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ANALYSIS OF A GENERAL AGE-DEPENDENT VACCINATION MODEL FOR A VERTICALLY TRANSMITTED DISEASE

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ABSTRACT

An SIR epidemic model of a general age-dependent vaccination for a vertically as well as horizontally transmitted disease is investigated when the total population is time dependent, and fertility, mortality and removal rates depend on age. We establish the existence and the uniqueness of the solution and obtain the asymptotic behaviour for the solution. For the steady state solution a critical vaccination coverage which will eventually eradicate the disease is determined.

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1. Introduction

Since infectious communicable diseases such as Malaria, AIDS and Cholera are the main causes of human death, modern scientific works are now to provide ways and means of controlling epidemics originating from these diseases.

A first step towards control of communicable diseases is the availability of vaccines that provide lasting protection. But then, importnat epidemiological questions remain to be answered, among which is the rate of vaccination that is sufficient to eradicate these diseases.

In this paper we describe and analyze a model for diseases that are vertically as well as horizontally transmitted. Vertical transmission is the passing of infection from a parent to a newborn via birth. Horizontal transmission is any transfer of infection that is not vertical i.e. via random contacts between individuals whether these contacts are direct or indirect through an intermediate host. For example AIDS is vertically as well as horizontally transmitted while Malaria is horizontally transmitted. And predict means of controlling such diseases.

Vertically transmitted diseases have seldom been considered in mathematical models of epidemics. Examples of such models are found in Anderson and May [1], Cooke and Busenberg [11], Busenberg and Cooke [5,6,7], Busenberg [9], Busenberg, Cooke and Pozio [8], Busenberg and Hadeler [10], Fine [14], Regniere [18] and El-Doma [12].

Several recent papers have dealt with age-dependent vaccination models. For example Hethecote [15], Katzman and Dietz [17], Anderson and May [2].

Our results in this paper generalize our previous results which are found in [12]. It is clear that vertical transmission present an added mathematical difficulty.

Under certain conditions we are able to show that in a decaying population the proportion of infective individuals at a certain age to the susceptibles at that same age approach zero for large time. On the other hand for a growing population the infective class and the removed class approach infinity while the susceptible class approach zero.

The steady state solution of the problem is determined and a threshold theorem is proved providing an inside look. It is shown that there is a parameter which basically consists of two parts: one part depends on the vertical transmission parameter q and the second part does not depend on q and it is the one that is present in the absence of vertical transmission. The size of this parameter determine the existence of endemic disease, a critical vaccination coverage that will eventually eradicate the disease with *minimum vaccination coverage and the disappearance of the disease.*

The organization of this paper is as follows: in section 2 we describe the model; in section 3 we reduce the model equations to several subsystems; in section 4 we establish the existence and the uniqueness of solution, and for this solution we obtain the asymptotic behaviour.

2. A general Age-Dependent Vaccination Model for a Vertically Transmitted Disease

We consider an age-structured population of variable size exposed to a communicable disease. The disease is both vertically and horizontally transmitted. We assume the following:

1) $s(a,t)$, $i(a,t)$ and $r(a,t)$, respectively, denote the age-density for susceptibles, infectives and immune individuals of age a at time t. Then

$\int_{a_1}^{a_2} s(a,t)da$ = total number of susceptibles at time t of ages between a_1 and a_2,

$\int_{a_1}^{a_2} i(a,t)da$ = total number of infectives at time t of ages between a_1 and a_2,

$\int_{a_1}^{a_2} r(a,t)da$ = total number of immune individuals of ages between a_1 and a_2. And that the total population consists entirely of susceptibles, infectives and immune individuals.

2) The horizontal transmission of the disease occurs according to the following mass action law: $ks(a,t)\int_0^\infty i(a,t)da$ where k is a constant which combines a multitude of environmental, social and epidemiological factors which play a role in transmitting the disease. The term $k\int_0^\infty i(a,t)da$ is often called the "force of infection".

3) The fertility rate $\beta(a)$ is non-negative, continuous, with compact support $[0,T]$, $(T \geq 0)$. And the birth rates are given by:

$$s(0,t) = \int_0^\infty \beta(a)[s(a,t) + (1-q)i(a,t) + r(a,t)]da, \ q \in (0,1)$$

and q is the probability of vertically transmitting the disease.

$$i(0,t) = q\int_0^\infty \beta(a)i(a,t)da$$
$$r(0,t) = 0$$

i.e. all newborns from susceptible and immune parents are susceptibles, but a portion q of newborns from infective parents are infective i.e. they acquire the disease via birth (vertical transmission).

4) The death rate $\mu(a)$ is the same for susceptibles, infectives and immunes, and $\mu(a)$ is non-negative, bounded, continuous and eventually non-decreasing.

5) The cure rate $\gamma(a)$ is a bounded continuous function of a, with compact support.

6) The vaccination rate $\nu(a)$ is a bounded continuous function of a.

7) The initial age-distributions: $s(a,0) = s_0(a)$, $i(a,0) = i_0(a)$, $r(a,0) = r_0(a)$ are assumed to be continuous and integrable functions of a in $[0,\infty)$.

These assumptions lead to the following system of nonlinear-integro-differential equations:

$$\begin{cases} \dfrac{\partial s(a,t)}{\partial a} + \dfrac{\partial s(a,t)}{\partial t} + (\mu(a)+\nu(a))s(a,t) = -ks(a,t)\displaystyle\int_0^\infty i(a,t)da \text{ for } a>0, \ t>0 \\ \dfrac{\partial i(a,t)}{\partial a} + \dfrac{\partial i(a,t)}{\partial t} + (\mu(a)+\gamma(a))i(a,t) = ks(a,t)\displaystyle\int_0^\infty i(a,t)da \text{ for } a>0, t>0 \\ \dfrac{\partial r(a,t)}{\partial a} + \dfrac{\partial r(a,t)}{\partial t} + \mu(a)r(a,t) = \nu(a)s(a,t) + \gamma(a)i(a,t) \text{ for } a>0, \ t>0 \\ s(0,t) = \displaystyle\int_0^\infty \beta(a)[s(a,t)+(1-q)i(a,t)+r(a,t)]da, \ t\geq 0 \\ i(0,t) = q\displaystyle\int_0^\infty \beta(a)i(a,t)da, \ t \geq 0 \\ r(0,t) = 0, \quad t \geq 0 \\ s(a,t) = s_0(a), \ i_0(a,0) = i(a) \ r(a,0) = r_0(a) \\ s(a,t), \ i(a,t), \ r(a,t) \to 0 \text{ as } a \to \infty \end{cases} \tag{2.1}$$

3. Formal Reduction of the Model

In this section, we develop some preliminary formal analysis of problem (2.1). We define $p(a,t)$ and $p_2(a,t)$ by:

$$p(a,t) = s(a,t) + i(a,t) + r(a,t)$$
$$p_2(a,t) = s(a,t) + i(a,t)$$

Then $p(a,t)$ satisfies the following McKendrick-Von Foerster equation:

$$\begin{cases} \dfrac{\partial p(a,t)}{\partial a} + \dfrac{\partial p(a,t)}{\partial t} + \mu(a)p(a,t) = 0, \ a>0, \ t>0 \\ p(0,t) = \displaystyle\int_0^\infty \beta(a)p(a,t)da \overset{\text{def}^n}{=} B(t), \ t\geq 0 \\ p(a,0) = p_0(a), \ a \geq 0, \ p(a,t) \to 0 \text{ as } a \to \infty \end{cases} \tag{3.1}$$

And $p_2(a,t)$ satisfies:

$$\begin{cases} \dfrac{\partial p_2(a,t)}{\partial a} + \dfrac{\partial p_2(a,t)}{\partial t} + (\mu(a)p + \nu(a))p_2(a,t) + (\gamma(a) \\ \qquad - \nu(a))i(a,t) = 0, \ a>0, \ t>0 \\ p_2(0,t) = \displaystyle\int_0^\infty \beta(a)p(a,t)da = B(t), \ t \geq 0 \\ p_2(a,0) = p_{20}(a), \ a \geq 0 \end{cases} \tag{3.2}$$

Here we note that since problem (3.1) is McKendrick-Von Foerster type its asymptotic behaviour is known and hence $B(t)$ is determined. And in turn it could be used in problem (3.2). Note that $r(a,t) = p(a,t) - p_2(a,t)$.

Now, we consider the following problem:

$$\begin{cases} \dfrac{\partial i(a,t)}{\partial a} + \dfrac{\partial i(a,t)}{\partial t} + (\mu(a) + \gamma(a))i(a,t) = k[p_2(a,t) - i(a,t)]\displaystyle\int_0^\infty i(a,t)da \\ i(0,t) = q\displaystyle\int_0^\infty \beta(a)i(a,t)da \\ i(a,0) = i_0(a) \end{cases} \tag{3.3}$$

In order to treat problem (3.3), we introduce the following transformation:

$$\omega(a,t) = i(a,t)/I(t), \quad I(t) = \int_0^\infty i(a,t)da \tag{3.4}$$

Then problem (3.3) can be rewritten in the following form:

$$\begin{cases} \dfrac{\partial \omega(a,t)}{\partial a} + \dfrac{\partial \omega(a,t)}{\partial t} + [\mu(a) + \gamma(a)]\omega(a,t) = kp_2(a,t) + \\ \qquad \left(\displaystyle\int_0^\infty [\mu(a) + \gamma(a) - q\beta(a)]\omega(a,t)da - kP_2(t)\right)\omega(a,t), \ a > 0, \ t > 0 \\ \omega(0,t) = q\displaystyle\int_0^\infty \beta(a)\omega(a,t)da, \ t \geq 0 \\ \displaystyle\int_0^\infty \omega(a,t)da = 1, \ a > 0, \ t \geq 0 \\ \omega(a,0) = \omega_0(a), \ \omega(a,t) \to 0 \text{ as } a \to \infty \ . \end{cases} \tag{3.5}$$

And $I(t)$ satisfies:

$$\begin{cases} \dfrac{dI(t)}{dt} = k[P_2(t) - I(t)]I(t) - \left(\displaystyle\int_0^\infty [\mu(a) + \gamma(a) - q\beta(a)]\omega(a,t)da\right) I(t) \\ I(0) = I_0 \neq 0 \end{cases} \tag{3.6}$$

where $P_2(t) = \int_0^\infty p_2(a,t)da$.

So, if we can determine the asymptotic behaviour of $\omega(a,t)$ and $I(t)$ from (3.5) and (3.6) then

$$i(a,t) = I(t)\omega(a,t)$$

and $s(a,t) = p_2(a,t) - i(a,t)$.

And hence the solution to problem (2.1) is determined.

4. The Asymptotic Behaviour

In this section, we further analyze problem (3.1), (3.2), (3.3), (3.5) and (3.6) to determine the asymptotic behaviour of $s(a,t)$, $i(a,t)$, $r(a,t)$, $S(t)$, $I(t)$ and $R(t)$ and their properties.

Now, let p^* be the unique real number which satisfies:

$$\int_0^\infty \beta(a)\pi(a)e^{-p^*a}da = 1, \quad \pi(a) = e^{-\int_0^a \mu(s)ds} \tag{4.1}$$

And define $\pi_2(a)$ by:

$$\pi_2(a) = e^{-\int_0^a (\mu(s)+\gamma(s))ds} \tag{4.2}$$

Now, let p_q^* be the unique real number which satisfies:

$$q\int_0^\infty \beta(a)\pi_2(a)e^{-p_q^*a}da = 1 \tag{4.3}$$

Then the following lemma describes the relationship between p_q^* and p^*.

Lemma (4.1)

Let p^* and p_q^* be the unique real numbers given by (4.1) and (4.3) respectively. Then $p_q^* < p^*$.

Proof.

Since $q \in (0,1)$ and $\beta(a) \not\equiv 0$, then $q\beta(a)\pi_2(a) < \beta(a)\pi_2(a) \leq \beta(a)\pi(a)$ $\forall a$ such that $\beta(a) \neq 0$. This implies that $e^{-p^*a} < e^{-p_q^*a}$ for otherwise $q\beta(a)\pi_2(a)e^{-p_q^*a} < \beta(a)\pi(a)e^{-p^*a}$ which is a contradiction, which in turn implies that $p_q^* < p^*$.

Here, we note that $p(a,t)$ has the following asymptotic behaviour (see Bellman and Cooke [3], Hoppensteadt [16] and Fellor [13]):

$$p(a,t) = \begin{cases} p_0(a,t)\pi(a)/\pi(a-t), \ a > t \\ [c + \theta(t-a)]\, e^{p^*(t-a)}\pi(a), \ a < t \end{cases} \tag{4.4}$$

where C is a constant and $\theta(t) \to 0$ as $t \to \infty$.

In the next result we establish the asymptotic behaviour of $\omega(a,t)$ starting from problem (3.5) when $p^* < 0$. We note that the existence and the uniqueness of solution $\omega(a,t)$ satisfying problem (3.5) has been established (see El-Doma [12]).

Theorem (4.1)

Suppose

1) $p^* < 0$

2) $\int_0^\infty \pi_2(a)e^{-(p^*+p_q^*)a}da < \infty$

3) $M = p^* - p_q^* + \sup_{[0,\infty)}(q\beta(a) - \mu(a) - \gamma(a)) < 0$

Then

$$\lim_{t\to\infty} \omega(a,t) = e^{-p_q^*a}\pi_2(a)/\int_0^\infty e^{-p_q^*a}\pi_2(a)da$$

Proof.

We start by noting that by integrating problem (3.5) along characteristics $a - t =$ constant, we obtain the following:

$$\omega(a,t) = \begin{cases} \omega_0(a-t)e^{-\int_0^t(\mu(a-t+s)+\gamma(a-t+s)-A(s))ds} \\ \qquad + k\int_0^t e^{-\int_\sigma^t(\mu(a-t+s)+\gamma(a-t+s)-A(s))ds} p_2(a-t+\sigma)d\sigma, \ a > t \\ \omega(0,t-a)\pi_2(a)e^{-\int_t^{t-a}A(s)ds} \\ +k\int_0^a e^{-\int_\sigma^a(\mu(s)+\gamma(s)-A(t-a+s))ds} p_2(\sigma, t-a+\sigma)d\sigma, \ a < t \end{cases} \tag{4.5}$$

where $A(t)$ is given by:

$$A(t) = \int_0^\infty [\mu(a) + \gamma(a) - q\beta(a)]\omega(a,t)da - kP_2(t) \tag{4.6}$$

Since $\omega(0,t) = q\int_0^\infty \beta(a)\omega(a,t)da$, by setting $B_q(t) = \omega(0,t)e^{-\int_0^t A(s)ds}$ we find that $B_q(t)$ satisfies the following:

$$B_q(t) = q\int_0^t \beta(a)\pi_2(a)B_q(t-a)da + G(t) \tag{4.7}$$

where $G(t)$ is given by:

$$G(t) = q\int_t^\infty \beta(a)\omega_0(a-t)e^{-\int_0^t(\mu(a-t+s)+\gamma(a-t+s))ds}da$$

$$+kq\int_0^t\int_0^\infty \beta(a+\sigma)e^{-\int_a^{a+\sigma}(\mu(s)+\gamma(s))ds} p_2(a, t-\sigma)e^{-\int_0^{t-\sigma}A(s)ds}dad\sigma \tag{4.8}$$

Since by assumption (3) $\beta(a)$ has compact support, we have the following:

$$q\int_0^\infty e^{-p_q^* t}\int_t^\infty \beta(a)\omega_0(a-t)e^{-\int_0^t(\mu(a-t+s)+\gamma(a-t+s))ds}dadt$$

$$\leq q\int_0^T e^{-p_q^* t}\int_t^T \beta(a)\omega_0(a-t)\pi_2(a)/\pi_2(a-t)\ dadt < \infty\ .$$

Also since $p_2(a,t) \leq p(a,t)$ and $\gamma(a)$ has compact support by assumption (5), then for a constant K we have the following:

$$kq\int_0^\infty e^{-p_q^* t}\int_t^\infty \beta(a)\left(\int_0^t \pi_2(a)/\pi_2(a-t+\sigma)p_2(a-t+\sigma,\sigma)e^{-\int_0^\sigma A(s)ds}\right)dadt$$

$$\leq Kkq\int_0^\infty e^{-p_q^* t}\int_t^\infty \beta(a)\pi_2(a)/\pi_2(a-t+\sigma)p_0(a-t)\left(\int_0^t e^{-\int_0^\sigma A(s)ds}\right)dadt < \infty$$

since by assumption (3) $\beta(a)$ has compact support.

Now, we estimate the following integral:

$$\int_0^t\int_0^a \beta(a)\pi_2(a)/\pi_2(\sigma)e^{-p_q^* t}p_2(\sigma, t-a+\sigma)\ e^{-\int_0^{t-a+\sigma}A(s)ds}d\sigma da$$

$$= \int_0^t\int_0^{t-\sigma} \beta(a+\sigma)\pi_2(a+\sigma)/\pi_2(\sigma)\ e^{-p_q^* t}p_2(\sigma, t-a)e^{-\int_0^{t-a}A(s)ds}dad\sigma$$

$$\leq \bar{K}\int_0^T\int_0^{T-\sigma} \beta(a+\sigma)\pi_2(a+\sigma)e^{-p^*(a+\sigma)}e^{\int_0^{t-a}(p^*-p_q^*-A(s))ds}dad\sigma$$

for a constant $\bar{K}$.

Since by assumption 1) of the theorem, $p^* < 0$, it follows that $e^{k\int_0^t P_2(s)ds} < D < \infty$, $\forall t \geq 0$ and therefore:

$$\int_0^\infty\int_0^t\int_0^a \beta(a)\pi_2(a)/\pi_2(\sigma)p_2(\sigma, t-a+\sigma)e^{-\int_0^{t-a+\sigma}A(s)ds}dad\sigma dt$$

$$\leq D\bar{K}\int_0^\infty\int_0^T\int_0^{T-\sigma} \beta(a+\sigma)\pi_2(a+\sigma)e^{-p^*\sigma}e^{-p_q^* a}e^{M(t-a)}dad\sigma dt < \infty$$

since by assumption 3) of the theorem $M < 0$.

Accordingly $B_q(t)$ have the following asymptotic behaviour (see Fellor [13]):

$$B_q(t) = [c_2 + \theta_2(t)]e^{p_q^* t} \tag{4.9}$$

where C_2 is a constant and $\theta_2(t) \to 0$ as $t \to \infty$.

Using $\int_0^\infty \omega(a,t)da = 1$, we obtain the following:

$$1 = \int_t^\infty \omega_0(a-t)e^{-\int_0^t(\mu(a-t+s)+\gamma(a-t+s))ds}e^{\int_0^t A(s)ds}da$$

$$+k\int_t^\infty\int_0^t e^{-\int_\sigma^t(\mu(a-t+s)+\gamma(a-t+s))ds}p_2(a-t+\sigma,\sigma)e^{\int_\sigma^t A(s)ds}d\sigma da$$

$$+\int_0^t \omega(0,t-a)e^{-\int_t^{t-a}A(s)ds}\pi_2(a)da$$

$$+k\int_0^t\int_0^a e^{-\int_\sigma^a(\mu(s)+\gamma(s))ds}p_2(\sigma,t-a+\sigma)e^{\int_\sigma^a A(s)ds}d\sigma da$$

Therefore for t sufficiently large we obtain the following:

$$1 = e^{\int_0^t(A(s)+p_q^*)ds}\int_t^\infty e^{-p_q^* t}\omega_0(a-t)\pi_2(a)/\pi_2(a-t)da$$

$$+e^{\int_0^t(A(s)+p_q^*)ds}\int_0^t [c_2 + \theta_2(t-a)]e^{-p_q^* a}\pi_2(a)da$$

$$+ke^{\int_0^t(A(s)+p_q^*)ds}\int_0^t\int_0^\infty \pi_2(a+\sigma)/\pi_2(a)p_2(a,t-\sigma)e^{-\int_0^{t-\sigma}A(s)ds}e^{-p_q^* t}dad\sigma$$

Then by assumption 2) of the theorem we find that

$$\int_0^\infty e^{-p_q^* a}\pi_2(a)da < \infty$$

and hence

$$0 < e^{\int_0^t(A(s)+p_q^*)ds} \leq D_1 < \infty\ \forall t \in [0,\infty)\ .$$

Accordingly,

$$e^{\int_0^t(A(s)+p_q^*)ds}\int_0^t \theta_2(t-a)e^{-p_q^* a}\pi_2(a)da \to 0$$

as $t \to \infty$ by the dominated convergence theorem.

Also

$$e^{\int_0^t (A(s)+p_q^*)ds}\int_t^\infty e^{-p_q^* t}\omega_0(a-t)\pi_2(a)/\pi_2(a-t)da$$

$$\le D_1\int_0^\infty e^{-p_q^* t}\omega_0(a)\pi_2(a+t)/\pi_2(a)da$$

$$= D_1\int_0^\infty \frac{\omega_0(a)\pi_2(a+t)e^{-p_q^*(a+t)}}{\pi_2(a)e^{-p_q^* a}}da \to 0$$

as $t \to \infty$, since

$$\frac{d}{da}\left(e^{-p_q^* a}\pi_2(a)\right) = -\left(p_q^* + \mu(a) + \gamma(a)\right)e^{-p_q^* a}\pi_2(a)$$

which is integrable since μ an γ are bounded functions and $e^{-p_q^* a}\pi_2(a)$ is integrable by assumption 2). So, $e^{-p_q^* a}\pi_2(a) \to 0$ as $t \to \infty$ which implies that $e^{-p_q^* a}\pi_2(a)$ is a monotone decreasing function of a if a is sufficiently large since $\mu(a)$ is eventually nondecreasing and $\gamma(a)$ has compact support. And hence the result is obtained by the dominated convergence theorem.

The term:

$$\int_t^\infty\int_0^t e^{-\int_\sigma^t(\mu(a-t+s)+\gamma(a-t+s))ds}p_2(a-t+\sigma,\sigma)e^{\int_\sigma^t A(s)ds}d\sigma da$$

$$= \int_t^\infty\int_0^t e^{-\int_{a-t+\sigma}^a(\mu(s)+\gamma(s))ds}p_2(a-t+\sigma,\sigma)e^{\int_\sigma^t A(s)ds}d\sigma da$$

$$\le KD_1\int_t^\infty\int_0^t p_0(a-t)\pi_2(a)/\pi_2(a-t)e^{p_q^*(\sigma-t)}d\sigma da$$

$$= KD_1\int_0^\infty\frac{p_0(a)\pi_2(a+t)e^{-p_q^*(t+a)}}{\pi_2(a)e^{-p_q^* a}}\left(\int_0^t e^{p_q^*\sigma}d\sigma\right)da \to 0$$

by the dominated convergence theorem and lemma (4.1).

The term:

$$\int_0^t\int_0^a e^{-\int_\sigma^a(\mu(s)+\gamma(s))ds}p_2(\sigma,t-a+\sigma)e^{\int_\sigma^a A(t-a+s)ds}$$

$$\le K\int_0^t\int_0^a \pi_2(a)[c+\theta(t-a)]e^{p^*(t-a)}e^{\int_{t-a+\sigma}^t A(s)ds}d\sigma da$$

$$\le KD_1 e^{p^* t}\int_0^t\int_0^a \pi_2(a)e^{-(p^*+p_q^*)a}[c+\theta(t-a)]e^{p_q^*\sigma}d\sigma da$$

$$= \frac{KD_1}{p_q^*}e^{p^* t}\int_0^t \pi_2(a)e^{-(p^*+p_q^*)a}[e^{p_q^* a}-1]da \to 0$$

by the dominated convergence theorem and assumption 2) of the theorem.

So, $c_3 e^{\int_0^t(A(s)+p_q^*)ds} \to \dfrac{1}{\int_0^\infty e^{-p_q^* a}\pi_2(a)da}$ and hence

$$\lim_{t\to a}\omega(a,t) = \omega_\infty(a) = \frac{e^{-p_q^* a}\pi_2(a)}{\int_0^\infty e^{-p_q^* a}\pi_2(a)da}.$$

As a consequence of theorem (4.1), we obtain a relationship between the size of the infectives relative to the size of the susceptibles, a relationship between the total population size and the size of the removed class for t sufficiently large.

Theorem (4.2)

Suppose

1) $p^* < 0$.

2) $\int_0^\infty \pi_2(a)e^{-(p^*+p_q^*)a}da < \infty$.

3) $M = p^* - p_q^* + \sup_{[0,\infty)}(q\beta(a) - \mu(a) - \gamma(a)) < 0$.

Then

$$\lim_{t\to\infty}\frac{i(a,t)}{s(a,t)} = 0\ ,\ \lim_{t\to\infty}\frac{p(a,t)}{r(a,t)} = [1-e^{-\int_0^a \nu(s)ds}]$$

$$\lim_{t\to\infty}\frac{P(t)}{R(t)} = \frac{\int_0^\infty e^{-p^* a}\pi(a)da}{\int_0^\infty e^{-p^* a}\pi(a)[1-e^{-\int_0^a\nu(s)ds}]da},\ \lim_{t\to\infty}\frac{p(a,t)}{s(a,t)} = e^{\int_0^a\nu(s)ds}$$

$$\lim_{t\to\infty}\frac{r(a,t)}{s(a,t)} = e^{\int_0^a\nu(s)ds}-1,\ \lim_{t\to 0}\frac{I(t)}{P(t)} = 0$$

Proof.

By assumptions 1), 2) and 3) of the theorem

$$\lim_{t\to\infty}\omega(a,t) = \omega_\infty(a) = \frac{e^{-p_q^* a}\pi_2(a)}{\int_0^\infty e^{-p_q^* a}\pi_2 da}$$

by theorem (4.1). First we look at

$$\lim_{t\to\infty}A(t) = \int_0^\infty(\mu(a)+\gamma(a)-q\beta(a))\frac{e^{-p_q^* a}\pi_2(a)}{\int_0^\infty e^{-p_q^* a}\pi_2(a)da}da$$

$$= \frac{1}{\int_0^\infty e^{-p_q a}\pi_2(a)da}\left[-e^{-p_q^* a}\pi_2(a)\Big|_0^\infty + (-p_q^*)\int_0^\infty e^{-p_q^* a}\pi_2(a)da - 1\right]$$

$$= \frac{1}{\int^\infty e^{-p_q^* a}\pi_a(a)da}\left[1-1-p_q^*\int_0^\infty e^{-p_q^* a}\pi_2(a)da\right] = -p_q^*.$$

i.e. $\lim_{t\to\infty}A(t) = -p_q^*$.

From (3.6) we find that $I(t)$ satisfies:

$$I(t) = \frac{I_0 e^{-\int_0^t A(s)ds}}{1+kI_0\int_0^t e^{-\int_0^\sigma A(s)ds}d\sigma} \tag{4.10}$$

From (4.10) we find that $I(t) \leq Ke^{p_q^* t}$ for some K constant. Since $\lim_{t\to\infty} i(a,t) = \lim_{t\to\infty} I(t)\omega(a,t)$

$$= \frac{e^{-p_q^* a}\pi_2(a)}{\int_0^\infty e^{-p_q^* a}\pi_2(a)} \lim_{t\to\infty} I(t) \leq K \frac{e^{-p_q^* a}\pi_2(a)}{\int_0^\infty e^{-p_q^* a}\pi_2(a)da} \lim_{t\to\infty} e^{-p_q^* t}$$

Now, $\lim_{t\to\infty} \frac{i(a,t)}{s(a,t)} = \lim_{t\to\infty} \frac{i(a,t)}{p_2(a,t)-i(a,t)}$.

By integrating problem (3.2) along characteristics $a - t =$ constant, we obtain:

$$\lim_{t\to\infty} p_2(a,t) = \lim_{t\to\infty} [c + \theta(t-a)]\pi(a)e^{-\int_0^a \nu(s)ds} e^{p^*(t-a)}$$

$$= \lim_{t\to\infty} c\pi(a)e^{-\int_0^a \nu(s)} e^{-p^* a} e^{p^* t}$$

Therefore

$$\lim_{t\to\infty} = \frac{i(a,t)}{s(a,t)} = \lim_{t\to\infty} \frac{1}{\frac{ce^{p^* t}e^{-p^* a}\pi(a)e^{-\int_0^a \nu(s)ds}}{i(a,t)} - 1}$$

$$\leq \lim_{t\to 0} \frac{1}{\frac{\frac{c}{k} e^{(p^*-p_q^*)t} - e^{(p^*-p_q^*)a} e^{-\int_0^a \nu(s)ds}}{e^{-\int_0^a \gamma(s)ds} \int_0^\infty e^{-p_q^* a}\pi_2(a)da} - 1} \to 0$$

as $t \to \infty$ by lemma (4.1), i.e. $\lim_{t\to\infty} \frac{i(a,t)}{s(a,t)} = 0$.

We note that, this result implies that $i(a,t)$, the density of the infectives dies out faster than $s(a,t)$, the density of the susceptibles.

It is easy to see from (3.2) and (4.4) that since

$$r(a,t) = p(a,t) - p_2(a,t) = ce^{-p^* a}\pi(a)[1 - e^{-\int_0^a \nu(s)ds}]e^{p^* t}$$

for t sufficiently large that:

$$\lim_{t\to\infty} \frac{r(a,t)}{p(a,t)} = [1 - e^{-\int_0^a \nu(s)ds}]$$

Since $p_2(a,t) = s(a,t) + i(a,t)$, $\lim_{t\to\infty} \frac{p_2(a,t)}{s(a,t)} = 1 - \lim_{t\to\infty} \frac{i(a,t)}{s(a,t)} = 1$, this in turn implies that $\lim_{t\to\infty} \frac{p(a,t)}{s(a,t)} = e^{\int_0^a \nu(s)ds}$. And therefore $\lim_{t\to\infty} \frac{r(a,t)}{s(a,t)} = \lim_{t\to\infty} \frac{p(a,t)-p_2(a,t)}{s(a,t)} = e^{\int_0^a \nu(s)ds} - 1$ and

$$\lim_{t\to\infty} \frac{R(t)}{P(t)} = \frac{\int_0^\infty e^{-p^* a}\pi(a)[1 - e^{-\int_0^a \nu(s)ds}]da}{\int_0^\infty e^{-p^* a}\pi(a)da}$$

Finally, since $I(t) \leq Ke^{p_q^* t}$, then

$$\lim_{t\to\infty} \frac{I(t)}{P(t)} \leq \lim_{t\to\infty} \frac{Ke^{(p_q^*-p^*)t}}{c\int_0^\infty \pi(a)e^{-p^* a}da} \to 0 \text{ as } t \to \infty \text{ by}$$

lemma (4.1).

In the following result, we discuss the steady state solution of the problem i.e. when $p^* = 0$. And we determine a critical vaccination coverage that will eventually eradicate the disease with minimum vaccination coverage.

Theorem (4.3)

Suppose that $p^* = 0$. Then if

$$ck\int_0^\infty \int_0^\infty \pi_2(a+\sigma)e^{\int_0^a (\gamma(s)-\nu(s))ds} da d\sigma$$

$$+ \frac{kcq\int_0^\infty \int_0^\infty \beta(a+\sigma)\pi_2(a+\sigma)e^{\int_0^a (\gamma(s)-\nu(s))ds}}{1 - q\int_0^\infty \beta(a)\pi_2(a)da} \int_0^\infty \pi_2(a)da > 1$$

then there exists a unique $I_\infty = \lim_{t\to\infty} I(t) > 0$ satisfying:

$$ck\int_0^\infty \int_0^\infty \pi_2(a+\sigma)e^{\int_0^a \gamma(s)-\nu(s))ds} e^{-kI_\infty a} da d\sigma +$$

$$+ \frac{kq\int_0^\infty \int_0^\infty \beta(a+\sigma)\pi_2(a+\sigma)e^{\int_0^a (\gamma(s)-\nu(s))ds}}{1 - q\int_0^\infty \beta(a)\pi_2(a)da} (c\int_0^\infty \pi_2(a)da - I_\infty)$$

$$= 1$$

And in this case, if we set V_∞ to satisfy:

$$V_\infty = \frac{kcqI_\infty \int_0^\infty \int_0^\infty \beta(a+\sigma)\pi_2(a+\sigma)e^{\int_0^a (\gamma(s)-\nu(s))ds} e^{-kI_\infty a} da d\sigma}{1 - q\int_0^\infty \beta(a)\pi_2(a)da + kqI_\infty \int_0^\infty \int_0^\infty \beta(a+\sigma)\pi_2(a+\sigma)e^{\int_0^a (\gamma(s)-\nu(s))ds} e^{-kI_\infty a} da d\sigma} \tag{4.11}$$

then $s_\infty(a) = (c - V_\infty)\pi(a)e^{-\int_0^a \nu(s)ds} e^{-kI_\infty a}$

$$i_\infty(a) = V_\infty \pi_2(a) + kI_\infty(c - V_\infty)\pi_2(a)\int_0^a e^{\int_0^\sigma (\gamma(s)-\nu(s))ds} e^{-kI_\infty \sigma} d\sigma \ .$$

Otherwise, $I_\infty = 0$, $i_\infty(a) = 0$ and $s_\infty(a) = c\pi(a)e^{-\int_0^a \nu(s)ds}$.

Proof:

We start by noting that from (2.1) $s(a,t)$ satisfies the following:

$$\begin{cases} \frac{\partial s(a,t)}{\partial a} + \frac{\partial s(a,t)}{\partial t} + (\mu(a) + \nu(a))s(a,t) = -ks(a,t)I(t), \ a > 0, \ t > 0 \\ s(0,t) = p(0,t) - i(0,t), \ t \geq 0 \\ s(a,0) = s_0(a), \ a \geq 0 \end{cases} \tag{4.12}$$

Then by integrating problem (4.12) along characteristics a-t = constant, we obtain the following:

$$s(a,t) = \begin{cases} s_0(a-t)e^{-\int_0^t(\mu(a-t+s)+\nu(a-t+s)+kI(s))ds}, \ a > t \\ \\ \left[p(0,t-a) - i(0,t-a)\right]\pi(a)e^{-\int_0^a \nu(s)ds}e^{-k\int_0^a I(t-a+s)ds}, \ a < t \end{cases} \tag{4.13}$$

We also note that by integrating problem (3.3) along characteristics a-t = constant, we obtain the following:

$$i(a,t) = \begin{cases} i_0(a-t)e^{-\int_0^t(\mu(a-t+s)+\gamma(a-t+s))ds} \\ + \quad k\int_0^t e^{-\int_\sigma^t(\mu(a-t+s)+\gamma(a-t+s))ds}s(a-t+\sigma,\sigma)I(\sigma)d\sigma, \ a > t \\ i(0,t-a)e^{-\int_0^a(\mu(s)+\gamma(s))ds} + k\int_0^a e^{-\int_\sigma^a(\mu(s)+\gamma(s))ds}s(\sigma,t-a+\sigma)I(t-a+\sigma)d\sigma, \ a < t \end{cases} \tag{4.14}$$

Now, by integrating (4.4) from 0 to ∞, we find that $I(t)$ satisfies the following:

$$I(t) = \int_0^\infty i_0(a)e^{-\int_0^t(\mu(a+s)+\gamma(a+s))ds}da + \int_0^t i(0,t-a)\pi_2(a)da$$

$$+k\int_0^t\int_0^\infty e^{-\int_a^{a+\sigma}(\mu(s)+\gamma(s))ds}s(a,t-\sigma)I(t-\sigma)dad\sigma \tag{4.15}$$

Here, we observe that

$$\lim_{t\to\infty}\int_0^\infty i_0(a)e^{-\int_0^t(\mu(a+s)+\gamma(a+s))ds}da \to 0$$

by the dominated convergence theorem and assumption 4) and 7). Therefore for t sufficiently large and using (4.13) and (4.14) we obtain the following:

$$I(t) = \int_0^t i(0,t-a)\pi_2(a)da$$

$$+k\int_0^t\int_0^\infty e^{-\int_a^{a+\sigma}(\mu(s)+\gamma(s))ds}[p(0,t-a-\sigma) - i(0,t-a-\sigma)]$$

$$\times e^{-\int_0^a(\mu(s)+\nu(s)+kI(t-a-\sigma+s))ds}dad\sigma \tag{4.16}$$

Now, if we set $V_\infty = \lim_{t\to\infty} i(0,t)$, $I_\infty = \lim_{t\to\infty} I(t)$ and $p^* = 0$, then using (4.16), I_∞ satisfies:

$$I_\infty = V_\infty\int_0^\infty \pi_2(a)da + ckI_\infty\int_0^\infty\int_0^\infty e^{-\int_a^{a+\sigma}(\mu(s)+\gamma(s))ds}e^{-\int_0^a(\mu(s)+\nu(s)+kI_\infty)ds}dad\sigma$$

$$-kI_\infty V_\infty\int_0^\infty\int_0^\infty e^{-\int_a^{a+\sigma}(\mu(s)+\gamma(s))ds}e^{-\int_0^a(\mu(s)+\nu(s)+kI_\infty)ds}dad\sigma \tag{4.17}$$

Setting $V(t) = i(0,t) = q\int_0^\infty \beta(a)i(a,t)da$ then $V(t)$ satisfies:

$$V(t) = q\int_t^\infty \beta(a)i(a-t,0)e^{-\int_0^t(\mu(a-t+s)+\gamma(a-t+s))ds}da + q\int_0^t \beta(a)V(t-a)\pi_2(a)da$$

$$+kq\int_0^t\int_0^\infty \beta(a+\sigma)e^{-\int_a^{a+\sigma}(\mu(s)+\gamma(s))da}s(a,t-\sigma)I(t-\sigma)dad\sigma \tag{4.18}$$

Letting $\lim_{t\to\infty} V(t) = V_\infty$, we get that V_∞ satisfies:

$$V_\infty = \frac{kqcI_\infty\int_0^\infty\int_0^\infty \beta(a+\sigma)\pi_2(a+\sigma)e^{\int_0^a(\gamma(s)-\nu(s))ds}e^{-kI_\infty a}dad\sigma)ds}{[1 - q\int_0^\infty \beta(a)\pi_2(a)da + kqI_\infty\int_0^\infty\int_0^\infty \beta(a+\sigma)\pi_2(a+\sigma)e^{\int_0^a(\gamma(s)-\nu(s))ds}e^{-kI_\infty a}dad\sigma]} \tag{4.19}$$

Substituting (4.19) into (4.17) we obtain the following result if $I_\infty \neq 0$.

$$1 = ck\int_0^\infty\int_0^\infty \pi_2(a+\sigma)e^{\int_0^a(\gamma(s)-\nu(s))ds}e^{-kI_\infty a}dad\sigma$$

$$+kq\int_0^\infty\int_0^\infty \beta(a+\sigma)\pi_2(a+\sigma)e^{\int_0^a(\gamma(s)-\nu(s))ds}e^{-kI_\infty a}dad\sigma\frac{(c\int_0^\infty \pi_2(a)da - I_\infty)}{1 - q\int_0^\infty \beta(a)\pi_2(a)da} \tag{4.20}$$

Since the right-hand side of (4.20) is a monotone decreasing function of I_∞, (4.20) has a unique solution $I_\infty > 0$ iff

$$ck\int_0^\infty\int_0^\infty \pi_2(a+\sigma)e^{\int_0^a(\gamma(s)-\nu(s))ds}dad\sigma$$

$$+\frac{kqc\int_0^\infty\int_0^\infty \beta(a+\sigma)\pi_2(a+\sigma)e^{\int_0^a(\gamma(s)-\nu(s))ds}dad\sigma\int_0^\infty \pi_2(a)da}{1 - q\int_0^\infty \beta(a)\pi_2(a)da} > 1 \tag{4.21}$$

And in this case (4.13) implies that $s_\infty(a) = (c - V_\infty)\pi(a)e^{-\int_0^a \nu(s)ds}e^{-kI_\infty a}$ and (4.14) implies that:

$$i_\infty(a) = \pi_2(a)[V_\infty + kI_\infty(c - V_\infty)\int_0^a e^{\int_0^\sigma(\gamma(s)-\nu(s)-kI_\infty)ds}d\sigma]$$

If $I_\infty = 0$, then $V_\infty = 0$ and therefore

$$s_\infty(a) = c\pi_2(a)e^{-\int_0^a \nu(s)ds}\ , i_\infty(a) = 0\ .$$

Here, we note that if $q = 0$ i.e. the case of no vertical transmission then, the condition for the existence of a unique positive equilibrium (endemic disease) is $ck\int_0^\infty\int_0^\infty \pi_2(a+\sigma)e^{\int_0^a(\gamma(s)-\nu(s))ds}dad\sigma > 1$, and in this case I_∞ satisfies:

$$1 = ck\int_0^\infty\int_0^\infty \pi_2(a+\sigma)e^{\int_0^a(\gamma(s)-\nu(s))ds}e^{-kI_\infty^a}dad\sigma$$

Here, we note that, if ν_c is such that

$$1 = ck\int_0^\infty\int_0^\infty \pi_2(a+\sigma)e^{\int_0^a(\gamma(s)-\nu_c)ds}dad\sigma$$

$$\frac{+kcq\int_0^\infty\int_0^\infty \beta(a+\sigma)e^{\int_0^a(\gamma(s)-\nu_c)ds}dad\sigma\int_0^\infty \pi_2(a)da}{1 - q\int_0^\infty \beta(a)\pi_2(a)da} \tag{4.22}$$

Then ν_c is the minimum vaccination coverage that will eventually eradicate the disease.

It is also worth noting that (4.22) could be used as a device for determining the effectiveness of a certain vaccination strategy; for example there are vaccination strategies that are followed for the eradication of important communicable diseases such as measles and rubella (see Hethecote [15], Katzman and Dietz [17]). Moreover, the effect of vertical transmission via its parameter q is obvious for making the feasibility of an endemic disease more possible by noting that the left-hand side of (4.21) is an increasing function of q.

Here, we remark that if $q = 1$ and $\gamma = 0$ and $k \neq 0$, then $V_\infty = c$ and $i_\infty(a) = c\pi(a)$ and $s_\infty(a) = r_\infty(a) = 0$. This result shows the power of vertical transmission in the sense that newborn susceptibles get infected as time goes by and vaccination alone will not stop the epidemic from becoming endemic. In the following result, we look at the behaviour of $I(t)$ when $p^* > 0$.

Theorem (4.4)
Suppose that $p^* > 0$. Then $I(t) \to \infty$ as $t \to \infty$ and $\lim_{t\to\infty} \frac{I(t)}{P_2(t)} = 1$.

Proof:
We start by considering the following problem for $r(a,t)$:

$$\begin{cases} \dfrac{\partial r(a,t)}{\partial a} + \dfrac{\partial r(a,t)}{\partial t} + \mu(a) r(a,t) = \nu(a) s(a,t) + \gamma(a) i(a,t), \ a > 0, \ t > 0 \\ r(0,t) = 0, \ t \geq 0 \\ r(a,0) = r_0(a), \ a \geq 0 \end{cases} \tag{4.23}$$

By integrating problem (4.23) along characteristics $t - a =$ constant, we obtain the following:

$$r(a,t) = \begin{cases} r_0(a-t) e^{-\int_0^t \mu(a-t+s)ds} + \displaystyle\int_0^t e^{-\int_\sigma^t \mu(a-t+s)ds} [\nu(a-t+s) s(a-t+\sigma,\sigma) \\ \qquad + \gamma(a-t+\sigma) i(a-t+\sigma,\sigma)] d\sigma, \ a > t \\ \displaystyle\int_0^a e^{\int_\sigma^a \mu(s)ds} [\nu(\sigma) s(\sigma, t-a+\sigma) + \gamma(\sigma) i(\sigma, t-a+\sigma)] d\sigma, \ a < t \end{cases} \tag{4.24}$$

Now, since $p(a,t) = s(a,t) + i(a,t) + r(a,t) \to \infty$ as $t \to \infty$, for $p^* > 0$, (4.24) imply that $p_2(a,t) \to \infty$ as $t \to \infty$, for otherwise $p(a,t)$ does not approach infinity as $t \to \infty$ which is a contradiction.

By setting $f(t) = e^{-\int_0^t A(s)ds}$, (4.10) implies that

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty} \frac{I_0 f(t)(-A(t))}{k I_0 f(t)} = \lim_{t\to\infty} \frac{(-A(t))}{k} .$$

Therefore, given $\varepsilon > 0$, $\exists \tau \in R$ such that $|I(t) + \frac{A(T)}{k}| < \varepsilon$ for $t > \tau$. But by (4.6)

$$-\frac{A(t)}{k} = P_2(t) - \frac{1}{k} \int_0^\infty [\mu(a) + \gamma(a) - q\beta(a)] \omega(a,t) da .$$

So,

$$-\frac{A(t)}{k} \frac{1}{P(t)} = 1 - \frac{1}{k P_2(t)} \int_0^\infty [\mu(a) + \gamma(a) - q\beta(a)] \omega(a,t) da .$$

And since $\int_0^\infty [\mu(a) + \gamma(a) - q\beta(a)] \omega(a,t) da \leq \sup_{a \in [0,\infty)} [\mu(a) + \gamma(a) - q\beta(0)] < \infty$, and $p_2(t) \to \infty$ as $t \to \infty$, we get that

$$\left| \frac{A(t)}{R} \frac{1}{p_2(t)} + 1 \right| < \varepsilon \text{ for } t > \tau .$$

This gives $\lim_{t\to\infty} \frac{I(t)}{P_2(t)} = 1$.

In the next result, we will show that in a growing population i.e. when $p^* > 0$, and for t sufficiently large, the total population consists essentially of infectives and immune individuals.

Theorem (4.5)
Suppose $p^* > 0$. Then $\lim_{t\to\infty} S(t) = 0$.

Proof:
Since $\lim_{t\to\infty} I(t) = -\lim_{t\to\infty} \frac{A(t)}{k}$

$$\therefore \lim_{t\to\infty} S(t) = \lim_{t\to\infty} P_2(t) + \lim_{t\to\infty} \frac{A(t)}{k}$$

i.e.

$$\lim_{t\to\infty} S(t) = \frac{1}{k} \lim_{t\to\infty} \int_0^\infty [\mu(a) + \gamma(a) - q\beta(a)] \omega(a,t) da \tag{4.25}$$

So, since $\lim_{t\to\infty} e^{\int_{t-a}^t A(s)ds} \to 0$ as $t \to \infty$ because $\mu(a)$ and $\beta(a)$ are bounded and $p_2(t) \to \infty$ as $t \to \infty$. And $\omega(0, t-a) = q \int_0^\infty \beta(a) \omega(a, t-a) da < \infty$. We observe that $\omega(0, t-a) \pi_2(a) e^{\int_{t-a}^t A(s)ds} \to 0$ as $t \to \infty$. By considering (4.5) we note that $e^{\int_\sigma^a A(t-a+s)ds} p_2(\sigma, t-a+\sigma) \to 0$ as $t \to \infty$ and therefore from (4.5) $\lim_{t\to\infty} \omega(a,t) = 0$ which implies that $\lim_{t\to\infty} S(t) = 0$ by using (4.25).

Accordingly, for t sufficiently large $P(t)$ behaves like $R(t) + I(t)$ and this means that newborns who are susceptibles either get infected or removed by vaccination to the immune class.

In the following result, we will show that in a growing population i.e. $p^* > 0$, the total number of infectives is proportional to the total population size.

Theorem (4.6)
Suppose that $p^* > 0$. Then

$$\lim_{t\to\infty} \frac{I(t)}{P(t)} = \frac{\int_0^\infty \pi_2(a) e^{-p^* a} da}{\int_0^\infty \pi(a) e^{-p^* a} da}$$

Proof:

From (4.14) we find that $I(t)$ satisfies:

$$I(t)=\int_0^\infty i_0(a)e^{-\int_0^t(\mu(a+s)+\gamma(a+s))ds}da+\int_0^t i(0,t-a)\pi_2(a)da$$

$$+k\int_0^t\int_0^a e^{-\int_\sigma^a(\mu(s)+\gamma(s))ds}s(\sigma,t-a+\sigma)I(t-a+\sigma)d\sigma da+$$

$$+k\int_t^\infty\int_0^t e^{-\int_\sigma^t(\mu(a-t+s)+\gamma(a-t+s))ds}s(a-t+\sigma,\sigma)I(\sigma)d\sigma da \qquad (4.26)$$

We note that by assumption 4)

$$\lim_{t\to\infty}\int_0^t i_0(a)e^{-\int_0^t(\mu(a+s)+\gamma(a+s))ds}da\to 0$$

by the dominated convergence theorem. Using (4.13) we see that:

$$\int_t^\infty\int_0^t e^{-\int_\sigma^t(\mu(a-t+s)+\gamma(a-t+s))ds}s(a-t+\sigma,\sigma)I(\sigma)d\sigma da=$$

$$\int_t^\infty\int_0^t e^{-\int_\sigma^t(\mu(a-t+s)+\gamma(a-t+s))ds}s_0(a-t)I(\sigma)\,e^{-\int_0^\sigma(\mu(a-t+s)+\nu(a-t+s)+kI(s))ds}dad\sigma$$

$$=\int_0^t\int_0^\infty e^{-\int_\sigma^t(\mu(a+\sigma)+\gamma(a+\sigma))ds}s_0(a)I(\sigma)e^{-\int_0^\sigma(\mu(a+s)+\nu(a+s)+kI_\infty(s))ds}dad\sigma$$

$$=\int_0^\infty e^{-\int_0^t\mu(a+s)ds}s_0(a)\left[\int_0^t e^{-\int_\sigma^t\gamma(a+s)ds}e^{-\int_0^\sigma\nu(a+s)ds}I(\sigma)e^{-k\int_0^\sigma I(s)ds}d\sigma\right]da$$

$$\le\int_0^\infty e^{-\int_0^t\mu(a+s)ds}s_0(a)\left[\int_0^t I(\sigma)e^{-k\int_0^\sigma I(s)ds}d\sigma\right]da\to 0$$

by assumption 4) and theorem (4.4).

Now, we note that by (4.4) we observe that $p(t)=ce^{p^*t}\int_0^\infty\pi(a)e^{-p^*a}da$ for t sufficiently large. So, since $i(0,t-a)=qI(t)\int_0^\infty\beta(s)\omega(s,t-a)ds$ then

$$\frac{\int_0^t i(0,t-a)\pi_2(a)da}{P(t)}\to\lim_{t\to\infty}q\int_0^\infty\frac{I(t)}{P(t)}\left(\int_0^\infty\beta(s)\omega(s,t-a)ds\right)da$$

$$=\lim_{t\to\infty}\frac{qI(t)}{P(t)}\int_0^\infty\left(\int_0^\infty\beta(s)\omega(s,t-a)ds\right)da\,.$$

But since by theorem (4.5) $\lim_{t\to\infty}\omega(a,t)\to0$ then by using (4.5)

$$\lim_{t\to 0}\frac{qI(t)}{P(t)}\int_0^\infty\left(\int_0^\infty\beta(s)\omega(s,t-a)ds\right)da\le\lim_{t\to q}q\int_0^\infty\left(\int_0^\infty\beta(s)\omega(s,t-a)ds\right)da\to0$$

by the dominated convergence theorem. Accordingly

$$\lim_{t\to\infty}\frac{I(t)}{P(t)}=lim_{t\to\infty}\frac{k\int_0^t\int_0^a e^{-\int_\sigma^a(\mu(s)+\gamma(s))ds}s(\sigma,t-a+\sigma)I(t-a+\sigma)d\sigma da}{P(t)}$$

$$=\lim_{t\to\infty}\frac{k\int_0^t\int_0^a e^{-\int_\sigma^a(\mu(s)+\gamma(s))ds}[p(0,t-a)-i(0,t-a)]\pi(\sigma)e^{-\int_0^\sigma\nu(s)ds}I(t-a+\sigma)e^{-k\int_0^\sigma I(t-a+s)ds}d\sigma da}{P(t)}$$

$$=k\lim_{t\to\infty}\frac{\int_0^t\int_0^a e^{-\int_\sigma^a(\mu(s)+\gamma(s))ds}p(0,t-a)\pi(\sigma)e^{-\int_0^\sigma\nu(s)ds}I(t-a+\sigma)e^{-k\int_0^\sigma I(t-a+s)ds}d\sigma ds}{P(t)}$$

$$-k\lim_{t\to\infty}\frac{\int_0^t\int_0^a e^{-\int_\sigma^a(\mu(s)+\gamma(s))ds}i(0,t-a)\pi(\sigma)e^{-\int_0^\sigma\nu(s)ds}I(t-a+\sigma)e^{-k\int_0^\sigma I(t-a+s)ds}d\sigma da}{P(t)}$$

Note that

$$\lim_{t\to\infty}\frac{\int_0^t\int_0^a e^{-\int_\sigma^a(\mu(s)+\gamma(s))ds}i(0,t-a)\pi(\sigma)e^{-\int_0^\sigma\nu(s)ds}I(t-a+\sigma)e^{-k\int_0^\sigma I(t-a+s)ds}d\sigma da}{P(t)}$$

$$=\lim_{t\to\infty}\frac{\int_0^t\pi(a)i(0,t-a)\left(\int_0^a e^{-\int_\sigma^a\gamma(s)ds}e^{-\int_0^\sigma\nu(s)ds}I(t-a+\sigma)e^{-k\int_0^\sigma I(t-a+s)ds}d\sigma\right)da}{P(t)}$$

$$\le\lim_{t\to\infty}\frac{\int_0^\infty\pi(a)i(0,t-a)\left(\int_0^a I(t-a+\sigma)e^{-k\int_0^\sigma I(t-a+s)da}d\sigma\right)da}{P(t)}\to0$$

as $t\to\infty$ by theorem (4.4) and the dominated convergence theorem. Therefore

$$\lim_{t\to\infty}\frac{I(t)}{P(t)}=\lim_{t\to\infty}\frac{k\int_0^t\int_0^a e^{-\int_\sigma^a(\mu(s)+\gamma(s))ds}p(0,t-a)\pi(\sigma)e^{-\int_0^\sigma\nu(s)ds}I(t-a+\sigma)e^{-k\int_0^\sigma I(t-a+s)ds}d\sigma da}{P(t)}$$

$$=k\lim_{t\to\infty}\frac{\int_0^\infty\int_0^a\pi(a)e^{-p^*a}e^{-\int_\sigma^a\gamma(s)ds}e^{-\int_0^\sigma\nu(s)ds}I(t-a+\sigma)e^{-\int_0^\sigma I(t-a+s)ds}d\sigma da}{\int_0^\infty\pi(a)e^{-p^*a}da}$$

Now,

$$k\lim_{t\to\infty}\int_0^\infty\int_0^a\pi(a)e^{-p^*a}e^{-\int_\sigma^a\gamma(s)ds}e^{-\int_0^\sigma\nu(s)ds}I(t-a+\sigma)e^{-k\int_0^\sigma I(t-a+s)ds}d\sigma da$$

$$=-\lim_{t\to\infty}\int_0^\infty\pi(a)e^{-p^*a}\left[e^{-\int_0^a\nu(s)ds}e^{-k\int_0^a I(t-a+s)ds}-e^{-\int_0^a\gamma(s)ds}\right]da$$

$$+\lim_{t\to\infty}\int_0^\infty\pi(a)e^{-p^*a}\left(\int_0^a e^{-\int_\sigma^a\gamma(s)ds}e^{-\int_0^\sigma\nu(s)ds}e^{-k\int_0^\sigma I(t-a+s)ds}[\gamma(\sigma)-\nu(\sigma)]d\sigma\right)da$$

$$=\int_0^\infty\pi_2(a)e^{-p^*a}da$$

by the dominated convergence theorem and theorem (4.4). Therefore we get the result:

$$\lim_{t\to\infty}\frac{I(t)}{P(t)}=\frac{\int_0^\infty\pi_2(a)e^{-p^*a}da}{\int_0^\infty\pi(a)e^{-p^*a}da}$$

As a consequence of theorem (4.6) and (4.5) the large time behaviour of $R(t)$ is given by:

$$R(t) = ce^{p^* t}\left[\int_0^\infty \pi(a)e^{-p^* a}da - \int_0^\infty \pi_2(a)e^{-p^* a}da\right]$$

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