

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DOE/ER/40757--058

DE-ER40757-058

CPP-94-35

January 13, 1995

Coordinates of the Quantum Plane as q -Tensor Operators in $\mathcal{U}_q(\mathfrak{su}(2) * \mathfrak{su}(2))$

L. C. Biedenharn and M. A. Lohe*

*Center for Particle Physics, Department of Physics, University of Texas at Austin,
Austin, TX 78712-1081*

Abstract

The relation between the set of transformations $\mathfrak{M}_q(2)$ of the quantum plane and the quantum universal enveloping algebra $\mathcal{U}_q(\mathfrak{u}(2))$ is investigated by constructing representations of the factor algebra $\mathcal{U}_q(\mathfrak{u}(2) * \mathfrak{u}(2))$. The non-commuting coordinates of $\mathfrak{M}_q(2)$, on which $U_q(2) * U_q(2)$ acts, are realized as q -spinors with respect to each $\mathcal{U}_q(\mathfrak{u}(2))$ algebra. The representation matrices of $U_q(2)$ are constructed as polynomials in these spinor components. This construction allows a derivation of the commutation relations of the non-commuting coordinates of $\mathfrak{M}_q(2)$ directly from properties of $\mathcal{U}_q(\mathfrak{u}(2))$. The generalization of these results to $\mathcal{U}_q(\mathfrak{u}(n))$ and $\mathfrak{M}_q(n)$ is also discussed.

1 Introduction

The concept of a quantum group [1, 2, 3] arose from physics as an abstraction from the problem of understanding common features of exactly solvable (two dimensional) models in quantum mechanics. The insights into quantized symmetry and non-commuting geometry that quantum groups afford are felt to have wider applicability to the real world of nature and to lead to new physics, but an understanding of the physical meaning of non-commuting coordinates has been lacking.

*Permanent address: Northern Territory University, PO Box 40146, Casuarina, NT Australia

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

We discuss here the simplest of quantum groups, the “ q -quantized angular momentum group” $SU_q(2)$ and demonstrate a new interpretation of non-commuting coordinates as q -tensor operators in the factor algebra $\mathcal{U}_q(\mathfrak{su}(2)*\mathfrak{su}(2))$, (defined in §4, with properties given in Lemma 1). We show that this interpretation can be used to derive, not postulate, the commutation relations of the non-commuting coordinates directly from the algebra—co-algebra structure without consideration of the duality between $SU_q(2)$ and $\mathcal{U}_q(\mathfrak{su}(2))$. We may think of this derivation as providing a constructive means of passing from the algebra to the corresponding “group” since it reduces, for $q = 1$, to a determination of the Lie group properties from those of the Lie algebra.

After constructing irreducible representations of $\mathcal{U}_q(\mathfrak{su}(2))$ in §2, and defining and reviewing the properties of $\mathfrak{M}_q(2)$ in §3, we construct representations of $\mathcal{U}_q(\mathfrak{su}(2)*\mathfrak{su}(2))$ in §4 in the space of polynomials \mathfrak{P}^4 of four complex variables z_j^i ($i, j = 1, 2$). Next, in §5, we review properties of q -tensor operators in $\mathcal{U}_q(\mathfrak{su}(2))$. Then, from the complex variables z_j^i , regarded as operators, and their adjoints we construct in §6 a matrix of q -tensor operators $\mathbf{t} = (t_j^i)$ which form q -spinor components with respect to each of the $\mathcal{U}_q(\mathfrak{su}(2))$ subalgebras. We show that these components form a realization of the coordinate elements of $\mathfrak{M}_q(2)$ from which the representation matrices of $SU_q(2)$ can be constructed. We also demonstrate abstractly – using the algebra of q -tensor operators – that these spinor components satisfy the non-commuting properties of $\mathfrak{M}_q(2)$. The concluding section, in which we discuss the generalization of these results to $\mathcal{U}_q(\mathfrak{u}(n))$ and $\mathfrak{M}_q(n)$, outlines arguments using q -tensor operators which generalize those for $n = 2$ and lead to a derivation of the noncommuting properties of the elements of $\mathfrak{M}_q(n)$, leading in turn to $GL_q(n)$ and $U_q(n)$.

2 Construction of Irreducible Representations

Let us describe the operators which we use to realize the irreducible representations of $\mathcal{U}_q(\mathfrak{su}(2))$. In the space \mathfrak{P} of polynomials of a complex variable z define the finite difference

operator D by

$$Df(z) \stackrel{\text{def}}{=} \frac{f(zq^{\frac{1}{2}}) - f(zq^{-\frac{1}{2}})}{z(q^{\frac{1}{2}} - q^{-\frac{1}{2}})} = z^{-1} \left(\frac{q^{\frac{N}{2}} - q^{-\frac{N}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \right) f(z), \quad f \in \mathfrak{F}, \quad q \in \mathbb{R}^+, \quad (1)$$

where we have written D in operator form using the “number operator” N defined by

$$N \stackrel{\text{def}}{=} z \frac{\partial}{\partial z}, \quad q^{\frac{N}{2}} f(z) = f(zq^{\frac{1}{2}}), \quad f \in \mathfrak{F}.$$

We use the same symbol z to denote the operator which acts on elements of \mathfrak{F} by multiplication by z , and the complex variable z . We will frequently use the operator equations

$$zD = [N]_q, \quad Dz = [N + 1]_q \quad (2)$$

where the function $[\]_q$ is defined by:

$$[n]_q \stackrel{\text{def}}{=} \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

We note that z and D can be considered equivalently as realizations of the q -boson operators a, \bar{a} introduced in [4, 5]; that is, we can write $af(z) = zf(z)$, $\bar{a}f(z) = Df(z)$, with commutation relations determined in \mathfrak{F} by (2).

We define an inner product on \mathfrak{F} by

$$(f, g) \stackrel{\text{def}}{=} f(D)g(z) \Big|_{z=0} \quad f, g \in \mathfrak{F}, \quad (3)$$

from which follows the orthogonality property

$$(z^m, z^n) = \delta_{m,n} [n]_q!, \quad n, m \in \mathbb{N},$$

where $[n]_q! \stackrel{\text{def}}{=} [n]_q [n-1]_q \dots [1]_q$. As a consequence of (3) we have $D^\dagger = z$ and $N^\dagger = N$.

We may construct unitary irreducible representations of $\mathcal{U}_q(\mathfrak{su}(2))$ in the space \mathfrak{F}^2 of homogeneous polynomials of degree $2j$ in two complex variables z_1, z_2 . This construction is well-known for $\mathfrak{su}(2)$ (see [6, Chapter 5], for a description using the language of boson creation and destruction operators), and similarly $\mathcal{U}_q(\mathfrak{su}(2))$ can be realized using two complex variables z_1, z_2 and their adjoint operators D_1, D_2 . The three generators J_+, J_-, J_z are given by

$$J_+ = z_1 D_2, \quad J_- = z_2 D_1, \quad J_z = \frac{1}{2}(N_1 - N_2), \quad (4)$$

and satisfy the defining relations of $\mathcal{U}_q(\mathfrak{su}(2))$:

$$[J_z, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = [2J_z]_q. \quad (5)$$

In addition, the hermiticity properties appropriate to unitary representations are satisfied, namely, $J_z^\dagger = J_z$, $J_+^\dagger = J_-$, $J_-^\dagger = J_+$. An orthonormal basis in \mathfrak{R}^2 is given by

$$|jm\rangle = ([j+m]_q! [j-m]_q!)^{-\frac{1}{2}} z_1^{j+m} z_2^{j-m}$$

where $-j \leq m \leq j$ and j is a half-integer. From this basis we may determine directly the matrix elements of the generators, and the dimension of the irreducible representation is $2j+1$. This construction has been previously carried out using q -boson operators (see for example [4]) and also using finite difference operators [7]. In §4 we use the realization (4) to construct generators $\Delta(\mathbf{J})$ using co-multiplication, where Δ is the non-commutative coproduct.

Symmetry transformations of q -angular momentum eigenstates are performed by the q -analog to the ordinary rotation matrices $\mathcal{D}^j(g)$, where $g \in SU(2)$. We denote the q -analog matrix by ${}^q\mathcal{D}^j$ and refer to it as the representation matrix of $SU_q(2)$ (for the irreducible representation labelled j) since the representation property ${}^q\mathcal{D}^j(g){}^q\mathcal{D}^j(g') = {}^q\mathcal{D}^j(gg')$ will be satisfied for all $g, g' \in SU_q(2)$. We will extend the definition so that ${}^q\mathcal{D}^j$ is defined on $\mathfrak{M}_q(2)$ (which itself is defined in the next section), and then the representation property will hold for all $g, g' \in \mathfrak{M}_q(2)$. Since $\mathcal{U}_q(\mathfrak{su}(2))$ is in fact a deformation of the Lie algebra of $SU(2)$ it is not immediately clear how to construct ${}^q\mathcal{D}^j$, since there is no direct means of constructing the general “group element” by exponentiation, or by integration. We will resolve this problem by generalizing the construction of irreducible representations and will determine ${}^q\mathcal{D}^j$ from considerations of an enveloping algebra more general than $\mathcal{U}_q(\mathfrak{su}(2))$.

3 The Quantum Plane

Following Faddeev, Reshetikhin and Takhtajan [3] one can define the quantum group $\mathcal{U}_q(\mathfrak{su}(2))$ by taking as a starting point the relations satisfied by the transition matrix T , usually written as

$RT_1T_2 = T_2T_1R$, where the R -matrix satisfies the Yang-Baxter relations, and hence defining the quantum matrix algebra $\mathfrak{M}_q(2)$. One introduces elements a, b, c, d of an algebra by postulating the commutation relations[†]

$$ab = q^{-\frac{1}{2}}ba, \quad ac = q^{-\frac{1}{2}}ca, \quad bd = q^{-\frac{1}{2}}db, \quad cd = q^{-\frac{1}{2}}dc, \quad (6)$$

$$bc = cb, \quad ad - da = -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})bc. \quad (7)$$

$\mathfrak{M}_q(2)$ then comprises the set of matrices $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with entries which satisfy (6,7), and which in fact forms a Hopf algebra. $\mathfrak{M}_q(2)$ has the following properties:

1. $\mathfrak{M}_q(2)$ is closed under matrix multiplication.
2. The element

$$\det_q T = ad - q^{-\frac{1}{2}}bc \quad (= da - q^{\frac{1}{2}}bc) \quad (8)$$

belongs to the center of $\mathfrak{M}_q(2)$. The quotient of $\mathfrak{M}_q(2)$ by the relation $\det_q(T) = 1$ can be regarded as a definition of the quantum group $SL_q(2)$, which is also a Hopf algebra. $SU_q(2)$ may be defined by adding a $*$ operation to the Hopf algebra.

3. The inverse matrix is given by

$$T^{-1} = (\det_q(T))^{-1} \begin{pmatrix} d & -q^{\frac{1}{2}}b \\ -q^{-\frac{1}{2}}c & a \end{pmatrix},$$

which satisfies $TT^{-1} = T^{-1}T = I$. Note that $T^{-1} \in \mathfrak{M}_{q^{-1}}(2)$, and so the set $\mathfrak{M}_q(2)$ with matrix multiplication does not form a group.

4. As indicated by Manin [8], the elements of $\mathfrak{M}_q(2)$ have a natural action of matrix multiplication on the coordinate vectors $\begin{pmatrix} x \\ y \end{pmatrix}$, which comprise the quantum plane. It is postulated that x and y each commute with a, b, c, d and, moreover, that

$$xy = q^{-\frac{1}{2}}yx. \quad (9)$$

This relation is preserved under the matrix action, that is, the elements $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ also satisfy (9), provided a, b, c, d satisfy (6,7).

[†]We have replaced q in [3] by $q^{\frac{1}{2}}$ and q^{-1} in [8] by $q^{-\frac{1}{2}}$ in accordance with our conventions.

The q -deformation $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$ of the universal enveloping algebra of $\mathfrak{sl}(2, \mathbb{C})$ is now defined, following Faddeev et al. [3], to be the dual $\text{Hom}(A, \mathbb{C})$ to the algebra $A = \mathfrak{M}_q(2)$ and co-multiplication on $\mathfrak{M}_q(2)$, which is the matrix multiplication of independent elements of $\mathfrak{M}_q(2)$, corresponds to co-multiplication on $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$ as given by Δ . In our approach $\mathfrak{M}_q(2)$ will appear as the algebra satisfied by q -spinor components with respect to $\mathcal{U}_q(\mathfrak{su}(2) * \mathfrak{su}(2))$ and properties of the representation matrices ${}^q\mathcal{D}$ will then follow.

4 Complementary Algebras

As indicated in the Introduction, we will derive the relations (6,7) using only properties of $\mathcal{U}_q(\mathfrak{su}(2))$ and its realization, but we need for our development a larger algebra than $\mathcal{U}_q(\mathfrak{su}(2))$. Consider first the $q = 1$ case and let us recall [6] that there is an elegant construction of all $U(2)$ irreducible representations in which four complex variables $\{z_j^i, i, j = 1, 2\}$ are introduced, assembled into a matrix $Z = (z_j^i)$, and from which one may construct *two* commuting $\mathfrak{u}(2)$ algebras, each of which is a subalgebra of $\mathfrak{u}(4)$ and acts in the space \mathfrak{P}^4 of homogeneous polynomials of degree $2k$ in four variables. We denote a factor group of this direct product group by $U(2) * U(2)$ in order to signify that the Casimir invariants of each algebra coincide. (Hence $\mathfrak{u}(2) * \mathfrak{u}(2)$ is a factor algebra, that is, the quotient of the direct product by the constraints which impose common invariants. The two algebras in the direct product are sometimes termed complementary).

We may summarize the main properties of this construction as follows (see [6, 9] for a detailed description):

1. Define a linear operator $\Theta_{(g, g')}$ on \mathfrak{P}^4 by

$$\Theta_{(g, g')}f(Z) = f(\tilde{g}Zg'), \quad (g, g') \in U(2) * U(2), \quad f \in \mathfrak{P}^4, \quad (10)$$

(where \tilde{g} is the transpose of g) then $(g, g') \rightarrow \Theta_{(g, g')}$ is a unitary representation of $U(2) * U(2)$.

2. \mathfrak{P}^4 carries symmetric representations of $U(4)$ but is reducible under the $U(2) * U(2)$ subgroup; the subspaces of \mathfrak{P}^4 which are invariant under the operators $\Theta_{(g, g')}$ are labelled by

integers $2j \leq 2k$.

3. The eigenvectors are denoted $|k, j; m, m'\rangle$, where j is the common invariant of the two commuting $SU(2)$ groups, with magnetic quantum numbers m and m' , and k is the eigenvalue of the commuting $U(1)$ generator; these eigenvectors are polynomials in z_j^i . Suitably normalized, they are precisely the representation matrix elements $\mathfrak{D}_{m,m'}^j(Z)$ expressed, not as functions of the group element, but as functions of Z , the coordinates on which $U(2) * U(2)$ acts.
4. The fundamental representation matrix $\mathfrak{D}^{\frac{1}{2}}(Z)$ is equal to Z itself. The rows of Z transform as spinor operators under right transformations by $g' \in U(2)$ (as implied in (10)) and the columns of Z transform as spinor operators by the left transformations of $g \in U(2)$.

Hence, we may calculate the representation matrices using only the properties of the Lie algebra, by determining the state of highest weight in \mathfrak{P}^4 with respect to both $\mathfrak{u}(2)$ algebras, and then using the lowering generators to calculate the full set of basis vectors. The $U(2)$ representation matrix is related to the $SU(2)$ matrix $\mathfrak{D}^j(Z)$, and the basis elements take the form

$$|k, j; m, m'\rangle = \left[\frac{(2j+1)}{(k-j)!(k+j+1)!} \right]^{\frac{1}{2}} (\det Z)^{k-j} \mathfrak{D}_{m,m'}^j(Z). \quad (11)$$

The explicit expression for $\mathfrak{D}^j(Z)$ is given below in (21) in the limit $q \rightarrow 1$. Hence, using only the properties of the Lie algebra and its realization we have determined, without integration, all representation matrices as functions of z_j^i , from which we may deduce the coordinate space realization of all representation matrices.

We now generalize the above construction to the quantum case by realizing $\mathcal{U}_q(\mathfrak{u}(2) * \mathfrak{u}(2))$ as follows, where the generators act in \mathfrak{P}^4 :

$$\begin{aligned} J_+ &= q^{\frac{1}{4}(N_1^2 - N_2^2)} z_1^1 D_2^1 + q^{\frac{1}{4}(N_2^2 - N_1^2)} z_1^2 D_2^2, & K_+ &= q^{\frac{1}{4}(N_2^1 - N_2^2)} z_1^1 D_1^2 + q^{\frac{1}{4}(N_1^2 - N_1^1)} z_2^1 D_2^2, \\ J_- &= q^{\frac{1}{4}(N_1^2 - N_2^2)} z_2^1 D_1^1 + q^{\frac{1}{4}(N_2^2 - N_1^1)} z_2^2 D_1^2, & K_- &= q^{\frac{1}{4}(N_2^1 - N_2^2)} z_1^2 D_1^1 + q^{\frac{1}{4}(N_2^2 - N_1^1)} z_2^2 D_2^1, \\ J_z &= \frac{1}{2}(N_1^1 - N_2^1 + N_1^2 - N_2^2), & K_z &= \frac{1}{2}(N_1^1 + N_2^1 - N_1^2 - N_2^2), \\ H &= \frac{1}{2}(N_1^1 + N_2^1 + N_1^2 + N_2^2). \end{aligned} \quad (12)$$

Here the finite difference operators D_j^i are defined by (1), the number operators are $N_j^i = z_j^i \partial / \partial z_j^i$, and we have constructed these generators from (4) using co-multiplication. The two sets $\{J_\pm, J_z\}$ and $\{K_\pm, K_z\}$ each generate $\mathcal{U}_q(\mathfrak{su}(2))$, which enlarges to $\mathcal{U}_q(\mathfrak{u}(2))$ upon adjoining H to each set. We summarize the properties of this realization in the following lemma:

Lemma 1 *The generators, $\{J_\pm, J_z\}$ and $\{K_\pm, K_z\}$, of the two $\mathcal{U}_q(\mathfrak{su}(2))$ algebras commute, and H , which generates $U(1)$, commutes with each of the $\mathcal{U}_q(\mathfrak{su}(2))$ algebras. The Casimir invariants of the commuting algebras coincide.*

We may now calculate the orthonormal basis vectors which have a form analogous to (11), and from which we may determine the q -analog representation matrices ${}^q\mathcal{D}$, but only as polynomials in the commuting variables z_j^i . In contrast to the case $q = 1$, however, the elements z_j^i do not form q -spinors under $\mathcal{U}_q(\mathfrak{su}(2))$, and it is on q -spinors that $SU_q(2)$ will act. Indeed, we have already seen that the natural action of elements of $\mathfrak{M}_q(2)$, which we wish to construct, is on non-commuting coordinates x, y satisfying (9), and it is these coordinates which will transform as spinors under the quantum group. Hence, we first discuss spinor operators in $\mathcal{U}_q(\mathfrak{su}(2))$, and then show that elements of $\mathfrak{M}_q(2)$ have a natural action on these q -spinors.

5 q -Tensor Operators

Let us denote by \mathfrak{M} the model space defined to be the direct sum of vector spaces carrying unitary irreducible representations of $\mathcal{U}_q(\mathfrak{su}(2))$, that is, $\mathfrak{M} = \sum_j \oplus V_j$, where V_j is the $2j + 1$ dimensional space carrying the irreducible representation labelled j . As discussed in [10], linear operators $\mathbf{T} : \mathfrak{M} \rightarrow \mathfrak{M}$ can be classified by their transformation properties under the quantum group, provided these operators are equivariant. For $\mathcal{U}_q(\mathfrak{su}(2))$, an irreducible tensor operator is a set of linear operators $\mathbf{T}_{jm} : \mathfrak{M} \rightarrow \mathfrak{M}$ which carries the irreducible representation j , with components labelled by m , and which satisfies equivariance. The action of \mathbf{T} in \mathfrak{M} must be compatible with the coproduct, and this leads to the concept of an induced action, discussed in [10]. We may summarize the relevant properties of an irreducible tensor operator in $\mathcal{U}_q(\mathfrak{su}(2))$ as follows:

1. A q -tensor operator of spin j , with components denoted \mathbf{T}_{jm} , satisfies:

$$\begin{aligned} (J_{\pm} \mathbf{T}_{jm} - q^{-\frac{m}{2}} \mathbf{T}_{jm} J_{\pm}) q^{-J_z/2} &= \sqrt{[j \pm m + 1]_q [j \mp m]_q} \mathbf{T}_{j, m \pm 1}, \\ [J_z, \mathbf{T}_{jm}] &= m \mathbf{T}_{jm}. \end{aligned} \quad (13)$$

2. Given any two tensor operators \mathbf{T}_{j_1, m_1} and \mathbf{T}'_{j_2, m_2} we can form a third tensor operator $\mathbf{T}''_{j, m}$ with a q -WCG (Wigner-Clebsch-Gordan) coupling:

$$\mathbf{T}''_{j, m} = \sum_{m_1, m_2} {}_q C_{m_1 m_2 m}^{j_1 j_2 j} \mathbf{T}_{j_1, m_1} \mathbf{T}'_{j_2, m_2},$$

where the q -WCG coefficients ${}_q C_{m_1 m_2 m}^{j_1 j_2 j}$ are known explicitly (see for example [11] and [12]).

3. We can form an invariant $\mathbf{T} \cdot \mathbf{T}'$ with the coupling

$$\mathbf{T} \cdot \mathbf{T}' = \sum_{m_1 + m_2 = 0} {}_q C_{m_1 m_2 0}^{j_1 j_2 0} \mathbf{T}_{j_1, m_1} \mathbf{T}'_{j_2, m_2},$$

which, with the help of the formula

$${}_q C_{m_1 m_2 0}^{j_1 j_2 0} = \delta_{j_1, j_2} \delta_{m_1, -m_2} \frac{(-1)^{j_1 - m_1} q^{\frac{m_1}{2}}}{\sqrt{[2j_1 + 1]}}$$

can be written, upon discarding a normalization, as

$$\mathbf{T} \cdot \mathbf{T}' = \sum_m (-1)^{j-m} q^{\frac{m}{2}} \mathbf{T}_{j, m} \mathbf{T}'_{j, -m}.$$

It may be verified directly from (13) that $\mathbf{T} \cdot \mathbf{T}'$ commutes with the generators of $\mathcal{U}_q(\mathfrak{su}(2))$.

For spinor operators we have $j = \frac{1}{2}$ and there are two components, corresponding to $m = \pm \frac{1}{2}$. As an example, the pair

$$\left(z_1 q^{-\frac{N_2}{4}}, z_2 q^{\frac{N_1}{4}} \right) \quad (14)$$

forms a spinor with respect to the generators realized as in (4); however, by contrast, (z_1, z_2) is not a spinor. Given any two spinors \mathbf{T}, \mathbf{T}' , we can form the invariant

$$\mathbf{T} \cdot \mathbf{T}' = q^{\frac{1}{4}} \mathbf{T}_{\frac{1}{2}, \frac{1}{2}} \mathbf{T}'_{\frac{1}{2}, -\frac{1}{2}} - q^{-\frac{1}{4}} \mathbf{T}_{\frac{1}{2}, -\frac{1}{2}} \mathbf{T}'_{\frac{1}{2}, \frac{1}{2}}. \quad (15)$$

We confine our discussion now to the realization (12), for which tensor operators act in \mathfrak{P}^4 , which realizes \mathfrak{M} , and we show that the matrix of operators \mathbf{t} , the rows and columns of which are q -spinors with respect to $\mathcal{U}_q(\mathfrak{su}(2) * \mathfrak{su}(2))$, must satisfy the relations (6,7) which define $\mathfrak{M}_q(2)$; furthermore, we will see that the fundamental operator-valued matrix ${}^q\mathfrak{D}^{\frac{1}{2}}$, expressed as a function of the q -spinors, is \mathbf{t} itself and so is an element of $\mathfrak{M}_q(2)$. (This shows, incidentally, that ${}^q\mathfrak{D}^{\frac{1}{2}}(\mathbf{t})$ cannot be identified with the matrix Z , because the rows and columns of Z do not form spinors with respect to $\mathcal{U}_q(\mathfrak{su}(2) * \mathfrak{su}(2))$.)

Let us denote the elements of the fundamental matrix ${}^q\mathfrak{D}^{\frac{1}{2}}$ by t_i^j ; hence we seek to identify elements of $\mathfrak{M}_q(2)$ with the matrix of q -spinor operators according to

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = {}^q\mathfrak{D}^{\frac{1}{2}}(\mathbf{t}) \stackrel{\text{def}}{=} \begin{pmatrix} t_1^1 & t_1^2 \\ t_2^1 & t_2^2 \end{pmatrix} = \mathbf{t}. \quad (16)$$

In the next section we prove

Lemma 2 *Each of the pairs (t_1^1, t_2^1) and (t_1^2, t_2^2) forms a q -spinor under $\mathcal{U}_q(\mathfrak{su}(2))$ generated by J_{\pm}, J_z , and each of the pairs (t_1^1, t_1^2) and (t_2^1, t_2^2) forms a q -spinor under $\mathcal{U}_q(\mathfrak{su}(2))$ generated by K_{\pm}, K_z , and the elements t_i^j satisfy the relations (6,7).*

We will also need an extension of the equality of irreducible representation labels, as expressed in Lemma 1, to a certain class of q -tensor operators. This extension is:

Lemma 3 *The q -tensor operators ${}^q\mathfrak{D}^{\frac{1}{2}}(\mathbf{t})$, which act in \mathfrak{P}^4 , transform as fundamental q -tensor operators with respect to each $\mathcal{U}_q(\mathfrak{su}(2))$ with identical operator pattern labels.*

PROOF: The elements of \mathfrak{P}^4 may equivalently be written as the operator elements of the \mathfrak{D} matrix acting to the right on the constant polynomial 1. Thus the action of ${}^q\mathfrak{D}^{\frac{1}{2}}$ on \mathfrak{P}^4 is equivalent to the product of ${}^q\mathfrak{D}$ -operators acting on the constant 1. This product of \mathfrak{D} -operators is exactly the q -analog of the Wigner definition of the q -WCG operators. By the extension of Wigner's theorem [10] the operators have the operator pattern labels restriction as stated.

6 Derivation of the Commutation Rules for the Coordinates of $\mathfrak{M}_q(2)$

We now outline the proof of Lemma 2 by two independent means, firstly by using abstract arguments based on coupling properties of q -spinor operators in \mathfrak{P}^4 and secondly, by determining an explicit realization of the fundamental q -spinors as operators in the variables z_j^i .

Consider any of the four spinors comprising the matrix \mathbf{t} , namely the two rows and the two columns of \mathbf{t} . From these four spinors we may form six invariants, using (15). For example, we may couple t_i^1 and t_j^1 to give a scalar with respect to the left algebra, but a tensor operator of rank 1 with respect to the right algebra $\mathcal{U}_q(\mathfrak{su}(2))$ generated by $\{\mathbf{K}\}$. Hence, using the q -WCG coefficients, we find

$$q^{\frac{1}{4}}t_1^1t_2^1 - q^{-\frac{1}{4}}t_2^1t_1^1 \quad (17)$$

is a tensor operator in $\mathcal{U}_q(\mathfrak{su}(2)*\mathfrak{su}(2))$ with (by construction), the irreducible representation label $j = 0$ in the left algebra $\mathcal{U}_q(\mathfrak{su}(2))$. However, in the right $\mathcal{U}_q(\mathfrak{su}(2))$ the tensor product can lead only to the irreducible representation $j = 1$, since, if we allow these tensor operators to act in \mathfrak{P}^4 , we obtain maximal states which “add” under the q -WCG tensor product. This result contradicts Lemma 3, *unless* the tensor product (17) is identically zero. If we now identify $a = t_1^1$ and $c = t_2^1$, as indicated in (16), then we have proved that the commutation relation $ac = q^{-\frac{1}{2}}ca$ in (6) is satisfied.

The same coupling scheme is valid for the remaining relations in (6) involving all four components of \mathbf{t} , and hence we have established these relations as well.

Next, let us consider the equations which relate the two forms of the determinant (8). We can view these relations as originating from the construction of a quadratic invariant operator in the q -tensor algebra of \mathbf{t} . There are two invariants—which differ at most by a constant—due to the two possibilities of coupling order. Tensor coupling in $\mathcal{U}_q(\mathfrak{su}(2)*\mathfrak{su}(2))$ involves q -WCG coefficients in both left and right $\mathcal{U}_q(\mathfrak{su}(2))$ algebras; due to the constraint on the two representation labels it suffices to couple in either space but, if the omitted coupling is not possible as in the cases above, the resulting q -tensor operator vanishes.

Let us carry out the construction of the two quadratic invariants by firstly coupling t_i^1 and t_j^2 to give an invariant I_1 , using the product of the two $\mathcal{U}_q(\mathfrak{su}(2))$ q -WCG coefficients, to obtain

$$I_1 = t_1^1 t_2^2 - q^{-\frac{1}{2}} t_2^1 t_1^2,$$

which we may identify with $ad - q^{-\frac{1}{2}}bc$. Secondly, we couple t_j^2 and t_i^1 to give an invariant I_2 :

$$I_2 = t_2^2 t_1^1 - q^{\frac{1}{2}} t_1^2 t_2^1,$$

which we identify with $da - q^{\frac{1}{2}}cb$.

In this coupling process we could equally well have coupled on the upper labels rather than the lower labels to form I_1 and I_2 . The q -WCG coefficients are unchanged, but the orders of the last two terms in each invariant is reversed. Thus we obtain

$$\begin{aligned} I_1 &= t_1^1 t_2^2 - q^{-\frac{1}{2}} t_1^2 t_2^1 \\ I_2 &= t_2^2 t_1^1 - q^{\frac{1}{2}} t_2^1 t_1^2. \end{aligned} \tag{18}$$

These results show that we must have $t_2^1 t_1^2 - t_1^2 t_2^1 = 0$ or $bc = cb$, another of the relations (7). Furthermore, we may also assert that the two invariants I_1, I_2 are proportional and, after evaluating the constant of proportionality by letting the invariants act on vectors of \mathfrak{P}^4 , we are led to the final relation in (7) or (8). Accordingly we have not only verified Lemma 2 directly from the q -tensor calculus of $\mathcal{U}_q(\mathfrak{su}(2) * \mathfrak{su}(2))$, but also demonstrated directly from the Hopf algebra axiom the commutation relations of (6) and (7).

The algebraic relations (6,7) suffice to prove that

$${}^q\mathcal{D}^{\frac{1}{2}}(\mathbf{t}) {}^q\mathcal{D}^{\frac{1}{2}}(\mathbf{t}') = {}^q\mathcal{D}^{\frac{1}{2}}(\mathbf{t}\mathbf{t}')$$

where $\mathbf{t}, \mathbf{t}' \in \mathfrak{M}_q(2)$ and where the elements of \mathbf{t} are assumed to commute with those of \mathbf{t}' . This result extends, by construction, to general irreducible representations and, moreover, the q -Wigner product law (which effects the reduction of the Kronecker product) is also valid. Thus the two fundamental product laws for the operator-valued representation matrices have been verified.

Next, we verify *independently* that q -spinors with respect to $\mathcal{U}_q(\mathfrak{su}(2)*\mathfrak{su}(2))$ satisfy the relations (6,7) by finding an explicit realization, which is not unique, of \mathfrak{t} in terms of the elements z_j^i and their adjoints D_j^i , which generalizes the $q = 1$ case for which \mathfrak{t} may be identified with the matrix Z . (Realizations of $\mathfrak{M}_q(2)$ have been obtained previously [14, 15], but differ from that described here). As previously indicated, we cannot identify \mathfrak{t} with Z for general q because the variables z_j^i do not form the components of a spinor under $\mathcal{U}_q(\mathfrak{su}(2)*\mathfrak{su}(2))$ and, moreover, commuting variables do not satisfy (6). There is apparently no simple way in which we can construct \mathfrak{t} from spinors such as $(z_1^1 q^{-N_2^1/4}, z_2^1 q^{N_1^1/4})$ considered in (14). One approach would be to construct \mathfrak{t} from its matrix elements using the q -WCG coefficients to determine invariant factors [6, see Chapter 5].

However, we will proceed in the following way: firstly, we note that it is sufficient to determine t_1^1 explicitly in terms of z_j^i , considered as operators, and their adjoints D_j^i since, given this element, the remaining elements t_2^1, t_1^2, t_2^2 may be determined using the commutation relations (13) satisfied by a spinor (putting $j = \frac{1}{2}, m = \pm\frac{1}{2}$). That is, t_2^1 can be determined from t_1^1 using the action of J_- , and t_1^2 can be determined using K_- . Then t_2^2 can be determined in either of two ways, by letting J_- act on t_1^2 , or by letting K_- act on t_2^1 . In order to determine t_1^1 we note, secondly, that $a = t_1^1$ must commute with $\det_q(T) = ad - q^{-\frac{1}{2}}bc$ which, up to an overall scalar multiple, we can determine directly. For $q = 1$, $\det(Z)$ is given by $\det(Z) = z_{12} = z_1^1 z_2^2 - z_2^1 z_1^2$ and is an invariant of $\mathfrak{u}(2) * \mathfrak{u}(2)$. The q -analog of $\det(Z)$, which we denote z_{12}^q , is given [13] by

$$z_{12}^q = q^{\frac{1}{4}(N_2^1 + N_1^2 + 1)} z_1^1 z_2^2 - q^{-\frac{1}{4}(N_1^1 + N_2^2 + 1)} z_2^1 z_1^2. \quad (19)$$

Lemma 4 z_{12}^q is an invariant of $\mathcal{U}_q(\mathfrak{u}(2)*\mathfrak{u}(2))$.

The proof is by direct calculation. Hence, z_{12}^q is proportional to $\det_q(T)$ and so, as noted above, will commute with all the entries of $\mathfrak{t} \in \mathfrak{M}_q(2)$, in particular with t_1^1 . Now we may construct t_1^1 from the operators which commute with z_{12}^q , namely $z_1 q^{-N_2^1/4}, q^{N_2^1 - N_1^2}$ and $q^{N_1^1 - N_2^2}$. We assume therefore the following form for t_1^1 :

$$t_1^1 = z_1 q^{-\frac{1}{4}N_2^1 + \alpha(N_2^1 - N_1^2) + \beta(N_1^1 - N_2^2)}$$

where α and β are constants to be determined. Because t_1^1 is the first component of a spinor with respect to each $\mathcal{U}_q(\mathfrak{su}(2))$ algebra, it must satisfy, according to (13), $J_+ t_1^1 = q^{-1/4} t_1^1 J_+$ and $K_+ t_1^1 = q^{-1/4} t_1^1 K_+$. These equations serve to determine $\alpha = \beta = \frac{1}{8}$. Therefore we obtain the required explicit representation for t_1^1 and hence (after a long calculation) we obtain also all the elements t_j^i comprising \mathfrak{t} . Of particular note is the form for t_2^2 , which involves D_1^1 and which is necessarily present in order to realize the full algebra given in (7). We may summarize the results as follows:

Lemma 5 *Let*

$$\begin{aligned}
t_1^1 &= z_1^1 q^{\frac{1}{8}(N_1^1 - N_2^1 - N_1^2 - N_2^2)} \\
t_1^2 &= z_1^2 q^{\frac{1}{8}(3N_1^1 - N_2^1 + N_1^2 - N_2^2)} \\
t_2^1 &= z_2^1 q^{\frac{1}{8}(3N_1^1 + N_2^1 - N_1^2 - N_2^2)} \\
t_2^2 &= z_2^2 q^{\frac{1}{8}(-N_1^1 + 3N_2^1 + 3N_1^2 + N_2^2)} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) z_2^1 z_1^2 D_1^1 q^{\frac{1}{8}(N_1^1 + N_2^1 + N_1^2 - N_2^2 - 2)}. \tag{20}
\end{aligned}$$

Then

1. for $q = 1$ the matrix \mathfrak{t} reduces to the matrix Z .
2. the columns of \mathfrak{t} each form a spinor with respect to the left $\mathcal{U}_q(\mathfrak{su}(2))$ algebra generated by $\{\mathbf{J}\}$, and the rows each form a spinor with respect to the right $\mathcal{U}_q(\mathfrak{su}(2))$ algebra generated by $\{\mathbf{K}\}$.
3. \mathfrak{t} is an element of $\mathfrak{M}_q(2)$, that is, if we write $\mathfrak{t} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the entries a, b, c, d satisfy the algebraic relations (6,7).
4. the q -determinant of \mathfrak{t} , defined in (8), is given by $\det_q(\mathfrak{t}) = q^{-\frac{3}{8}} z_{12}^q$ where z_{12}^q is given by (19), and commutes with each element t_j^i of \mathfrak{t} .

The proof of these results is again by direct calculation.

We may now define representations of $SU_q(2)$ analogously to (10), in which the operator Θ acts on the space $\hat{\mathfrak{P}}^4$ of homogeneous polynomials in the noncommuting operators t_j^i by

left and right transformations (noting that $\tilde{T} \in \mathfrak{M}_q(2)$ if $T \in \mathfrak{M}_q(2)$). Hence we may define the representation matrices ${}^q\mathfrak{D}$ as elements of $\hat{\mathfrak{P}}^4$, that is, as polynomials in the four spinor components t_j^i . The basis vectors in \mathfrak{P}^4 of $\mathcal{U}_q(\mathfrak{su}(2)*\mathfrak{su}(2))$ may be obtained from these matrices, when \mathfrak{t} is realized as in (20), by allowing ${}^q\mathfrak{D}$ to act on the constant polynomial 1. These matrices are given explicitly as elements of $\hat{\mathfrak{P}}^4$ by:

$${}^q\mathfrak{D}_{m,m'}^j(\mathfrak{t}) = \left([j+m]_q! [j-m]_q! [j+m']_q! [j-m']_q! \right)^{\frac{1}{2}} \\ \times \sum_{\boxed{\alpha}} q^{\frac{1}{4}(\alpha_1+\alpha_2)(\alpha_2+\alpha_1^2)} q^{\frac{1}{2}(\alpha_2\alpha_1^2)} \frac{(t_1^1)^{\alpha_1} (t_2^1)^{\alpha_2} (t_1^2)^{\alpha_1^2} (t_2^2)^{\alpha_2^2}}{[\alpha_1^1]_q! [\alpha_2^1]_q! [\alpha_1^2]_q! [\alpha_2^2]_q!}, \quad (21)$$

where the sum is over the square array $\boxed{\alpha}$ of nonnegative integers (α_j^i) satisfying

$$\alpha_1^1 + \alpha_1^2 = j + m', \quad \alpha_2^1 + \alpha_2^2 = j - m', \\ \alpha_1^1 + \alpha_2^1 = j + m, \quad \alpha_1^2 + \alpha_2^2 = j - m.$$

For $q \rightarrow 1$, (21) reduces to the operator-valued representation matrix first given in [6] in terms of boson operators. Forms similar to (21) have been given by other authors [16, 17] in different contexts, but without a tensor operator interpretation.

7 Generalization to the Quantum Group $U_q(n)$ and the Quantum Hyperplane

The construction given in §4-6 for $\mathcal{U}_q(\mathfrak{u}(2))$ generalizes directly to $\mathcal{U}_q(\mathfrak{u}(n))$ including the factor algebra $\mathcal{U}_q(\mathfrak{u}(n)*\mathfrak{u}(n))$ and, in particular, there exist generalizations of the algebraic relations (12). Lemmas 1, 2 and 3 generalize as well. $\mathfrak{M}_q(n)$ has been defined by Faddeev et al. [18] and comprises the set of $n \times n$ matrices with elements t_j^i satisfying

$$t_j^i t_l^i = q^{-\frac{1}{2}} t_l^i t_j^i \quad (l > j), \quad t_j^i t_j^k = q^{-\frac{1}{2}} t_j^k t_j^i \quad (k > i), \quad (22)$$

$$[t_j^k, t_l^i] = 0 \quad (k > i, l > j), \quad [t_j^i, t_l^k] = -(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) t_l^i t_j^k \quad (k > i, l > j) \quad (23)$$

and may be endowed with a Hopf algebra structure, which in turn defines $GL_q(n)$. Other properties of $n = 2$ also generalize, such as the existence of a central element, the q -determinant.

Matrices in $\mathfrak{M}_q(n)$ act on vectors with n non-commuting components which comprise the quantum hyperplane.

Now let us consider properties of q -spinors with respect to $\mathcal{U}_q(\mathfrak{u}(n) * \mathfrak{u}(n))$ and show that the components t_j^i satisfy the commutation relations of elements of $\mathfrak{M}_q(n)$, following exactly the methods used in §6, as we now sketch abstractly. Consider two q -tensor components t_j^i and t_l^i , $j < l$, which are chosen to lie in the same column (here i). The product of these two elements necessarily belongs to the carrier space \mathfrak{B} of representations of $\mathcal{U}_q(\mathfrak{u}(n) * \mathfrak{u}(n))$ with Gel'fand-Weyl labels $[20 \dots 0] \times [20 \dots 0]$ — since this is enforced by the condition that the two indices i are the same — and to a unique vector in this space. The product of these two elements *in opposite order* leads to exactly this same unique vector but with a different numerical coefficient. Using tables of the fundamental q -WCG coefficients [19] one finds the following explicit result for the commutation relation: $t_j^i t_l^i = q^{-\frac{1}{2}} t_l^i t_j^i$, $j < l$, that is, the first set of relations in (22). Similarly, by exchanging columns for rows exactly the same abstract relations hold, and we derive (again using q -WCG tables of [19]) the second set of relations in (22).

Consider also the (generic) products $t_j^k t_l^i$, ($i < k, j < l$), and $t_j^i t_l^k$, ($i < k, j < l$), and each taken in the two possible orders. The resulting *four* products, when acting in \mathfrak{B} , are each numerical multiples of two abstract vectors in \mathfrak{B} , one vector in $[20 \dots 0] \times [20 \dots 0]$, the second vector in $[110 \dots 0] \times [110 \dots 0]$. Thus there exist *two* linear relations between the *four* products, as determined by the q -WCG coefficients. These relations are:

$$t_l^i t_j^k = t_j^k t_l^i$$

and

$$t_l^k t_j^i - t_j^i t_l^k = \left(q^{-\frac{1}{2}} - q^{\frac{1}{2}} \right) t_l^i t_j^k,$$

which are the relations (23). This abstract argument shows that the basic results of this paper are *generic* and demonstrate the importance of the q -tensor operator concept in understanding the quantum hyperplane, and hence also the relationship of $U_q(n)$ to $\mathcal{U}_q(\mathfrak{u}(n))$.

As a concluding remark, we note that our construction of explicit q -spinor operators in §6 leads naturally to a possible physical interpretation of non-commuting coordinates [20]. Ex-

pectation values of physical operators, such as Hamiltonians, which can be expressed in terms of q -tensors, can be evaluated using q -coherent states to give real values (which depend on q), even when the underlying coordinates are noncommuting. The basis of this interpretation is the operator realization of non-commuting coordinates using q -tensor operators.

8 Acknowledgement

This work was supported in part by grant DE-FG03-93ER40757.

References

- [1] Drinfeld, V. G., *Proc. Intern. Congress of Mathematicians*, Berkeley, Vol. 1, 798 (1986).
- [2] Jimbo, M., *Lett. Math. Phys.* **11**, 247 (1986).
- [3] Faddeev, L. D., Reshetikhin, N. Yu., and Takhtajan, L. A., “Quantization of Lie Groups and Lie Algebras”, in *Algebraic Analysis*, eds. M. Kashiwara and T. Kawai, Academic, Boston (1989) 129-139 (reprinted in “Yang-Baxter Equation in Integrable Systems”, ed. M. Jimbo, World Scientific (1990); Faddeev, L. D., Takhtajan, L. A., *Lecture Notes in Physics* **246**, 166 (1986).
- [4] Biedenharn, L. C., *J. Phys. A: Math. Gen.* **22**, L873 (1989).
- [5] Macfarlane, A. J., *J. Phys. A: Math. Gen.* **22**, 4581 (1989).
- [6] Biedenharn, L. C., and Louck, J. D., “Angular Momentum in Quantum Physics”, *Encyclopedia of Mathematics and Its Applications*, Vol. 8, Reading, MA: Addison-Wesley, (1981).
- [7] Alvarez-Gaumé, L., Gomez, C., and Sierra, G., *Nucl. Phys.* **B330**, (1990) 347-398.
- [8] Manin, Yu. I., “Quantum Groups and Non-Commutative Geometry”, Centre de Recherches Mathématiques, Université de Montréal (1988).
- [9] Louck, J. D., *Am. J. Phys.* **38**, 3 (1970).

- [10] Biedenharn, L. C., and Tarlini, M., *Lett. Math. Phys.* **20**, 271 (1990).
- [11] Kirillov, A. N. and Reshitikhin, N. Yu., “Representations of the Algebra $U_q(\mathfrak{sl}(2))$, q -Orthogonal Polynomials and Invariants of Links”, USSR Academy of Sciences (preprint) (1988).
- [12] Biedenharn, L. C., and Lohe, M. A., in *Differential Geometry, Group Representations, and Quantization*, ed. J. D. Hennig, W. Lücke and J. Tolar, *Lecture Notes in Physics* **379**, Springer-Verlag, 193 (1991).
- [13] Biedenharn, L. C. and Lohe, M. A., *Comm. Math. Phys.* **146**, 483 (1992).
- [14] Mo-Lin Ge, Xu-Feng Liu, and Chang-Pu Sun, *J. Math. Phys.* **33**, 2541 (1992).
- [15] Damaskinsky, E. V., and Sokolov, M. A., “Some remarks on the Gauss decomposition for quantum group $GL_q(n)$ with application to q -bosonization”, preprint hep-th/9407024 (1994).
- [16] Masuda, T., Mimachi, K., Nakagami, Y., Noumi, M., and Ueno, K., *C. R. Acad. Sci. Paris*, **307**, Série I 559 (1988).
- [17] Nomura, M., *J. Phys. Soc. Jap.* **59**, 4260 (1990).
- [18] Faddeev, L. D., Reshetikhin, N. Yu., and Takhtajan, L. A., *Leningrad J. Math.* **1**, 193 (1990).
- [19] Gould, M., and Biedenharn, L. C., *J. Math. Phys.* **33**, 3613 (1992).
- [20] Biedenharn, L. C. and Lohe, M. A., “Quantum Groups as a Model of Quantized Space,” Clausthal Proceedings (1994) (to be published).