignoring translations, are $SU(2)/G$ [20,3]. When $\pi_1(Q)$ is nontrivial, there are as many ways of quantising the system as there are unitary irreducible representations (UIRR's) of G [3]. These uncertainties correspond to uncertainties in the choice of BC's for the wave functions or equivalently the domain of the Hamiltonian. The distinct quantisations can be so different that wave functions are spinorial in one and tensorial in another [3,20].

Now, it is often the case that molecules and nuclei described by these different UIRR's do occur in nature. In their microscopic structure, they differ in their nuclear and/or electronic constituents. One can say that quantisation ambiguities in the Born-Oppenheimer approximation or in the collective nuclear models reflect the possibility of differing microscopic constituents for the same rigid body. Indeed the particular quantisation appropriate for a molecular or nuclear species is chosen by chemists and nuclear physicists by appealing to its microscopic description.

In a similar way, the fundamental group $\pi_1(Q)$ of the configuration space Q for the two flavour Skyrme model is \mathbf{Z}_2 [3]. It therefore has a two-fold uncertainty of quantisation leading respectively to a spinorial and a tensorial soliton [3]. Here too this ambiguity reflects the possibility of differing microscopic constituents for solitons and can be resolved by postulating a specific microscopic structure. This microstructure is provided by QCD. The soliton is spinorial if the number of colours is odd, and tensorial if it is even.

These examples suggest that quantisation ambiguities may indicate an underlying microstructure and its associated novel degrees of freedom.

Let us now turn to gravity. Here the fundamental group $\pi_1(Q)$ of the configuration space Q is extremely complex in the presence of geons [21,1-3] leading to enormous quantisation uncertainties. Our experience in molecular, nuclear and particle physics now suggests strongly that *gravity too has an underlying microscopic structure with its novel degrees of freedom, and that these ambiguities merely reflect the fact that it is an effective theory of several differing microscopic theories.*

The preceding arguments, one based on the spin-statistics connection and the second on quantisation ambiguities, lend encouragement to radical attempts at deriving Einstein gravity [22] and even spacetime as a manifold [7] from deeper microscopic models.

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