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**INTERNATIONAL CENTRE FOR
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MAXWELL-DIRAC AND THE
KLEIN-GORDON-DIRAC EQUATIONS**



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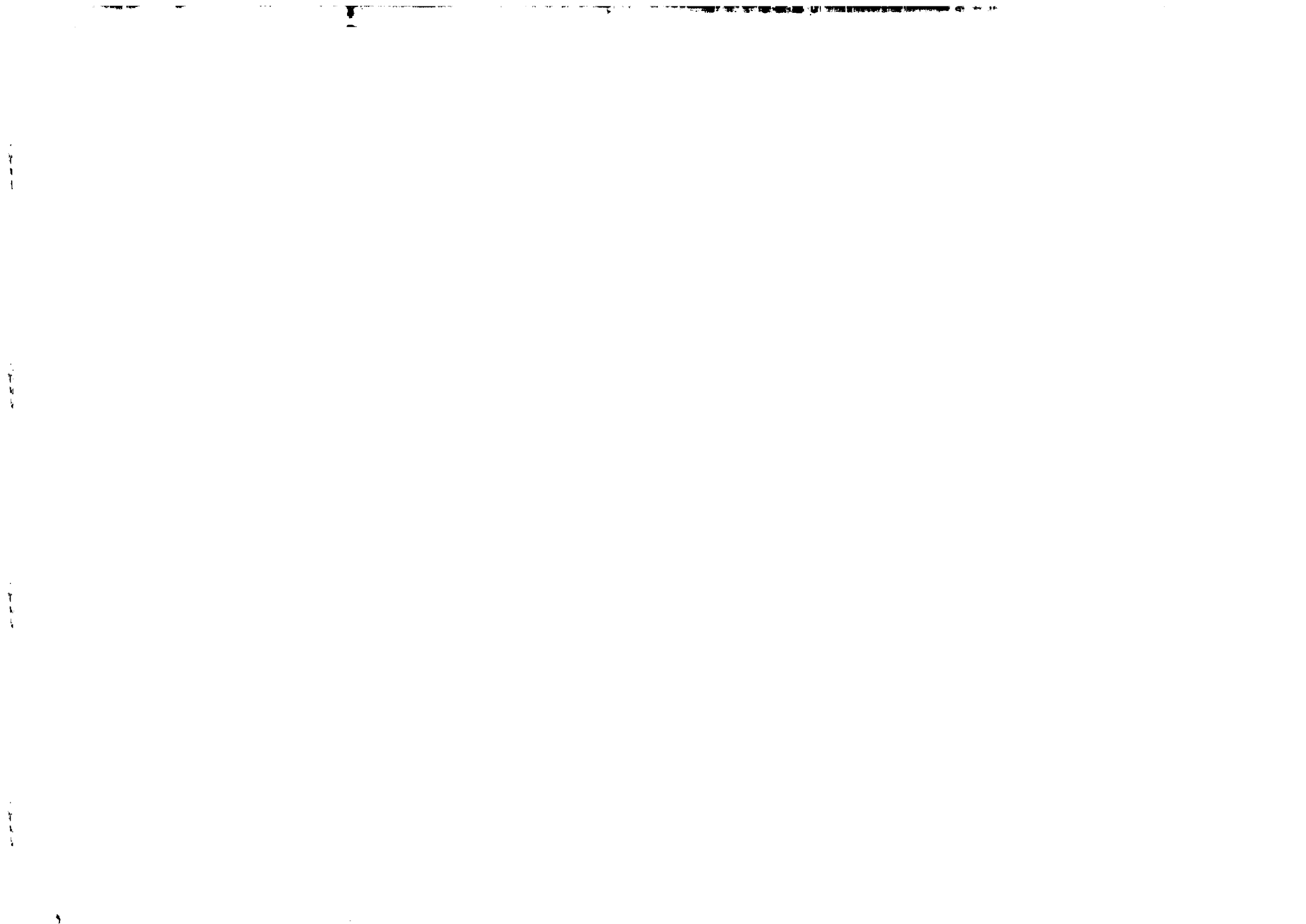
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**STATIONARY SOLUTIONS OF THE MAXWELL-DIRAC
AND THE KLEIN-GORDON-DIRAC EQUATIONS**

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ABSTRACT

The Maxwell-Dirac system describes the interaction of an electron with its own electromagnetic field. We prove the existence of soliton-like solutions of Maxwell-Dirac in (3+1)-Minkowski space-time. The solutions obtained are regular, stationary in time, and localized in space. They are found by a variational method, as critical points of an energy functional. This functional is strongly indefinite and presents a lack of compactness. We also find soliton-like solutions for the Klein-Gordon-Dirac system, arising in the Yukawa model.

I. Introduction.

The Maxwell-Dirac equations, which describe the interaction of an electron with its own electromagnetic field, play a major role in quantum electrodynamics. They can be written as follows

$$(M-D) \quad \begin{cases} (i\gamma^\mu \partial_\mu - \gamma^\mu A_\mu)\psi - m\psi = 0 & \text{in } \mathbb{R} \times \mathbb{R}^3 \\ \partial_\mu A^\mu = 0 \quad , \quad 4\pi \partial_\mu \partial^\mu A^\nu = J^\nu & \text{in } \mathbb{R} \times \mathbb{R}^3 \end{cases}$$

where $\nu, \mu \in \{0, 1, 2, 3\}$, $m > 0$, (\cdot, \cdot) is the usual hermitian product in \mathbb{C}^4 , $\psi(x_0, x) \in \mathbb{C}^4$ for $(x_0, x) \in \mathbb{R} \times \mathbb{R}^3$ and $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_{4 \times 4}(\mathbb{C})$, $\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \in M_{4 \times 4}(\mathbb{C})$, $\bar{\psi} = \gamma^0 \psi$, $J^\mu = (\bar{\psi}, \gamma^\mu \psi)$, $J_0 = J^0$, $J_k = -J^k$, $k = 1, 2, 3$, and σ^k are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Solutions of (M-D) that are stationary in time, and localized in space, are called soliton-like solutions of Maxwell-Dirac. They can be viewed as representations of extended particles. Their existence has been an open problem for a long time (see e.g. [17], p.235). It is the aim of this paper to find such solutions. We also find soliton-like solutions for the Klein-Gordon-Dirac equations arising in the so-called Yukawa model (see [7] and [4]). These equations are

$$(KG-D) \quad \begin{cases} (i\gamma^\mu \partial_\mu - \chi)\psi - m\psi = 0 & \text{in } \mathbb{R} \times \mathbb{R}^3 \\ \partial^\mu \partial_\mu \chi + M^2 \chi = \frac{1}{4\pi} (\bar{\psi}, \psi) & \text{in } \mathbb{R} \times \mathbb{R}^3 . \end{cases}$$

The above systems have been studied for a long time and many results are available concerning the Cauchy problem for (M-D). The first result about the local existence and uniqueness of solutions of (M-D) was obtained by L. Gross in [18]. Later developments were made by Chadam [10] and Chadam and Glassey [11] in 1+1 and 2+1 space-time dimensions and in 3+1 dimensions when the magnetic field is 0. Choquet-Bruhat studied in [12] the case of spinor fields of zero mass and Maxwell-Dirac equations in the Minkowski space were studied by Flato, Simon and Taffin in [15]. In [16], Georgiev obtained a class of initial values for which the Maxwell-Dirac equations have a global solution. This was performed by using a technique of Klainerman (see [21-24]), which gives L^∞ a priori estimates via the Lorentz invariance of the equations,

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and a generalized version of the energy inequalities. In this respect, see also [20]. The same method was used by Bachelot [1] to obtain a similar result for (KG-D). Finally, recent results of Beals and Bezard yield the existence and uniqueness of weak solutions for initial data satisfying the natural energy estimates.

As far as the existence of stationary solutions (soliton-like) of (M-D) is concerned, there is a pioneering work by Wakano ([31]) in which an approximation of (M-D) is studied:

Assuming that the electrostatic potential is predominant, the extreme case in which $A_0 \neq 0, A_1 = A_2 = A_3 \equiv 0$ is considered (Coulomb-Dirac). The approximate problem (C-D) can be reduced to a system of three coupled differential equations by using the spherical spinors. Wakano obtained numerical evidence for the existence of stationary solutions of (C-D). Further work in this direction (see [27, 29]) yielded the same kind of numerical results for some modified Maxwell-Dirac equations which include some nonlinear self-coupling.

Recently, Garret Lisi (see [32]) obtained numerical solutions for the whole system of Maxwell-Dirac equations. This was done by using an axially symmetric ansatz.

In the present paper we make no approximation on the electromagnetic potential, and we show that for $0 < \omega < m$ there are exact solutions

$(\psi, A) : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}^4 \times \mathbb{R}^4$ of (M-D) of the form

$$(1.1) \quad \begin{cases} \psi(x_0, x) = e^{i\omega x_0} \varphi(x) & , \quad \varphi : \mathbb{R}^3 \rightarrow \mathbb{C}^4 \\ A^\mu(x_0, x) = J^\mu * \frac{1}{|x|} = \int_{\mathbb{R}^3} \frac{dy}{|x-y|} J^\mu(y) . \end{cases}$$

We prove this result by using a variational method which was introduced by Esteban and Séré in [14] (see also [13]) to deal with some class of nonlinear Dirac equations in which the nonlinear coupling is local, the so-called Soler model (for more details on this model, see e.g. [27, 9, 26, 2, 3, 8]). This variational method was inspired by earlier works on periodic and homoclinic orbits of hamiltonian systems ([6, 5, 19, 30, 28]).

In order to state the main results contained in this paper, let us note that

If (ψ, A) is a solution of (M-D) of the form (1.1), then (φ, A) is a solution of

$$(1.2) \quad \begin{cases} i\gamma^k \partial_k \varphi - m\varphi - \omega\gamma^0 \varphi - \gamma^\mu A_\mu \varphi = 0 & \text{in } \mathbb{R}^3 \\ -4\pi\Delta A_0 = J^0 = |\varphi|^2 & , \quad -4\pi\Delta A_k = -J^k & \text{in } \mathbb{R}^3 . \end{cases}$$

The solutions of (1.2) are given by the critical points $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$ of the functional

$$(1.3) \quad \begin{aligned} I_\omega(\varphi) = & \int_{\mathbb{R}^3} \frac{1}{2} (i\gamma^0 \gamma^k \partial_k \varphi, \varphi) - \frac{m}{2} (\bar{\varphi}, \varphi) - \frac{\omega}{2} |\varphi|^2 \\ & - \frac{1}{4} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{J^\mu(x) J_\mu(y)}{|x-y|} dx dy . \end{aligned}$$

Our main result concerning the Maxwell-Dirac equations is the following.

Theorem 1. *For any $\omega \in (0, m)$ there exists a non-zero critical point φ_ω of I_ω in $H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$. φ_ω is a smooth function of x , exponentially decreasing at infinity together with all its derivatives. The fields $\psi(x_0, x) = e^{i\omega x_0} \varphi_\omega$, $A^\mu(x_0, x) = J_\omega^\mu * \frac{1}{|x|}$ are solutions of the Maxwell-Dirac system (M-D).*

The stationary solutions of the Klein-Gordon-Dirac equations are given by critical points $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$ of the functional

$$(1.4) \quad \begin{aligned} J_\omega(\varphi) = & \int_{\mathbb{R}^3} \frac{1}{2} (i\gamma^0 \gamma^k \partial_k \varphi, \varphi) - \frac{m}{2} (\bar{\varphi}, \varphi) - \frac{\omega}{2} |\varphi|^2 \\ & - \frac{1}{4} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{(\bar{\varphi}, \varphi)(x)(\bar{\varphi}, \varphi)(y)}{|x-y|} e^{-M|x-y|} dx dy . \end{aligned}$$

About this problem we will prove the following.

Theorem 2. *There are infinitely many critical points of J_ω for any $\omega \in (0, m)$. These critical points have the form*

$$(1.5) \quad \varphi(x) = \begin{pmatrix} v(r) & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ iu(r) & \begin{pmatrix} \cos \theta \\ e^{i\phi} \sin \theta \end{pmatrix} \end{pmatrix} ,$$

where (r, θ, ϕ) are the spherical coordinates of $x \in \mathbb{R}^3$.

When φ is a critical point of J_ω , it is a smooth function of x , and it decreases exponentially at infinity together with all its derivatives. Moreover the fields

$$(1.6) \quad \begin{cases} \psi(x_0, x) = e^{i\omega x_0} \varphi(x) \\ \chi(x_0, x) = (\bar{\varphi}, \varphi) * \frac{e^{-M|x|}}{|x|} \end{cases}$$

are stationary solutions of the Klein-Gordon-Dirac equations.

We are going to give the proof of Theorem 1, and we will only give some indications about the proof of Theorem 2 which is similar and even easier. Note that the ansatz (1.5), which simplifies the analysis, cannot be used for (M-D). This is why we obtain multiplicity for (KG-D), but only existence for (M-D). The multiplicity problem for (M-D) seems very difficult, because of a lack of compactness of the functional I_ω .

Theorem 1 is proved by a linking argument, similar to the one used for the nonlinear Dirac equation in [14]. In §2, we treat the linking argument which gives us a min-max. In §3, we study the convergence of suitably chosen Palais-Smale sequences, obtained thanks to this min-max. The combination of the results in §2,3 proves Theorem 1.

II. The linking argument.

The functional I_ω may be written as

$$(2.1) \quad \begin{aligned} I_\omega(\varphi) &= \frac{1}{2} \int (\varphi, D\varphi) - \frac{\omega}{2} \|\varphi\|_{L^2}^2 - \frac{1}{4} \int J^\mu A_\mu \\ &= \frac{1}{2} (\|P_+\varphi\|_E^2 - \|P_-\varphi\|_E^2) - \frac{\omega}{2} \|\varphi\|_{L^2}^2 - \frac{1}{4} \int J^\mu A_\mu, \end{aligned}$$

where $D = i\gamma^0\gamma^k\partial_k - m\gamma^0$, $E = H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$, $|D| = (D^2)^{1/2}$, $P_\pm = \frac{|D|\mp 1}{2} (|D| \pm D)$, $(f|g)_E = \int (f, |D|g)$, $\bar{\varphi} = \gamma^0\varphi$, $J^\mu = (\bar{\varphi}, \gamma^\mu\varphi)$, $A^\mu = J^\mu * \frac{1}{|\cdot|}$, $J^\mu A_\mu = J^0 A^0 - \sum_{k=1}^3 J^k A^k$. Let us also define the functional

$$(2.2) \quad Q(\varphi) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{(\bar{\varphi}, \varphi)(x) (\bar{\varphi}, \varphi)(y)}{|x-y|} dx dy.$$

Q is non-negative, continuous and convex on E , and vanishes only when $(\bar{\varphi}, \varphi)(x) = 0$ a.e. in \mathbb{R}^3 .

Lemma 2.1 For any $\varphi \in E$, the following inequalities hold :

$$(2.3) \quad J^\mu A_\mu(x) \geq 0 \quad , \quad \text{a.e. in } \mathbb{R}^3.$$

$$(2.4) \quad \int_{\mathbb{R}^3} J^\mu A_\mu \geq Q(\varphi)$$

$$(2.5) \quad A_0 \geq \left(\sum_{k=1}^3 |A_k|^2 \right)^{1/2}.$$

Proof. For any $\xi = (\xi^0, \xi^1, \xi^2, \xi^3) \in \mathbb{R}^4$, and $\varphi \in \mathbb{C}^4$, we have

$$(2.6) \quad \begin{aligned} & \left| \xi^0 (\bar{\varphi}, \varphi) + \sum_{k=1}^3 \xi^k (\bar{\varphi}, \gamma^k \varphi) \right|^2 \\ &= \left| \left(\bar{\varphi}, \left[\xi_0 \mathbf{1} + \sum_{k=1}^3 \xi^k \gamma^k \right] \varphi \right) \right|^2 \\ &\leq |\bar{\varphi}|_{\mathbb{C}^4}^2 \left(\varphi, \left(\xi_0 \mathbf{1} - \sum_{k=1}^3 \xi^k \gamma^k \right) \left(\xi_0 \mathbf{1} + \sum_{k=1}^3 \xi^k \gamma^k \right) \varphi \right) \\ &= |\xi|_{\mathbb{R}^4}^2 |\varphi|_{\mathbb{C}^4}^4. \end{aligned}$$

Here, we have used the formulas $(\gamma^k)^\dagger = -\gamma^k$, $(\gamma^0)^\dagger \gamma^0 = \mathbf{1}$, and $\gamma^k \gamma^\ell + \gamma^\ell \gamma^k = -2\delta_{k\ell}$, $1 \leq k, \ell \leq 3$.

As a consequence,

$$(2.7) \quad (\bar{\varphi}, \varphi)^2 + \sum_{k=1}^3 (\bar{\varphi}, \gamma^k \varphi)^2 \leq |\varphi|_{\mathbb{C}^4}^4.$$

So, taking $\varphi(x) \in E$, $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$, $\xi^0 = \pm(\bar{\varphi}, \varphi)(y)$, $\xi^k = (\bar{\varphi}, \gamma^k \varphi)(y)$, we get from (2.6) and (2.7) :

$$(2.8) \quad \begin{aligned} & \pm (\bar{\varphi}, \varphi)(y) (\bar{\varphi}, \varphi)(x) + \sum_{k=1}^3 (\bar{\varphi}, \gamma^k \varphi)(y) (\bar{\varphi}, \gamma^k \varphi)(x) \\ &\leq |\xi| |\varphi(x)|_{\mathbb{C}^4}^2 \leq |\varphi(y)|_{\mathbb{C}^4}^2 |\varphi(x)|_{\mathbb{C}^4}^2. \end{aligned}$$

Hence

$$(2.9) \quad J^\mu(x) J_\mu(y) \geq |(\bar{\varphi}, \varphi)(y) (\bar{\varphi}, \varphi)(x)|.$$

By integration, (2.3) and (2.4) follow immediately from (2.9).

Now, denoting $J = (J^1, J^2, J^3)$ and $A = (A^1, A^2, A^3)$, (2.7) implies that

$$(2.10) \quad |J(y)| \leq J^0(y) \quad \text{for all } y \in \mathbb{R}^3.$$

So, for any $x \in \mathbb{R}^3$,

$$|A(x)| = \left| \int_{\mathbb{R}^3} \frac{J(y)}{|x-y|} dy \right| \leq \int_{\mathbb{R}^3} \frac{J_0(y)}{|x-y|} dy = A_0,$$

and this is the inequality (2.5). \square

The inequality (2.3) of Lemma 2.1 is fundamental in our linking argument. It will also be used in §3 for the a priori estimates on Palais-Smale sequences.

Before starting the study of the linking, let us remark that the signs in front of $\int (i\gamma^0\gamma^k\partial_k\varphi, \varphi)$ and $m\int(\bar{\varphi}, \varphi)$ in the functional are not important. They can be changed by replacing $\varphi(x)$ by $\varphi(-x)$, and $\varphi = (\varphi_1, \varphi_2) \in \mathbb{C}^2 \times \mathbb{C}^2$ by $\bar{\varphi} = (\varphi_2, \varphi_1)$. What is important is that the signs in front of $\frac{\omega}{2}\|\varphi\|_{L^2}^2$ and $\frac{1}{4}\int J^\mu A_\mu$ are the same. We do not know if there are stationary solutions in the case $\omega \in (-m, 0)$. The importance of these signs will appear in the next Lemma, which is similar to Lemma 2.1 in [14].

Lemma 2.2. *Take $\mu > 0$. There is a non-zero function $e_+ \in E_+ = P_+E$ such that, if $P_+\varphi = e_+$, then*

$$(2.11) \quad \frac{1}{2} \int_{\mathbb{R}^3} (\varphi, D\varphi) - \frac{1}{4} Q(\varphi) \leq \frac{\mu}{2} \|\varphi\|_{L^2}^2.$$

An immediate consequence of Lemmas 2.1 and 2.2 is

Corollary 2.3. *Given $0 < \mu < m$, there is a non-zero function $e_+ \in E_+$ such that, if $\omega \in (\mu, m)$ and $P_+\varphi = e_+$, then $I_\omega(\varphi) < 0$.*

Proof of Lemma 2.2. We consider the vector $X = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{C}^4$ and a function $\theta \in C^\infty(\mathbb{R}_+, \mathbb{R})$ such that $\theta \equiv 1$ on $[0, \frac{1}{2}]$ and $\theta \equiv 0$ on $[1, +\infty)$.

We define $\phi \in E$ by its Fourier transform,

$$\hat{\phi}(\xi) = \theta(|\xi|)X.$$

Given $\lambda > 0$, let

$$\epsilon(\lambda) = \frac{P_+[\phi(\lambda x)]}{\|P_+[\phi(\lambda x)]\|_E}.$$

For any $\eta > 0$, there is $\lambda(\eta)$, small enough, such that, if $0 < \lambda < \lambda(\eta)$, then

$$(2.12) \quad m\|e(\lambda)\|_{L^2}^2 \geq m(e(\lambda), \gamma^0 e(\lambda))_{L^2} \geq 1 - \frac{\eta}{2}$$

$$(2.13) \quad (\forall \varphi_- \in P_-E) \quad |(\varphi_-, \gamma^0 e(\lambda))| \leq \frac{\eta}{2} \|\varphi_-\|_{L^2}.$$

For $\mu > 0$, we choose $\eta(\mu) > 0$ and $R(\mu) > 0$ such that

$$e_+ = R(\mu) e(\lambda \circ \eta(\mu))$$

satisfies (2.11).

Consider $\varphi = \varphi_- + e_+$. There are two possibilities :

First case : $\|\varphi_-\|_E^2 \geq R^2(1 - \frac{\mu}{2m})$. Then

$$(2.14) \quad \begin{aligned} \frac{1}{2} \int (\varphi, D\varphi) - \frac{1}{4} Q(\varphi) &\leq \frac{1}{2} (R^2 - \|\varphi_-\|_E^2) \leq \frac{\mu R^2}{4m} \\ &\leq \frac{\mu \|e_+\|_{L^2}^2}{4(1-\eta)} \leq \frac{\mu}{2} \|\varphi\|_{L^2}^2, \quad \text{for } \eta \leq \frac{1}{2}. \end{aligned}$$

Second case : $\|\varphi_-\|_E^2 \leq R^2(1 - \frac{\mu}{2m})$. Let us prove that in this case, for λ small enough,

$$(2.15) \quad Q(\varphi) \geq C(\lambda) \|\varphi\|_E^4,$$

for some constant $C(\lambda) > 0$ independent of R, φ_- .

We proceed by contradiction. Since Q is 4-homogeneous and non-negative, if (2.15) is not true then there is a sequence $\varphi_n = (\varphi_n)_- + e_+$, with $(\varphi_n)_- \in E_-$, $\|(\varphi_n)_-\|_E \leq 1$, and

$$Q(\varphi_n) \rightarrow 0.$$

After extraction, $(\varphi_n)_-$ has a weak limit φ_-^* , with $\varphi_-^* \in E_-$ and $\|\varphi_-^*\|_E \leq 1$. But the functional Q is continuous in E and convex, so it is weakly lower semi-continuous. So

$$Q(\varphi^*) = 4\pi \int_{\mathbb{R}^3} |\nabla((\bar{\varphi}^*, \varphi^*) * \frac{1}{|x|})|^2 dx = 0,$$

where $\varphi^* = \varphi_-^* + e(\lambda)$.

As a consequence, $(\bar{\varphi}^*, \varphi^*) = -4\pi\Delta((\bar{\varphi}^*, \varphi^*) * \frac{1}{|\cdot|}) = 0$ a.e. in \mathbb{R}^3 . But from (2.12) and (2.13) :

$$\begin{aligned} \int (\bar{\varphi}^*, \varphi^*) &\geq (c(\lambda), \gamma^0 e(\lambda))_{L^2} - 2|(\varphi_-^*, \gamma^0 e(\lambda))_{L^2}| - |(\varphi_-^*, \gamma^0 \varphi_-^*)_{L^2}| \\ &\geq 1 - \frac{\eta}{2} - \eta - \left(1 - \frac{\mu}{2m}\right) \geq \frac{\mu}{2m} - \frac{3\eta}{2}. \end{aligned}$$

So, for $\eta \leq \frac{\mu}{6m}$, we find a contradiction, and there is $C(\lambda)$ such that (2.15) is true.

Finally, we take $\eta(\mu) = \min(\frac{1}{2}, \frac{\mu}{6m})$ and $R(\mu) = [\frac{2}{C(\lambda(\mu))}]^{1/2}$. (2.11) immediately follows from (2.14), (2.15). \square

We now describe the linking procedure. We will actually do it for a class of functionals containing I_ω , this being useful below. For the proofs, we refer to [14].

We define

$$\mathcal{N}_- = \{\varphi = \varphi_- + \lambda c_+, \varphi_- \in E_-, \|\varphi_-\|_E \leq \|\varepsilon_+\|_E, \lambda \in [0, 1]\}$$

and

$$\partial\mathcal{N}_- = \{\varphi = \varphi_- + \lambda \varepsilon_+ \in \mathcal{N}_- / \text{either } \|\varphi_-\|_E = \|\varepsilon_+\|_E, \text{ or } \lambda = 1\},$$

with c_+ associated to $\mu > 0$ as in Lemma 2.2 and Corollary 2.3.

Now, for every $\varepsilon \geq 0$ we define $I_{\omega, \varepsilon}$, for $\varepsilon \geq 0$:

$$(2.16) \quad I_{\omega, \varepsilon}(\varphi) = I_\omega(\varphi) - \frac{2\varepsilon}{5} \|\varphi\|_{L^{5/2}}^{5/2}, \quad \varepsilon > 0, \mu < \omega < m.$$

The critical points of $I_{\omega, \varepsilon}(\varphi)$ satisfy

$$(2.17) \quad \begin{cases} i\gamma^k \partial_k \varphi - m\varphi - \omega\gamma^0 \varphi - \gamma^\mu A_\mu \varphi - \varepsilon\gamma^0 |\varphi|^{3/2} \varphi = 0 \\ -4\pi\Delta A_0 = J^0 = |\varphi|^2, \quad -4\pi\Delta A_k = -J^k. \end{cases}$$

Let θ be a smooth function satisfying $\theta(s) = 0$ for $s \leq -1$, $\theta(s) = 1$ for $s \geq 0$. The gradient being defined by $-\nabla I_{\omega, \varepsilon} = -|D|^{-1} I'_{\omega, \varepsilon}$, let us consider the flow for positive times t , $\eta_{\omega, \varepsilon}^t$, of a modified gradient :

$$(2.18) \quad \begin{cases} \eta^0 = Id_E \\ \partial_t \eta_{\omega, \varepsilon}^t = -(\theta(I_{\omega, \varepsilon}) \nabla I_{\omega, \varepsilon}) \circ \eta_{\omega, \varepsilon}^t. \end{cases}$$

Let $\rho > 0$, and $\Sigma_+ = \{\varphi \in P_+ E / \|\varphi\|_E = \rho\}$. If we fix ρ small enough, one easily sees that $I_{\omega, \varepsilon}(\varphi) \geq 0$ for any $\varphi \in E_+$ such that $\|\varphi\|_E \leq \rho$, and also $\nu = \inf_{\varphi \in \Sigma_+} I_{\omega, \varepsilon}(\varphi) > 0$.

But from Corollary 2.3, $I_\omega|_{\partial\mathcal{N}_-} < 0$, hence for $\varepsilon \geq 0$, $I_{\omega, \varepsilon}|_{\partial\mathcal{N}_-} < 0$ and

$$(2.19) \quad \eta_{\omega, \varepsilon}^t(\partial\mathcal{N}_-) \cap \Sigma_+ = \emptyset, \quad \forall t \geq 0.$$

Then, using the same degree arguments as in [14], it is possible to prove the following lemma (note that the arguments in [14] were inspired by [19]).

Lemma 2.4. *For any $0 < \mu < \omega < m$, $\varepsilon > 0$ and Σ_+ , \mathcal{N}_- constructed as above, the set $\eta_{\omega, \varepsilon}^t(\mathcal{N}_-) \cap \Sigma_+$ is non-empty, for all $t \geq 0$. Then the min-max level defined by*

$$(2.20) \quad c_{\omega, \varepsilon} = \inf_{t \geq 0} I_{\omega, \varepsilon} \circ \eta_{\omega, \varepsilon}^t(\mathcal{N}_-)$$

is strictly positive, and $c_{\omega, \varepsilon} \rightarrow c_\omega > 0$ as $\varepsilon \rightarrow 0$.

Moreover, for any ω, ε fixed, there is a sequence $(\varphi_{\omega, \varepsilon}^n)_{n \geq 0} \subset E$ such that

$$(2.21) \quad \begin{cases} I_{\omega, \varepsilon}(\varphi_{\omega, \varepsilon}^n) \xrightarrow{n \rightarrow \infty} c_{\omega, \varepsilon}, \\ (1 + \|\varphi_{\omega, \varepsilon}^n\|_E) \nabla I_{\omega, \varepsilon}(\varphi_{\omega, \varepsilon}^n) \xrightarrow{n \rightarrow \infty} 0. \end{cases}$$

III. A priori estimates on particular Palais-Smale sequences.

We start this section with a new inequality related to the quantity $A^\mu J_\mu$:

Lemma 3.1. *There is a constant $C > 0$ such that if $\psi \in \mathbb{T}^d$ and $A \in \mathbb{R} \times \mathbb{R}^3$, with $A^0 \geq (\sum_{k=1}^3 (A^k)^2)^{1/2}$,*

$$(3.1) \quad |\gamma^\mu A_\mu \psi| \leq C \sqrt{A^0} \sqrt{A^\mu J_\mu}.$$

Proof. Let $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$. Then from $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$ we get

$$\begin{cases} J^0 = |\psi|^2 = |\psi_1|^2 + |\psi_2|^2 \\ J^k = (\gamma^0 \gamma^k \psi, \psi) = 2 \operatorname{Re}(\sigma^k \psi_1, \psi_2) \quad , \quad k = 1, 2, 3. \end{cases}$$

Given any vector $\xi \in \mathbb{R}^3$, let

$$\sigma(\xi) = \sigma^1 \xi_1 + \sigma^2 \xi_2 + \sigma^3 \xi_3 ,$$

σ^i being the Pauli matrices. Note that $\sigma(\xi)$ is self-adjoint, and that $\sigma(\xi)^2 = |\xi|^2 I$. So when ξ is normalized, $\sigma(\xi)$ is unitary and involutive.

Recalling the notation $A = (A_1, A_2, A_3)$, we have

$$\begin{aligned} A_\nu J^\nu &= A_0 |\psi|^2 + A_k (\gamma^0 \gamma^k \psi, \psi) = \\ &= (A_0 - |A|) |\psi|^2 + |A| |\psi|^2 + A_k (\gamma^0 \gamma^k \psi, \psi) = \\ &= (A_0 - |A|) |\psi|^2 + |A| \left(|\psi_1|^2 + |\psi_2|^2 + 2 \operatorname{Re} \left(\sigma \left(\frac{A}{|A|} \right) \psi_1, \psi_2 \right) \right) \\ &= (A_0 - |A|) |\psi|^2 + |A| \left| \sigma \left(\frac{A}{|A|} \right) \psi_1 + \psi_2 \right|^2. \end{aligned}$$

The above computation shows that

$$(3.2) \quad A_\nu J^\nu = (A_0 - |A|) |\psi|^2 + |A| \left| \sigma \left(\frac{A}{|A|} \right) \psi_1 + \psi_2 \right|^2 \geq 0.$$

On the other hand, we have

$$(3.3) \quad \begin{aligned} \gamma^\nu A_\nu \psi &= A_0 \begin{pmatrix} \psi_1 \\ -\psi_2 \end{pmatrix} + \begin{pmatrix} \sigma(A) \psi_2 \\ -\sigma(A) \psi_1 \end{pmatrix} = \\ &= (A_0 - |A|) \begin{pmatrix} \psi_1 \\ -\psi_2 \end{pmatrix} + |A| \begin{pmatrix} \psi_1 + \sigma \left(\frac{A}{|A|} \right) \psi_2 \\ -\sigma \left(\frac{A}{|A|} \right) \psi_1 - \psi_2 \end{pmatrix} \end{aligned}$$

and (3.3) together with (3.2) implies (3.1). \square

We now prove the following a priori estimates on $\varphi_{\omega, \varepsilon}^n$, the functions defined in Lemma 2.4 :

Lemma 3.2. ω being fixed in $(0, m)$, for all $\varepsilon > 0$ there exists a positive constant $N(\varepsilon)$ such that the sequences $\varphi_{\omega, \varepsilon}^n$ of Lemma 2.4 satisfy the estimate

$$(3.4) \quad \|\varphi_{\omega, \varepsilon}^n\|_{H^{1/2}} \leq N(\varepsilon),$$

for n large enough.

Proof. From (2.21), we have

$$I_{\omega, \varepsilon}(\varphi_{\omega, \varepsilon}^n) - \frac{1}{2} (\nabla I_{\omega, \varepsilon}(\varphi_{\omega, \varepsilon}^n), \varphi_{\omega, \varepsilon}^n)_E = c_{\omega, \varepsilon} + o(1).$$

From (2.1), (2.3) and (2.16), this is equivalent to

$$(3.5) \quad \frac{1}{4} \|J^\mu A_\mu(\varphi_{\omega, \varepsilon}^n)\|_{L^1} + \frac{\varepsilon}{10} \|\varphi_{\omega, \varepsilon}^n\|_{L^{5/2}}^{5/2} = c_{\omega, \varepsilon} + o(1).$$

So $\varphi_{\omega, \varepsilon}^n$ is bounded in $L^{5/2}$. We also have

$$(3.6) \quad \varphi_{\omega, \varepsilon}^n = D^{-1}(\gamma^0 \gamma^\mu A_\mu(\varphi_{\omega, \varepsilon}^n) \varphi_{\omega, \varepsilon}^n) + \varepsilon D^{-1} |\varphi_{\omega, \varepsilon}^n|^{\frac{1}{2}} \varphi_{\omega, \varepsilon}^n + r_n,$$

with $\|r_n\|_E \rightarrow 0$. From (3.5), $|\varphi_{\omega, \varepsilon}^n|^{\frac{1}{2}} \varphi_{\omega, \varepsilon}^n$ is bounded in $L^{5/3}$.

Moreover, $A^0 \geq A^k$ for $k = 1, 2, 3$, and

$$\|A_0\|_{L^{15/2}} = \left\| |\varphi|^2 * \frac{1}{|x|} \right\|_{L^{15/2}} \leq C \|\varphi\|_{L^{5/4}}^2 = C(\|\varphi\|_{L^{5/2}})^2.$$

So, by Hölder's inequality,

$$(3.7) \quad \|\gamma^\mu A_\mu \varphi\|_{L^{15/8}} \leq C' \|\varphi\|_{L^{5/2}}^3.$$

Now, the operator D^{-1} sends L^p spaces into $W^{1,p}$ spaces, so (3.6) implies that $\varphi_{\omega, \varepsilon}^n$ is bounded in $W^{1, 15/8} + W^{1, 5/3} + E$. Using the Sobolev embedding

$$W^{1,p}(\mathbb{R}^3) \subset H^{1/2}(\mathbb{R}^3), \quad \frac{3}{2} \leq p \leq 2,$$

we get (3.4). \square

We now define two kinds of norms that will be useful below.

Definition 3.3. • For any $q \geq 1$ and $\varphi = \mathbb{R}^3 \rightarrow B$ measurable, where $(B, |\cdot|)$ is a Banach space we write

$$(3.8) \quad U_q(\varphi) = \sup_{\ell \in \mathbb{Z}^3} \left(\int_{Q_\ell} |\varphi(x)|^q dx \right)^{\frac{1}{q}},$$

where $Q_\ell = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \ell_i \leq x_i \leq \ell_i + 1, i = 1, 2, 3\}$. U_q is a norm on the Banach space "uniform $L_{\text{loc}}^q(\mathbb{R}^3, B)$ ". Clearly, $U_q \leq \|\cdot\|_{L^q}$.

• For any $q \geq 1$ and $\varphi = \mathbb{R}^3 \rightarrow B$ measurable, let us define

$$(3.9) \quad N_q(\varphi) = \left(\sum_{\ell \in \mathbb{Z}^3} \left(\int_{Q_\ell} |\varphi(x)|^q dx \right)^{\frac{2}{q}} \right)^{\frac{1}{2}}.$$

N_q is a norm in the Banach space of functions on which N_q is finite. Note that $N_2 = \|\cdot\|_{L^2(\mathbb{R}^3, B)}$.

Let us now obtain estimates for critical points of the functionals $I_{\omega, \varepsilon}$. We begin with

Lemma 3.4. *If φ is a solution of $I'_{\omega, \varepsilon}(\varphi) = 0$, then*

$$(3.10) \quad I_{\omega, \varepsilon}(\varphi) = \frac{1}{4} \|J^\mu A_\mu(\varphi)\|_{L^1} + \frac{\varepsilon}{10} \|\varphi\|_{L^{5/2}}^{5/2},$$

and

$$(3.11) \quad \begin{aligned} \int_{\mathbb{R}^3} (i\gamma^0 \gamma^k \partial_k \varphi_\varepsilon, \varphi_\varepsilon) dx &= \\ &= \frac{3}{2} \int_{\mathbb{R}^3} \left(m \overline{\varphi_\varepsilon} \varphi_\varepsilon + \omega |\varphi_\varepsilon|^2 + \frac{4}{5} \varepsilon |\varphi_\varepsilon|^{5/2} + \frac{5}{6} A^\mu J_\mu \right) dx. \end{aligned}$$

Proof.

These formulas are standard in the theory of variational problems.

(3.10) is equivalent to the obvious equality $I_{\omega, \varepsilon}(\varphi) = I_{\omega, \varepsilon}(\varphi) - I'_{\omega, \varepsilon}(\varphi) \cdot \varphi$.

(3.11) is a ‘‘Pohozaev identity’’. To obtain it, one has to multiply the equation

$$D\varphi - \omega\varphi - \varepsilon|\varphi|^{1/2}\varphi - \gamma^0 \gamma^\mu \left(J^\mu \star \frac{1}{|x|} \right) \varphi = 0$$

by $x \cdot \nabla \varphi$, and then integrate it by parts (see the details of a similar proof in [14], Proposition 3.1). \square

Let us give another technical lemma :

Lemma 3.5. *Let $\omega \in (0, m)$. Then there exists a constant $K > 0$, depending only on ω, m , such that for any $\varepsilon \in [0, 1]$, $\varphi \in E$, $\Omega \in E'$, and any solution $\varphi \in E$ of*

$$(3.12) \quad D\varphi - \omega\varphi - \gamma^0 \gamma^\mu A_\mu(\varphi)\varphi - \varepsilon|\varphi|^{\frac{1}{2}}\varphi = \Omega,$$

the following inequality holds, with the notation $\psi = \varphi - (D - \omega I)^{-1} \Omega$.

$$(3.13) \quad \begin{aligned} N_{15/8}(\psi) + N_{15/8}(\nabla \psi) \leq \\ K \left(\sqrt{U_{15}(A_0(\varphi)) \|A_\mu J^\mu(\varphi)\|_{L^1(\mathbb{R}^3)}} + \varepsilon \sqrt{U_{5/2}(\varphi)} N_3(\varphi) \right). \end{aligned}$$

As a consequence, there are two constants $\kappa, C > 0$, independent of ε , such that, if

$$(3.14) \quad \varepsilon^2 U_{5/2}(\varphi) + U_{15}(A_0(\varphi)) \leq \kappa,$$

or if

$$(3.15) \quad \varepsilon^2 U_{5/2}(\varphi) + \|A_\mu J^\mu(\varphi)\|_{L^1} \leq \kappa,$$

then

$$(3.16) \quad \|\varphi\|_E \leq C \|\Omega\|_{E'}.$$

Moreover, if $\Omega = 0$ and $\varphi \neq 0$, then

$$(3.17) \quad I_{\omega, \varepsilon}(\varphi) \geq \frac{\kappa}{4}, \quad \|\varphi\|_E \geq \left(\frac{\kappa}{2} \right)^{\frac{1}{2}}.$$

Proof.

We write $\psi_1 = (D - \omega I)^{-1}(\gamma^0 \gamma^\mu A_\mu \varphi)$, $\psi_2 = (D - \omega I)^{-1}(|\varphi|^{1/2} \varphi)$. Then, $\psi = \psi_1 + \varepsilon \psi_2$. By the same kind of arguments as in the proof of lemma 3.2, we find, for $\frac{1}{q} = \frac{1}{5} + \frac{1}{p}$, $p \geq 1$, the following estimate:

$$(3.18) \quad N_q(\psi_2) + N_q(\nabla \psi_2) \leq C U_{5/2}(\varphi)^{1/2} N_p(\varphi).$$

For reasons that will appear later, we are interested in exponents p, q such that $E \subset L^2 \cap L^p$, and $W^{1, q} \subset H_{\text{loc}}^{1/2}$, the second Sobolev inclusion being compact. We take $p = 3$ and $q = 15/8$, but this is not the only possible choice.

Using (3.1), we obtain, for $r \geq 1$ and $\frac{1}{q} = \frac{1}{2} + \frac{1}{2r}$,

$$(3.19) \quad N_q(\psi_1) + N_q(\nabla \psi_1) \leq C' \sqrt{U_r(A_0)} \|J_\mu A^\mu\|_{L^1(\mathbb{R}^3)}.$$

The choice $q = 15/8$ leads to $r = 15$.

Combining (3.18) and (3.19), we immediately get (3.13).

Now, if $\varphi \in E$, then $A_0 = |\varphi|^2 * \frac{1}{|\cdot|}$ is in $L^q(\mathbb{R}^3)$ for all $q > \frac{3}{2}$, and in particular $\sqrt{U_\varepsilon(A_0)} \leq C\|\varphi\|_E$. We also have $N_3(\varphi) \leq C\|\varphi\|_E$ and $\|\psi\|_E \leq N_{15/8}(\psi) + N_{15/8}(\nabla\psi)$. If (3.15) is true, (3.16) follows from these estimates combined with (3.13).

Now, using (3.10) and the obvious estimate $I_{\varepsilon,\omega} \leq \frac{1}{2}\|\varphi\|_E^2$, one finds $\|A_\mu J^\mu\|_{L^1} \leq 2\|\varphi\|_E^2$. All this, together with (3.14) and the Sobolev inequalities above, implies (3.16).

Finally, from (3.10) and the fact that (3.15) implies (3.16), we get (3.17). \square

Let us now prove a result on the behavior of bounded critical sequences of the functionals $I_{\omega,0}$ and $I_{\omega,\varepsilon}$. This will be extremely useful for obtaining critical points of these functionals. In some sense, here we will deal with the possible losses of compactness of these sequences.

Proposition 3.6. *Let $\omega \in (0, m)$ and $\varepsilon \geq 0$ be fixed. Let $\{\varphi_n\}_{n \in \mathbb{N}} \subset E$ be a sequence in E such that*

$$(3.20) \quad 0 < \inf_n \|\varphi_n\|_E \leq \sup_n \|\varphi_n\|_E < +\infty$$

and $I'_{\omega,\varepsilon}(\varphi_n) \xrightarrow{n \rightarrow +\infty} 0$ in E' . Then we can find a finite integer $p \geq 1$, p non-zero solutions ϕ^1, \dots, ϕ^p of (2.17) in E and p sequences $\{x_n^i\}_{n \in \mathbb{N}} \subset \mathbb{R}^3$, $i = 1, \dots, p$ such that for $i \neq j$, $|x_n^i - x_n^j| \xrightarrow{n \rightarrow +\infty} +\infty$, and, up to extraction of a subsequence,

$$(3.21) \quad \left\| \varphi_n - \sum_{i=1}^p \phi^i(\cdot - x_n^i) \right\|_E \xrightarrow{n \rightarrow +\infty} 0.$$

Proof. The existence of ϕ^1, \dots, ϕ^p follows from a classical concentration-compactness argument (see [25]) applied to the concentration sequence

$$\rho_n = |\varphi_n|^2 + |\varphi_n|^3 + |A_n^0|^{15},$$

with $A_n^0 = |\varphi_n|^2 * \frac{1}{|\cdot|}$. The key point is that, since (3.14) implies (3.16), vanishing cannot occur.

Note that from (3.17), $p \leq \left(\frac{2}{\kappa}\right)^{\frac{1}{2}} \sup_n \|\varphi_n\|_E$. \square

We have now all we need to prove the existence of solutions for the problem (2.17) when $\varepsilon > 0$. This will provide us with approximate solutions for the exact Maxwell-Dirac problem, i.e. (2.17) with $\varepsilon = 0$.

Lemma 3.7. *Let $0 < \omega < m$ and $0 < \varepsilon \leq 1$. Then there is $\varphi_\varepsilon \in E$ such that $I'_{\omega,\varepsilon}(\varphi_\varepsilon) = 0$ and*

$$(3.22) \quad \frac{\kappa}{4} \leq I_{\omega,\varepsilon}(\varphi_\varepsilon) \leq c_{\omega,\varepsilon},$$

where κ is the same as in Lemma 3.5.

Proof. Just apply Lemma 3.2 and Proposition 3.6 to the sequence $(\varphi_{\omega,\varepsilon}^n)_{n \geq 0}$ of Lemma 2.4. As function φ_ε , take one of the functions ϕ^i obtained from Proposition 3.6. From (3.17), for any i we have $I_{\omega,\varepsilon}(\phi^i) \geq \frac{\kappa}{4}$. From (3.21),

$$\sum_{i=1}^p I_{\omega,\varepsilon}(\phi^i) = c_{\omega,\varepsilon}.$$

The inequality (3.22) is thus satisfied. \square

Proof of Theorem 1. Let us consider the sequence $\varphi_{1/n}$ defined by Lemma 3.7, for $\varepsilon = 1/n$. Our aim is to prove that, passing to the limit in this sequence with the help of Proposition 3.6 (case $\varepsilon = 0$), we can get a non-zero solution of the Maxwell-Dirac system, of the form (1.1).

We know that $\varphi_{1/n}$ is an exact solution of (2.17), which implies that $\|\varphi_{1/n}\|_E \geq \left(\frac{\kappa}{2}\right)^{\frac{1}{2}}$. Moreover, (2.17) may be written as

$$I'_\omega(\varphi_{1/n}) = \frac{1}{n} |\varphi_{1/n}|^{1/2} \varphi_{1/n}.$$

So to check that $\varphi_{1/n}$ satisfies the hypotheses of Proposition 3.6 (for $\varepsilon = 0$), we just need to find an upper estimate on $\|\varphi_{1/n}\|_E$. This will end the proof of Theorem 1.

First of all, from Lemma 2.4, $c_{\omega,1/n} \rightarrow c_\omega > 0$. Hence, $I_{\omega,1/n} \leq C$. Then, the representation formula (2.1), together with (3.10) and (3.11) yield the existence of $C > 0$, independent of n , such that

$$(3.23) \quad \left| \int_{\mathbb{R}^3} m \bar{\varphi}_{1/n} \varphi_{1/n} + \omega |\varphi_{1/n}|^2 dx \right| \leq C.$$

We now prove that $\|\varphi_{1/n}\|_E$ is uniformly bounded. Assume that this does not hold, i.e., $\|\varphi_{1/n}\|_E \xrightarrow{n \rightarrow +\infty} +\infty$. Then define the normalized function

$$\tilde{\varphi}_n = \|\varphi_{1/n}\|_E^{-1} \varphi_{1/n}.$$

This function satisfies

$$(3.24) \quad D\tilde{\varphi}_n - \omega\tilde{\varphi}_n - \gamma^0\gamma^\mu A_\mu(\varphi_{1/n})\tilde{\varphi}_n - \frac{1}{n}|\varphi_{1/n}|^{1/2}\tilde{\varphi}_n = 0.$$

Now we use inequality (3.13) and $U_{15}(A_0(\varphi)) \leq C\|\varphi\|_E^2$ to obtain

$$\begin{aligned} & N_{\frac{15}{8}}(\tilde{\varphi}_n) + N_{\frac{15}{8}}(\nabla\tilde{\varphi}_n) \\ & \leq \frac{K}{n}U_{5/2}(\varphi_{1/n})^{1/2}\|\tilde{\varphi}_n\|_E + KU_{15}(A^0(\tilde{\varphi}_n))^{1/2}\|A^\mu J_\mu(\varphi_{1/n})\|_{L^1(\mathbb{R}^3)}^{1/2} \\ & \leq \frac{K}{n}\sqrt{U_{5/2}(\varphi_{1/n})} + K'\sqrt{\|A^\mu J_\mu(\varphi_{1/n})\|_{L^1(\mathbb{R}^3)}}. \end{aligned}$$

This inequality, combined with (3.10) and Lemma 2.4, implies that

$$N_{\frac{15}{8}}(\tilde{\varphi}_n) + N_{\frac{15}{8}}(\nabla\tilde{\varphi}_n) \leq C$$

for some constant $C > 0$ independent of n . The sequence $\{\tilde{\varphi}_n\}$ is thus relatively compact in $H_{loc}^{1/2}(\mathbb{R}^3)$.

Let us now apply a classical concentration-compactness argument to the sequence

$$(3.25) \quad \rho_n = |\tilde{\varphi}_n|^2 + |\tilde{\varphi}_n|^3 + |A^0(\tilde{\varphi}_n)|^{15} + \frac{1}{n}|\varphi_{1/n}|^{5/2} + J^\mu A_\mu(\varphi_{1/n}).$$

That vanishing cannot occur follows from (3.15) in Lemma 3.5. Then, by the relative compactness of the sequence $\{\tilde{\varphi}_n\}$ in E , we infer the existence of $\psi_1, \dots, \psi_q \in E$ and $y_i^n, i = 1, \dots, q, (q \leq +\infty)$, with $|y_i^n - y_j^n| \xrightarrow{n \rightarrow +\infty} +\infty$ if $i \neq j$, such that, after extraction,

$$(3.26) \quad \left\| \tilde{\varphi}_n - \sum_{i=1}^q \psi_i(\cdot - y_i^n) \right\|_E \xrightarrow{n \rightarrow +\infty} 0.$$

Since $\int A_\mu^n J_\mu^n dx$ is uniformly bounded,

$$(3.27) \quad \begin{aligned} 0 &= \lim_{n \rightarrow +\infty} \int \tilde{A}_n^\mu \tilde{J}_\mu^n dx = \\ &= \lim_{n \rightarrow +\infty} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{(\tilde{\varphi}_n, \gamma^\mu \tilde{\varphi}_n)(x) (\tilde{\varphi}_n, \gamma^\mu \tilde{\varphi}_n)(y)}{|x-y|} dx dy. \end{aligned}$$

From (3.26) and (3.27) we easily deduce that

$$(3.28) \quad 0 = \sum_{i=1}^q \iint \frac{(\tilde{\psi}_i, \gamma^\mu \psi_i)(x) (\tilde{\psi}_i, \gamma^\mu \psi_i)(y)}{|x-y|} dx dy.$$

From (3.23) and (3.26) we obtain

$$(3.29) \quad \sum_{i=1}^q \int_{\mathbb{R}^3} m\tilde{\psi}_i \psi_i + \omega|\psi_i|^2 = 0.$$

Now, from (2.2), (2.4) and (3.28), we see that for $i = 1, \dots, q$

$$\iint \frac{(\tilde{\psi}_i, \psi_i)(x) (\tilde{\psi}_i, \psi_i)(y)}{|x-y|} dx dy = 4\pi \int \left| \nabla \left((\tilde{\psi}_i, \psi_i) * \frac{1}{|x|} \right) \right|^2 dx = 0.$$

This implies that $(\tilde{\psi}_i, \psi_i) = -4\pi\Delta((\tilde{\psi}_i, \psi_i) * \frac{1}{|x|}) = 0$ a.e. in \mathbb{R}^3 . Using this fact in (3.29), we find $\psi_i = 0$ a.e. in \mathbb{R}^3 for $i = 1, \dots, q$. But, recalling that $\|\tilde{\varphi}_n\|_E = 1$, we deduce from (3.26) the formula

$$(3.30) \quad \sum_{i=1}^q \|\psi_i\|_E = 1.$$

This is a contradiction. \square

Sketch of proof of Theorem 2. Let us first consider the existence problem. The linking argument introduced in Section II to deal with Maxwell-Dirac equations can be applied with almost no changes to deal with the Klein-Gordon-Dirac system. This provides us with a sequence $(\varphi_{\omega,\varepsilon}^n)_{n \geq 0}$ such that for all $\varepsilon \geq 0$,

$$\begin{aligned} J_{\omega,\varepsilon}(\varphi_{\omega,\varepsilon}^n) &\xrightarrow{n \rightarrow \infty} c_{\omega,\varepsilon}, \\ (1 + \|\varphi_{\omega,\varepsilon}^n\|_E) \nabla J_{\omega,\varepsilon}(\varphi_{\omega,\varepsilon}^n) &\xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

with $c_{\omega,\varepsilon} \rightarrow_{\varepsilon \rightarrow 0^+} c_\omega > 0$, $J_{\omega,\varepsilon}(\varphi) = J_\omega(\varphi) - \frac{2\varepsilon}{5} \|\varphi\|_{L^{5/2}}^{5/2}$.

All the intermediate lemmata proved to achieve the proof of Theorem 1 can be carried out in this case in a similar way. However some changes have to be introduced. Let us enumerate and justify the most important ones.

Inequality (3.13) has to be replaced by the simpler one

$$N_{15/8}(\psi) + N_{15/8}(\nabla\psi) \leq K \left(U_6(\lambda) N_{30/11}(\varphi) + \varepsilon \sqrt{U_{5/2}(\varphi)} N_3(\varphi) \right)$$

for all $\varphi \in E$ satisfying

$$(3.12') \quad D\varphi - \omega\varphi - \lambda|\varphi| - \varepsilon|\varphi|^{5/2} = \Omega,$$

and for $\psi = \varphi - (D - \omega I)^{-1}\Omega$.

Moreover, in order to have the equivalent of (3.23) one has to note the following two changes : the identity (3.11) is now modified to yield

$$\begin{aligned} & \int_{\mathbb{R}^3} (i\gamma^0 \gamma^k \partial_k \varphi_\varepsilon, \varphi_\varepsilon) dx - \frac{3}{2} \int_{\mathbb{R}^3} (m\bar{\varphi}_\varepsilon \varphi_\varepsilon + \omega|\varphi_\varepsilon|^2 + \frac{4}{5} \varepsilon |\varphi_\varepsilon|^{5/2}) dx = \\ & = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (\bar{\varphi}_\varepsilon, \varphi_\varepsilon)(x) (\bar{\varphi}_\varepsilon, \varphi_\varepsilon)(y) e^{-M|x-y|} \left(\frac{5}{4|x-y|} - \frac{M}{4} \right) dx dy. \end{aligned}$$

Now, in order to use this to obtain the equivalent of (3.23) we remark that for all $f \in L^2(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ one has

$$M \iint_{\mathbb{R}^3 \times \mathbb{R}^3} f(x)f(y)e^{-M|x-y|} dx dy \leq 2 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{f(x)f(y)}{|x-y|} e^{-M|x-y|} dx dy.$$

Indeed, if $g = (-\Delta + M^2)^{-1}f$, we have

$$\begin{aligned} \iint \frac{f(x)f(y)}{|x-y|} e^{-M|x-y|} dx dy &= \frac{1}{4\pi} \int_{\mathbb{R}^3} (|\nabla g|^2 + M^2 g^2) \\ &\geq \frac{M^2}{4\pi} \int g^2 = \frac{M^2}{4\pi} (f, (-\Delta + M^2)^{-2} f)_{L^2} \\ &= \frac{M}{2} \iint f(x)f(y)e^{-M|x-y|} dx dy. \end{aligned}$$

The last equality comes from $(-\Delta + M^2)e^{-M|x|} = 2M \frac{e^{-M|x|}}{|x|}$.

To obtain the multiplicity result, we may work with J_ω restricted to the space E_* of functions of the form (1.5). This gives much more compactness to the functional. For instance, we always have $p = 1$ in (3.21) of Proposition 3.6, because the ansatz (1.5) breaks the translation invariance. Then we proceed as in [14], proof of Theorem 1. \square

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