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Affine Lie Algebraic Origin of Constrained KP Hierarchies

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ABSTRACT

We present an affine $sl(n+1)$ algebraic construction of the basic constrained KP hierarchy. This hierarchy is analyzed using two approaches, namely linear matrix eigenvalue problem on hermitian symmetric space and constrained KP Lax formulation and we show that these approaches are equivalent.

The model is recognized to be the generalized non-linear Schrödinger (GNLS) hierarchy and it is used as a building block for a new class of constrained KP hierarchies. These constrained KP hierarchies are connected via similarity-Bäcklund transformations and interpolate between GNLS and multi-boson KP-Toda hierarchies. Our construction uncovers origin of the Toda lattice structure behind the latter hierarchy.

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ABSTRACT

We present an affine $sl(n+1)$ algebraic construction of the basic constrained KP hierarchy. This hierarchy is analyzed using two approaches, namely linear matrix eigenvalue problem on hermitian symmetric space and constrained KP Lax formulation and we show that these approaches are equivalent.

The model is recognized to be the generalized non-linear Schrödinger (GNLS) hierarchy and it is used as a building block for a new class of constrained KP hierarchies. These constrained KP hierarchies are connected via similarity-Bäcklund transformations and interpolate between GNLS and multi-boson KP-Toda hierarchies. Our construction uncovers origin of the Toda lattice structure behind the latter hierarchy.

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1 Introduction

This paper examines a class of the constrained KP hierarchies and their mutual relations, both from the matrix and pseudo-differential point of view. The constrained KP hierarchy structure we uncover can be denoted (for n pairs of bosonic fields) by a symbol $\mathsf{CKP}_n^{(k)}$ with index k ranging from 0 to n and is associated with the Lax operators

$$
\begin{aligned}
L_n^{(k)} &\equiv \partial + a_1^{(k)}\left(\partial - S_k^{(k)}\right)^{-1} + a_2^{(k)}\left(\partial - S_{k-1}^{(k-1)}\right)^{-1}\left(\partial - S_k^{(k)}\right)^{-1} + \ldots \\
&+ a_k^{(k)}\left(\partial - S_1^{(1)}\right)^{-1}\cdots\left(\partial - S_k^{(k)}\right)^{-1} + \sum_{i=k+1}^{n} a_i^{(k)}(\partial - S_i)^{-1}\left(\partial - S_1^{(1)}\right)^{-1}\cdots\left(\partial - S_k^{(k)}\right)^{-1}
\end{aligned}
\tag{1.1}
$$

all satisfying Sato's KP evolution equations. The basic model is the one corresponding to $k=0$ with the Lax operator [1]:

$$
L_n \equiv L_n^{(0)} = \partial + \sum_{i=1}^{n} a_i(\partial - S_i)^{-1} \tag{1.2}
$$

and the remaining models with $k \geq 1$ can be derived from it via the similarity gauge transformations at the Lax operator level. Because of the form of the Lax operator (1.2) and its abelian first bracket structure we call this $\mathsf{CKP}_n^{(0)}$ model the n-generalized two-boson KP hierarchy. We identify it, at the matrix hierarchy level, with the $sl(n+1)$ generalized non-linear Schrödinger (GNLS) hierarchy [2]. Therefore this model can also be denoted as GNLS_n. We discuss GNLS_n model using the structure of the underlying hermitian symmetric space and derive its recursion operator as well as its Hamiltonian structure. We also provide an affine Lie algebraic foundation for GNLS_n hierarchy by fitting it into the AKS $\hat{sl}(n+1)$ framework [3], generalizing the work of [4] for the AKNS model. In view of the similarity relations connecting this basic $\mathsf{CKP}_n^{(0)}$ model and all the other constrained KP models our construction uncovers the $sl(n+1)$ structure behind all the $\mathsf{CKP}_n^{(k)}$ hierarchies.

The other extreme of the sequence $\mathsf{CKP}_n^{(k)}$ is the important case of $k=n$ in which we recognize the n-boson KP-Toda hierarchy [5, 6], which recently appeared in a study of Toda hierarchies and matrix models [7, 8]. For one bose pair with $n=1$ there is only one model, namely the two-boson KP hierarchy [9, 10] or $\mathsf{CKP}_1^{(0)}$, equivalent to the AKNS model.

In section 2 we recapitulate the Zakharov-Shabat-AKNS (ZS-AKNS) scheme in the framework of the hermitian symmetric spaces. We work with the algebra $\mathcal{G}$ containing an element $E = 2\mu_n \cdot H/\alpha_n^2$, which allows the decomposition $\mathcal{G} = \mathrm{Ker}(\mathrm{ad}E) \oplus \mathrm{Im}(\mathrm{ad}E)$.

We derive a general formula for the recursion and the Hamiltonian operators for a particular case where the linear spectral problem $\lambda E + A^0\Psi = \partial\Psi$ defining ZS-AKNS scheme involves a matrix $A^0 \in \mathrm{Ker}(\mathrm{ad}E)$. For $sl(n+1)$ A^0 can be written in the matrix form as

$$
A^0 = \begin{pmatrix}
0 & \cdots & 0 & \cdots & q_1 \\
0 & 0 & \cdots & 0 & q_2 \\
0 & & \ddots & & \vdots \\
\vdots & & & \ddots & q_n \\
r_1 & r_2 & \cdots & r_n & 0
\end{pmatrix} \tag{1.3}
$$

In this case we deal with the GNLS_n hierarchy.

We also analyze the ZS-AKNS formalism in case of $sl(n+1)$ algebra connected with the linear spectral problem with A^0:

$$A^0 = \begin{pmatrix} 0 & q_1 & 0 & \dots & 0 \\ 0 & 0 & q_2 & \dots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & & & \ddots & q_n \\ r_1 & r_2 & \cdots & r_n & 0 \end{pmatrix} \tag{1.4}$$

which belongs this time to $\mathrm{Ker}(\mathrm{ad}E) \oplus \mathrm{Im}(\mathrm{ad}E)$. We give a prescription how to obtain the recursion operator in this case, shown in Appendix B (for $n = 2$) to be equivalent to the multi-boson KP-Toda hierarchy recursion operator.

In section 3 we derive the equations of motion for the GNLS_n model using AKS approach [3]. This provides the Lie algebraic foundation for this model, which will turn out to be a basic model of the constrained KP hierarchy. In section 4 we prove equivalence of ZS-AKNS matrix and KP Lax formulations for the GNLS_n hierarchy. Our proof shows that respective recursion operators coincide leading to identical flow equations for both hierarchies.

In section 5 we construct the general $\mathsf{CKP}_n^{(k)}$ models in terms of the GNLS_n hierarchy. We base the construction on the similarity-Bäcklund transformation. The structure of this transformation ensures that all encountered models satisfy Sato's KP flow equation. In particular we obtain a proof for the multi-boson KP-Toda model being the constrained KP hierarchy and recover its discrete Toda structure [11].

In Appendix A we collect few Lie algebraic definitions and properties.

2 ZS-AKNS Scheme and Recursion Operators

Consider the linear matrix problem:

$$A\Psi = \partial\Psi \tag{2.1}$$

$$B_m\Psi = \partial_{t_m}\Psi \qquad m = 2,3,\dots \tag{2.2}$$

for

$$A = \lambda E + A^0 \qquad \text{with} \qquad E = \frac{2\mu_a \cdot H}{\alpha_a^2} \tag{2.3}$$

where μ_a is a fundamental weight and α_a are simple roots of $\mathcal{G}$. The element E is used to decompose the Lie algebra $\mathcal{G}$ as follows:

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{M} = \mathrm{Ker}(\mathrm{ad}E) \oplus \mathrm{Im}(\mathrm{ad}E) \tag{2.4}$$

where the $\mathcal{K} \equiv \mathrm{Ker}(\mathrm{ad}E)$ has the form $\mathcal{K}' \times u(1)$ and is spanned by the Cartan subalgebra of $\mathcal{G}$ and step operators associated to roots not containing α_a. Moreover $\mathcal{M} \equiv \mathrm{Im}(\mathrm{ad}E)$ is the orthogonal complement of $\mathcal{K}$.

In fact, if $2\mu_a.\alpha/\alpha_a^2 = \pm 1, 0$, the above decomposition defines an hermitian symmetric space $\mathcal{G}/\mathcal{K}$ [2]. This special choice of E will play a crucial role in what follows.

The compatibility condition for the linear problem (2.1) leads to the Zakharov-Shabat (Z-S) integrability equations

$$\partial_m A - \partial B_m + [A\,,\,B_m] = 0 \quad ; \quad \partial \equiv \partial_x \quad \partial_{t_m} \equiv \partial_m \tag{2.5}$$

Inserting the decomposition of A from (2.3) into (2.5) we get

$$\partial_m A^0 - \partial B_m + \lambda[E\,,\,B_m] + [A^0\,,\,B_m] = 0 \tag{2.6}$$

We search for solutions of (2.6) of the form

$$B_m = \sum_{i=0}^{m} \lambda^i B_m^i \tag{2.7}$$

To determine the coefficients of the expansion in (2.6) we first find from (2.7) the following identity:

$$\partial_m A^0 - \lambda \partial_{m-1} A^0 + \partial X(\lambda) + \lambda[E\,,\,X(\lambda)] + [A^0\,,\,X(\lambda)] = 0 \tag{2.8}$$

where $X(\lambda) = B_m - \lambda B_{m-1}$. By comparing powers of λ in (2.8) we can split $X(\lambda)$ as

$$X(\lambda) = O_m(1) + Y_m(\lambda) \quad ; \quad Y_m(\lambda) \in \mathcal{K} \tag{2.9}$$

where $O_m(1)$ is an arbitrary element of algebra $\mathcal{G}$ independent of λ. Therefore, all dependence on λ is contained in $Y_m(\lambda)$, an element of $\mathcal{K}$.

We now consider two distinct cases defined according to whether A^0 is entirely in $\mathcal{M}$ or not.

2.1 $\mathbf{A^0 \in Im(adE)}$

Since now $A^0 \in \mathcal{M}$ equation (2.8) yields for the λ dependent element Y_m in $\mathcal{K}$ the condition $\partial Y_m(\lambda) = 0$ and hence this component of $X(\lambda)$ can be chosen to vanish. With this choice, equation (2.9) becomes a so-called congruence relation:

$$B_m = \lambda B_{m-1} + O_m(1) = \sum_{i=0}^{m} O_{m-i}(1)\lambda^i \tag{2.10}$$

Plugging (2.10) into (2.6) we find that the coefficients $O_j(1)$ with $1 \le j \le m-1$ satisfy the recursion relation

$$\partial O_j(1) - \left[A^{(0)}\,,\,O_j(1)\right] = [E\,,\,O_{j+1}(1)] \tag{2.11}$$

There is a compact way to rewrite all these recurrence relations for all m. It is provided by a single equation:

$$\partial B = \left[\lambda E + A^{(0)}\,,\,B\right] \tag{2.12}$$

where B is now an infinite series:

$$B \equiv \sum_{j=0}^{\infty} O_j(1)\lambda^{-j} \tag{2.13}$$

Since $Y_m = 0$, the equation (2.8) splits into two simple expressions for coefficients of λ^i, $i = 0, 1$

$$
\begin{aligned}
\partial_{m-1} A^0 &= [E, O_m(1)] && (2.14)\\
\partial_m A^0 &= \partial O_m(1) - [A^0, O_m(1)] && (2.15)
\end{aligned}
$$

Equation (2.14) can easily be inverted to yield expression for $O_m^{\mathcal{M}}(1)$.

From the relation (A.5) the solution of (2.14) is

$$O_m^{\mathcal{M}}(1) = [E, \partial_{m-1} A^0] \tag{2.16}$$

From (2.15) we find now

$$\partial O_m^{\mathcal{K}}(1) = [A^0, O_m^{\mathcal{M}}(1)] \quad \rightarrow \quad O_m^{\mathcal{K}}(1) = \partial^{-1}[A^0, [E, \partial_{m-1} A^0]] \tag{2.17}$$

displaying the non-local character of $O_m^{\mathcal{K}}(1)$. Plugging now these two results into (2.15) we find

$$\partial_m A^0 = [E, \partial\partial_{m-1} A^0] - [A^0, \partial^{-1}[A^0, [E, \partial_{m-1} A^0]]] = \mathcal{R}\partial_{m-1} A^0 \tag{2.18}$$

where we have defined a recursion operator [2]

$$\mathcal{R} \equiv \left(\partial - ad_{A^0}\, \partial^{-1} ad_{A^0}\right) ad_E \tag{2.19}$$

Since $\mathcal{M}$ is spanned by $2n$ step operators associated to roots containing α_a only once, i.e. $\alpha = \alpha_a + \ldots$ we denote $E_{\pm(\alpha_a + \ldots)} = E_{\pm}^a$, where $a = 1, \ldots, n$. We thus parametrize this model by

$$A^{(0)} = \sum_{a=1}^{n} \left(q_a E_+^a + r_a E_-^a\right) \tag{2.20}$$

With this parametrization we have

$$
\begin{aligned}
&\partial_n \begin{pmatrix} r_i \\ q_l \end{pmatrix} = \mathcal{R}_{(i,l),(j,m)} \begin{pmatrix} r_j \\ q_m \end{pmatrix} = && (2.21)\\
&\begin{pmatrix} -\partial + r_s \partial^{-1} q_k R^{i*}_{sj-k} & r_s \partial^{-1} r_k R^{i*}_{sk-m} \\ -q_s \partial^{-1} q_k R^l_{sk-j} & \partial - q_s \partial^{-1} r_k R^l_{sm-.} \end{pmatrix} \partial_{n-1} \begin{pmatrix} r_j \\ q_m \end{pmatrix}
\end{aligned}
$$

where R^l_{sm-k} and R^{i*} are defined in (A.8). We now find conserved Hamiltonians and the first Hamiltonian bracket structure associated to the Z-S equations (2.5) in the Hamiltonian form:

$$\partial_m A^0 = \{ H_m, A^0 \} \tag{2.22}$$

Lemma: $\partial_n \operatorname{Tr}\left(\mathbf{A^{0^2}}\right) = -2\partial \operatorname{Tr}(\mathbf{E O_{n+1}(1)})$

The proof follows from the following calculation:

$$
\begin{aligned}
\partial_n \operatorname{Tr}\left(A^0 A^0\right) &= 2 \operatorname{Tr}\left([E, [E, A^0]]\partial_n A^0\right) = 2 \operatorname{Tr}\left(E[A^0, [\partial_n A^0, E]]\right)\\
&= -2\partial \operatorname{Tr}\left(E O_{n+1}^{\mathcal{K}}(1)\right) = -2\partial \operatorname{Tr}(E O_{n+1}(1)) && (2.23)
\end{aligned}
$$

Consider now the Hamiltonian density $\mathcal{H}_1 = \sum_{i=1}^{n} q_i r_i = \mathrm{Tr}(A^0)^2$ and impose the basic relation $\mathcal{H}_n = \partial^{-1}\partial_n \mathcal{H}_1$ for higher Hamiltonians. It follows now from the above Lemma that $\mathcal{H}_n = -2\,\mathrm{Tr}(EO_{n+1}(1))$, which coincides with the result of Wilson [12] (see also [13]). The Z-S equations can be obtained from the Hamiltonians defined above and the first bracket structure given by

$$P_1\delta(x-y) = \begin{pmatrix} \{r_i, r_j\} & \{r_i, q_m\} \\ \{q_l, r_j\} & \{q_l, q_m\} \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\delta(x-y) \tag{2.24}$$

Expressing the recursion operator $\mathcal{R}$ in terms of the first two bracket structures through

$$\mathcal{R} = P_2 P_1^{-1} \tag{2.25}$$

we may obtain the closed expression for the second bracket structure of ZS-AKNS hierarchy

$$P_2 = \begin{pmatrix} r_s\partial^{-1} r_k R^{i*}_{sk-m} & \left(\partial - r_s\partial^{-1} q_k R^{i*}_{sj-k}\right) \\ \left(\partial - q_s\partial^{-1} r_k R^{l}_{sm-k}\right) & q_s\partial^{-1} q_k R^{l}_{sk-j} \end{pmatrix} \tag{2.26}$$

This already suggests that the model is bihamiltonian.

We now consider $\mathcal{G} = sl(n+1)$ with roots $\alpha = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j$ for some $i, j = 1, \ldots, n$, $E = 2\mu_n.H\Omega\alpha_n^2$, μ_n is the n^{th} fundamental weight and H_a, $a = 1, \ldots, n$ are the generators of the Cartan subalgebra. This decomposition generates the symmetric space $sl(n+1)/sl(n) \times u(1)$ (see appendix A for details).

In matrix notation we have:

$$E = \frac{1}{n+1}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & -n \end{pmatrix} \tag{2.27}$$

This model is defined by $A^0 \in \mathcal{M}$ and is parametrized with fields q_a and r_a, $a = 1, \ldots, n$

$$A^0 = \sum_{a=1}^{n}\left(q_a E_{(\alpha_a + \ldots + \alpha_n)} + r_a E_{-(\alpha_a + \ldots + \alpha_n)}\right) \tag{2.28}$$

which in the matrix form can be written as (1.3). The successive flows (2.18) related by the recursion operator (2.21) are given by

$$\partial_n \begin{pmatrix} r_i \\ q_l \end{pmatrix} = \mathcal{R}_{(i,l)(j,m)} \begin{pmatrix} r_j \\ q_m \end{pmatrix} = \tag{2.29}$$

$$\begin{pmatrix} (-\partial + r_k\partial^{-1}q_k)\delta_{ij} + r_i\partial^{-1}q_j & r_i\partial^{-1}r_m + r_m\partial^{-1}r_i \\ -q_l\partial^{-1}q_j - q_j\partial^{-1}q_l & (\partial - q_k\partial^{-1}r_k)\delta_{lm} - q_l\partial^{-1}r_m \end{pmatrix}\partial_{n-1}\begin{pmatrix} r_j \\ q_m \end{pmatrix}$$

From (2.26) an explicit expression for the second bracket is found in this case to be:

$$P_2 = \begin{pmatrix} r_i\partial^{-1}r_j + r_j\partial^{-1}r_i & (\partial - \sum_k r_k\partial^{-1}q_k)\delta_{im} - r_i\partial^{-1}q_m \\ (\partial - \sum_k q_k\partial^{-1}r_k)\delta_{lj} - q_l\partial^{-1}r_j & q_l\partial^{-1}q_m + q_m\partial^{-1}q_l \end{pmatrix} \tag{2.30}$$

2.2 $A^0 \in \mathrm{Im(adE)} + \mathrm{Ker(adE)}$

The non-zero component $A^0_{\mathcal{K}}$ in A^0 leads to non-zero terms in equation (2.8) in $\mathcal{K}$ given by

$$\partial_m A^0_{\mathcal{K}} - \lambda\partial_{m-1}A^0_{\mathcal{K}} + \partial Y_m(\lambda) + +[A^0_{\mathcal{K}}, Y_m(\lambda)] + [A^0_{\mathcal{M}}, O^{\mathcal{M}}_m(1)] = 0 \tag{2.31}$$

where $O^{\mathcal{M}}_n(1)$ is the component of $O_n(1)$ in the subspace $\mathcal{M}$.

From equation (2.31) we see that $Y_m(\lambda) = \lambda Y_m \in \mathcal{K}$. The λ-dependent components in subspaces $\mathcal{M}$ and $\mathcal{K}$ read respectively:

$$-\partial_{m-1}A^0_{\mathcal{M}} + [E, O^{\mathcal{M}}_m(1)] + [A^0_{\mathcal{M}}, Y_m] = 0 \tag{2.32}$$

$$-\partial_{m-1}A^0_{\mathcal{K}} - \partial Y_m + [A^0_{\mathcal{K}}, Y_m] = 0 \tag{2.33}$$

while the λ independent part leads to

$$\partial_m A^0_{\mathcal{M}} - \partial O^{\mathcal{M}}_m(1) + [A^0_{\mathcal{M}}, O^{\mathcal{K}}_m(1)] + [A^0_{\mathcal{K}}, O^{\mathcal{M}}_m(1)] = 0 \tag{2.34}$$

$$\partial_m A^0_{\mathcal{K}} - \partial O^{\mathcal{K}}_m(1) + [A^0_{\mathcal{M}}, O^{\mathcal{M}}_m(1)] + [A^0_{\mathcal{K}}, O^{\mathcal{K}}_m(1)] \tag{2.35}$$

From (2.33) we find

$$Y_m = -\sum_{j=1}^{\infty}\left(\partial^{-1}\mathrm{ad}_{A^0_{\mathcal{K}}}\right)^{j-1}\partial^{-1}\partial_{m-1}A^0_{\mathcal{K}} \tag{2.36}$$

while (2.32) gives

$$O^{\mathcal{M}}_m(1) = \mathrm{ad}_E\partial_{m-1}A^0 + \mathrm{ad}_E\mathrm{ad}_{A^0_{\mathcal{M}}}\left(\sum_{j=1}^{\infty}\left(\partial^{-1}\mathrm{ad}_{A^0_{\mathcal{K}}}\right)^{j-1}\partial^{-1}\partial_{m-1}A^0_{\mathcal{K}}\right) \tag{2.37}$$

It is difficult to present a closed form solution for $O^{\mathcal{K}}_m(1)$ in general case. However there is a simple argument for the existence of an unique solution. Notice that (2.34) provides $n^2 - n + 1$ algebraic equations (which are not equations of motion) whilst (2.35) provide $n - 1$ such equations making a total of n^2 algebraic equations to solve $n^2 = \dim\mathcal{K}$ unknowns in $O^{\mathcal{K}}_m$. Explicit examples can be worked out case by case (see Appendix B for $n = 2$).

Let us consider a particular class of models which are connected to the Toda Lattice hierarchy. The models are defined by a Drinfeld-Sokolov linear system of the type

$$\begin{pmatrix} \partial - \lambda/(n+1) & q_1 & \cdots & 0 & 0 \\ 0 & \partial - \lambda/(n+1) & q_2 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & \partial - \lambda/(n+1) & q_n \\ r_1 & r_2 & \cdots & r_n & \partial + n\lambda/(n+1) \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \\ \psi_{n+1} \end{pmatrix} = 0 \tag{2.38}$$

with

$$A^0 = \sum_{a=1}^{n} q_a E_{(\alpha_a)} + r_a E_{-(\alpha_a + \ldots + \alpha_n)} \tag{2.39}$$

which in the matrix form can be written as in (1.4).

We now argue that a more general case, with non zero upper triangular elements may be reduced to (1.4) after suitable redefinition of fields and basis vectors. For this purpose consider the generalized DS linear system

$$\begin{pmatrix} \partial & q_1 & a_{1,3} & \cdots & a_{1,n-1} \\ 0 & \partial & q_2 & \cdots & a_{2,n-1} \\ \vdots & & \ddots & q_{n-1} & a_{n,n-1} \\ 0 & 0 & \cdots & \partial & q_n \\ r_1 & r_2 & \cdots & r_n & \partial + \lambda \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \\ \psi_{n+1} \end{pmatrix} = 0 \tag{2.40}$$

From where we find

$$\partial \psi_n + q_n \psi_{n+1} = 0 \tag{2.41}$$

and

$$\partial \psi_{n-1} + q_{n-1} \psi_n + a_{n,n-1} \psi_{n+1} = 0 \tag{2.42}$$

The elimination of ψ_{n+1} in favor of ψ_n in (2.41) after reintroducing it in (2.42) we get

$$\partial \tilde{\psi}_{n-1} + \tilde{q}_{n-1} \psi_n = 0 \tag{2.43}$$

where we have redefined

$$\tilde{\psi}_{n-1} = (\psi_{n-1} - \frac{a_{n,n-1}}{q_n} \psi_n) \tag{2.44}$$

and

$$\tilde{q}_{n-1} = q_{n-1} + \partial(\frac{a_{n,n-1}}{q_n}) \tag{2.45}$$

We have therefore removed away $a_{n,n-1}$ from the linear system. After a succession of such redefinitions, we are able to remove all $a_{i,j}$ from the linear system showing that (2.40) can be reduced to (2.38). It can also be shown that the similar construction works in the corresponding Lax formalism.

3 AKS and the Generalized Non Linear Schrödinger Equations

We now derive the GNLS equations from the AKS formalism [3] revealing an affine Lie algebraic structure underlying the integrability of the hierarchy studied by Fordy and Kulish [2] in connection with the symmetric space $sl(n+1)/sl(n) \times u(1)$. This section provides a generalization of the work of Flaschka, Newell and Ratiu [4] of the $sl(2)$ case. The main idea lies in associating an affine Lie algebraic structure (loop algebra $\hat{\mathcal{G}} \equiv \mathcal{G} \otimes \mathbf{C}[\lambda, \lambda^{-1}]$ of $\mathcal{G}$) to an Hamiltonian structure leading to the same equations of motion as in (2.29). Here we only consider $\mathcal{G} = sl(n+1)$ leading to GNLS$_n$ hierarchy. For the AKS treatment of general symmetric spaces see [14].

The AKS formalism will provide us with a set of Casimir-like Hamiltonians in involution. The standard construction is based on the following functions

$$\phi_k(X) \equiv \frac{1}{2} \mathrm{Res} \left(\lambda^{k-1} \mathrm{Tr}(X^2) \right) \quad ; \quad X = \sum_j X_j \lambda^j \in \hat{\mathcal{G}} \tag{3.1}$$

where $k \in \mathbb{Z}_+$. Define now a subalgebra $\hat{\mathcal{G}}_0^-$ of $\hat{\mathcal{G}}$ as $\hat{\mathcal{G}}_0^- \equiv \sum_{j\geq 0} \mathcal{G} \otimes \lambda^{-j}$. On the subalgebra $\hat{\mathcal{G}}_0^-$ there is a natural Poisson bracket:

$$\{ f , g \}(X) \equiv -\Big\langle X \Big| [\pi_+ \nabla f(X) , \pi_+ \nabla g(X)] \Big\rangle \quad ; \quad X \in \hat{\mathcal{G}}_0^- \tag{3.2}$$

where $\langle X | Y \rangle = \sum_{i+j=0} \mathrm{Tr}\, X_i Y_j$ and π_+ is the canonical projection $\pi_+ : \hat{\mathcal{G}} \to \hat{\mathcal{G}}_0^+ \equiv \sum_{j\geq 0} \mathcal{G} \otimes \lambda^j$.

A calculation yields $\nabla \phi_k(X) = S^k X$ where S^k is the shift $S^k : X = \sum X_j \lambda^j \to \sum X_j \lambda^{j+k}$. Consequently $[\nabla \phi_k(X) , X] = 0$, meaning that ϕ_k is ad-invariant. The main AKS theorem [3] ensures now that $\phi_k(X)$ restricted to $\hat{\mathcal{G}}_0^-$ are in involution: $\{ \phi_k , \phi_{k'} \}(X) = 0$ and generate commuting Hamiltonian flows. For the parametrization of $\hat{\mathcal{G}}_0^-$ given by $\sum_{j\geq 0} X_j \lambda^{-j}$ with $X_j = h_a^{(j)} H_a + E_{\pm\alpha}^{(j)} E_{\pm\alpha}$ we obtain:

$$\phi_k = \sum_{i=0}^{k} \frac{1}{2} g^{ab} h_a^{(i)} h_b^{(k-i)} + \sum_{\alpha > 0} E_\alpha^{(i)} E_{-\alpha}^{(k-i)} \tag{3.3}$$

where $g^{ab} = \frac{1}{2} \alpha_b^2 K_{ab}^{-1}$, K_{ab} is the Cartan matrix for $sl(n+1)$, $E_{\pm\alpha}$ are step operators associated to the roots $\pm\alpha$. Summation over repeated indices is understood. The Poisson brackets (3.2) for $h_a^{(i)}$ and $E_\alpha^{(i)}$ (considered as functions on $\hat{\mathcal{G}}_0^-$) are

$$\begin{aligned} \{h_a^{(i)}, h_b^{(j)}\} &= 0 \\ \{h_a^{(i)}, E_\alpha^{(j)}\} &= K_{\alpha a} E_\alpha^{(i+j)} \end{aligned}$$

$$\{E_\alpha^{(i)}, E_\beta^{(j)}\} = \begin{cases} \epsilon(\alpha,\beta) E_{\alpha+\beta}^{(i+j)} & \alpha+\beta \quad \text{is a root} \\ \sum_{a=1}^{n} n_a h_a^{(i+j)} & \alpha + \beta = 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

where $K_{\alpha a} = \frac{2\alpha \cdot \alpha_a}{\alpha_a^2}$, $\alpha = \sum n_a \alpha_a$ and $\alpha_a, a = 1, \ldots, n$ are the simple roots of $sl(n+1)$ [3].

Hamilton equations in components take the form

$$\frac{\partial T^{(j)}}{\partial_{t_k}} = \{\phi_k, T^{(j)}\} \quad ; \quad T^{(j)} = h_a^{(j)}, E_\alpha^{(j)} \tag{3.5}$$

where the bracket is the Lie-Poisson bracket (3.4).

The equations of motion are derived from (3.5) together with the restrictions for the following leading terms

$$\begin{aligned} h_a^{(1)} &= 0 \\ E_\alpha^{(0)} &= 0, \quad \text{for all roots } \alpha \\ E_{\alpha_n+\ldots+\alpha_a}^{(1)} &= q_a, \quad a = 1, \ldots, n \\ E_{-(\alpha_n+\ldots+\alpha_a)}^{(1)} &= r_a, \quad a = 1, \ldots, n \\ E_{\pm(\alpha_a+\ldots+\alpha_b)}^{(1)} &= 0, \quad a, b = 1, \ldots, n-1 \end{aligned} \tag{3.6}$$

[3]For the $sl(n+1)$ algebra the root system is given in terms of the simple roots as $\alpha = \alpha_a + \alpha_{a+1} + \ldots + \alpha_{a+l}$ with $a = 1, \ldots, n$ and $l = 0, 1, \ldots, n-1$ (see appendix A for explanation and notation).

which subsequently will determine the model.

From eqn (3.5) we find the evolution of the step operator $E_\beta^{(j)}$ to be

$$
\begin{aligned}
\frac{\partial E_\beta^{(j)}}{\partial t_k} &= -\frac{1}{2}\sum_{i=0}^{k}\sum_{a=1}^{n} n_a \left(h_a^{(i)} E_\beta^{(k+j-i)} + h_a^{(k-i)} E_\beta^{(i+j)}\right) \\
&+ \sum_{i=0}^{k}\sum_{\alpha\neq\beta,\alpha>0} \left(\epsilon(\alpha,\beta) E_{\alpha+\beta}^{(i+j)} E_{-\alpha}^{(k-i)} + \epsilon(-\alpha,\beta) E_\alpha^{(i)} E_{\beta-\alpha}^{(j+k-i)}\right) \\
&+ \sum_{i=0}^{k}\sum_{a=1}^{n} n_a \left(\theta(\beta) E_\beta^{(i)} h_a^{(j+k-i)} + \theta(-\beta) h_a^{(i+j)} E_\beta^{(k-i)}\right) \qquad (3.7)
\end{aligned}
$$

for $\beta = \sum n_a \alpha_a$ and θ the Heaviside function. It is easy to see that $\partial E_\beta^{(0)}/\partial_{t_k} = 0$ for all k once we impose the condition (3.6) on the right hand side of (3.7). This shows that $E_\beta^{(0)} = 0$ from (3.6) is preserved by the flows. If we now consider $k = j = 1$ we find after using the conditions (3.6)

$$
\frac{\partial E_\beta^{(1)}}{\partial t_1} = -\sum_{a=1}^{n} n_a h_a^{(0)} E_\beta^{(2)} \qquad (3.8)
$$

Consider now a general root not containing α_n of the form $\beta = \sum_{a=1}^{n} n_a \alpha_a = \alpha_l + \alpha_{l+1} + \cdots + \alpha_s$ with $s \leq n-1$. Since, in this case $n_n = 0$, from the condition (3.6), i.e. $E_\beta^{(1)} = 0$ we may choose $h_l^{(0)} = 0$ for $l = 1, \ldots, n-1$ allowing therefore a nontrivial "evolution" for $E_\beta^{(2)}$. For $\beta = \alpha_n + \ldots + \alpha_a$, and renaming $\partial_{t_1} = \partial_x$ we find

$$
E_{\alpha_n + \ldots + \alpha_a}^{(2)} = \partial_x q_a \qquad (3.9)
$$

after choosing a normalization $h_n^{(0)} = -1$. In a similar manner we have

$$
E_{-(\alpha_n + \ldots + \alpha_a)}^{(2)} = -\partial_x r_a \qquad (3.10)
$$

Let us now consider the evolution of $h_a^{(j)}$ given from (3.5) by

$$
\frac{\partial h_a^{(j)}}{\partial t_k} = \sum_{i=0}^{k}\sum_{\alpha>0} K_{\alpha,a}\left(E_\alpha^{(i)} E_{-\alpha}^{(j+k-i)} - E_\alpha^{i+j} E_{-\alpha}^{(k-i)}\right) \qquad (3.11)
$$

Splitting the sum over the positive roots in terms of its simple root content, i.e. $\sum_{\alpha>0} = \sum_{s=0}^{n-1}\sum_{b=1}^{n-s}$ we find

$$
\begin{aligned}
\frac{\partial h_a^{(j)}}{\partial t_k} &= \sum_{i=0}^{k}\sum_{s=0}^{n-1}\sum_{b=1}^{n-s}\left(E_{(\alpha_b+\ldots+\alpha_{b+s})}^{(i)} E_{-(\alpha_b+\ldots+\alpha_{b+s})}^{(j+k-i)} - E_{-(\alpha_b+\ldots+\alpha_{b+s})}^{(k-i)} E_{(\alpha_b+\ldots+\alpha_{b+s})}^{(j+i)}\right) \times \\
&\times \quad (K_{b,a} + K_{b+1,a} + \ldots + K_{b+s,a}) \qquad (3.12)
\end{aligned}
$$

It is easy to verify that $\partial h_a^{(1)}/\partial t_k = 0$ for $a = 1, \ldots, n$ after imposing the condition (3.6) on the right hand side of (3.12). Hence the flows preserve the condition $h_a^{(1)} = 0$ from (3.6). For $h_l^{(2)}$ we find

$$
\frac{\partial h_l^{(2)}}{\partial t_1} = \sum_{b=1}^{n}(q_b E_{-(\alpha_b+\ldots+\alpha_n)}^{(2)} - r_b E_{(\alpha_b+\ldots+\alpha_n)}^{(2)})(\delta_{l,b} - \delta_{l,b-1}) = \partial_x(q_{l+1} r_{l+1}) - \partial_x(q_l r_l) \qquad (3.13)
$$

for $l = 1, \ldots, n-1$. Due to explicit form of the Cartan matrix K_{ab}, $h_n^{(2)}$ requires an independent calculation yielding

$$\frac{\partial h_n^{(2)}}{\partial t_1} = \sum_{l=1}^{n-1} q_l E^{(2)}_{-(\alpha_l+\ldots+\alpha_n)} - r_l E^{(2)}_{(\alpha_l+\ldots+\alpha_n)}) + 2(q_n E^{(2)}_{-\alpha_n} - r_n E^{(2)}_{\alpha_n})$$
$$= -\sum_{a=1}^{n} \partial_x(q_a r_a) - \partial_x(q_n r_n) \quad (3.14)$$

By plugging (3.9), (3.10) in (3.14) and choosing the integration constants to vanish we get

$$h_l^{(2)} = \begin{cases} q_{l+1} r_{l+1} - q_l r_l & l = 1, \ldots, n-1 \\ -\sum_{a=1}^{n} q_a r_a - q_n r_n & l = n \end{cases} \quad (3.15)$$

We now determine the evolution of the step operators $E_\beta^{(2)}$. From (3.7) we find

$$\frac{\partial E^{(2)}_{\alpha_l+\alpha_{l+1}+\ldots+\alpha_s}}{\partial t_1} = \epsilon(\alpha_{s+1} + \ldots + \alpha_n, \alpha_l + \ldots + \alpha_s) r_{s+1} E^{(2)}_{\alpha_l+\ldots+\alpha_n}$$
$$+ \epsilon(-(\alpha_l + \ldots + \alpha_n), \alpha_l + \ldots + \alpha_s) q_l E^{(2)}_{-(\alpha_{s+1}+\ldots+\alpha_n)} \quad (3.16)$$

Using eqns (3.9) and (3.10) and the relation (A.2) implying

$$\epsilon(\alpha_{s+1} + \ldots + \alpha_n, \alpha_l + \ldots + \alpha_s) = -\epsilon(-(\alpha_l + \ldots + \alpha_n), \alpha_l + \ldots + \alpha_s) \quad (3.17)$$

we may integrate eqn (3.16) again choosing the integration constants to vanish. Thus we find

$$E^{(2)}_{\alpha_l+\alpha_{l+1}+\ldots+\alpha_s} = \epsilon(\alpha_{s+1} + \ldots + \alpha_n, \alpha_l + \ldots + \alpha_s) q_l r_{s+1} \quad (3.18)$$

In a similar manner we get

$$E^{(2)}_{-(\alpha_l+\alpha_{l+1}+\ldots+\alpha_s)} = \epsilon(\alpha_{s+1} + \ldots + \alpha_n, \alpha_l + \ldots + \alpha_s) r_l q_{s+1} \quad (3.19)$$

Using equation (3.7),(3.9) and (3.10) it is not difficult to see that

$$\partial_x^2 q_a = \frac{\partial E^{(2)}_{\alpha_n+\alpha_{n-1}+\ldots+\alpha_a}}{\partial t_1} = E^{(3)}_{\alpha_n+\alpha_{n-1}+\ldots+\alpha_a} + 2q_a \sum_{b=1}^{n} q_b r_b \quad (3.20)$$

and

$$-\partial_x^2 r_a = \frac{\partial E^{(2)}_{-(\alpha_n+\alpha_{n-1}+\ldots+\alpha_a)}}{\partial t_1} = -E^{(3)}_{-(\alpha_n+\alpha_{n-1}+\ldots+\alpha_a)} - 2r_a \sum_{b=1}^{n} q_b r_b \quad (3.21)$$

Finally, from (3.7) we obtain

$$\frac{\partial q_a}{\partial t_2} = \frac{\partial E^{(1)}_{\alpha_a+\alpha_{a+1}+\ldots+\alpha_n}}{\partial t_2} = E^{(3)}_{\alpha_a+\alpha_{a+1}+\ldots+\alpha_n} \quad (3.22)$$

and

$$\frac{\partial r_a}{\partial t_2} = \frac{\partial E^{(1)}_{-(\alpha_a+\alpha_{a+1}+\ldots+\alpha_n)}}{\partial t_2} = -E^{(3)}_{-(\alpha_a+\alpha_{a+1}+\ldots+\alpha_n)} \quad (3.23)$$

where we made use of relations (A.2) and (A.3) between the ϵ's.

Equations (3.21) and (3.20) yield the following equations of motion after inserting (3.22) and (3.23)

$$\begin{aligned} \frac{\partial q_a}{\partial t_2} &= \partial_x^2 q_a - 2q_a \sum_{b=1}^{n} q_b r_b \\ \frac{\partial r_a}{\partial t_2} &= -\partial_x^2 r_a + 2r_a \sum_{b=1}^{n} q_b r_b \end{aligned} \tag{3.24}$$

for $a = 1, \ldots, n$. These are the GNLS equations [2].

4 The n-Generalized Two-boson KP Hierarchy and its Equivalence to sl(n + 1) GNLS Hierarchy

In this section we will establish a connection between GNLS matrix hierarchy for the hermitian symmetric space $sl(n+1)$ and the constrained KP hierarchy.

The connection is first established between linear systems defining both hierarchies. Recall first the linear problem (2.1), which for $A^0 \in \mathrm{Ker}(\mathrm{ad}E)$ is parametrized according to (2.20):

$$\begin{pmatrix} \partial - \lambda/(n+1) & 0 & \cdots & 0 & q_1 \\ 0 & \partial - \lambda/(n+1) & 0 & \cdots & q_2 \\ \vdots & & \ddots & & \vdots \\ 0 & & & \partial - \lambda/(n+1) & q_n \\ r_1 & r_2 & \cdots & r_n & \partial + n\lambda/(n+1) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \\ \psi_{n+1} \end{pmatrix} = 0 \tag{4.1}$$

Perform now the phase transformation:

$$\psi_k \longrightarrow \bar{\psi}_k = \exp\left(-\frac{1}{n+1}\int \lambda dx\right)\psi_k \qquad k = 1, \ldots, n+1 \tag{4.2}$$

We now see that thanks to the special form of E in $A = \lambda E + A^0$, (4.1) takes a simple and equivalent form:

$$\begin{pmatrix} \partial & 0 & \cdots & 0 & q_1 \\ 0 & \partial & 0 & \cdots & q_2 \\ \vdots & & \ddots & & \vdots \\ 0 & & & \partial & q_n \\ r_1 & r_2 & \cdots & r_n & \partial + \lambda \end{pmatrix} \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \vdots \\ \bar{\psi}_n \\ \bar{\psi}_{n+1} \end{pmatrix} = 0 \tag{4.3}$$

The linear problem (4.3) after elimination of $\bar{\psi}_k$, $k = 1, \ldots, n$ takes a form of the scalar eigenvalue problem in terms of a single eigenfunction $\bar{\psi}_{n+1}$:

$$-\left[\partial - \sum_{k=1}^{n} r_k \partial^{-1} q_k\right]\bar{\psi}_{n+1} = \lambda \bar{\psi}_{n+1} \tag{4.4}$$

This defines the pseudo-differential operator

$$L_n = \partial - \sum_{k=1}^{n} r_k \partial^{-1} q_k \tag{4.5}$$

which can also be rewritten as

$$L_n = \partial + \sum_{i=1}^{n} a_i (\partial - S_i)^{-1} \tag{4.6}$$

by a simple substitution:

$$r_i = a_i e^{\int S_i} \quad ; \quad q_i = -e^{-\int S_i} \quad ; \quad i = 1, \ldots, n \tag{4.7}$$

The pseudo-differential operator of the form (4.5) defines a constrained KP hierarchy [1, 15, 16].

4.1 On the Constrained KP Hierarchies

Recall that KP flows with respect to infinite many times t_m are defined by:

$$\partial_{t_m} L = [(L^m)_+ \, , L] \tag{4.8}$$

for the general KP Lax operator:

$$L = \partial + \sum_{i=0}^{\infty} u_i (\{t_m\}) \partial^{-1-i} \tag{4.9}$$

The subscripts $\pm$ mean here that we only take non-negative/negative powers of the differential operators ∂.

The KP hierarchy (4.8) allows a straightforward reduction to the so-called k-th order KdV hierarchy by imposing the condition

$$\left(L^k\right)_- = 0 \tag{4.10}$$

This condition is preserved by the flows in (4.8) due to

$$\partial_{t_m} \left(L^k\right)_- = [(L^m)_+ \, , L^k]_- = [(L^m)_+ \, , \left(L^k\right)_-]_- \tag{4.11}$$

As noted in [1, 15, 16] condition (4.10) can be made less restrictive by allowing $\left(L^k\right)_-$ to be

$$\left(L^k\right)_- = -r\partial^{-1} q \tag{4.12}$$

since also this condition will be preserved by the KP flows, with the flows of r and q given by:

$$\begin{aligned} \partial_{t_m} r &= (L^m)_+ r & (4.13) \\ \partial_{t_m} q &= -(L^m)^*_+ q & (4.14) \end{aligned}$$

where the upperscript denotes adjoint operation w.r.t. ∂. In this way one can obtain as a special case of (4.12) the two-boson Lax operator:

$$L_1 = \partial - r\partial^{-1}q = \partial + a\,(\partial - S)^{-1} \tag{4.15}$$

defining a consistent reduction of KP hierarchy with flows defined by $\partial_{t_m} L_1 = [(L_1^m)_+ \, , \, L_1]$ and local bi-Hamiltonian structure [9, 10]. For the linkage of this restriction of the KP hierarchy to the additional symmetries of the KP model see [17].

Obviously in expansion of $L_1 = \partial + \sum_i u_i \partial^{-1-i}$, the coefficient u_0 is equal to $-rq$. This can be understood as a symmetry constraint [1, 15, 16], which naturally generalizes to $u_0 = -\sum_{k=1}^{n} r_k q_k$ in the case of many bosons. This leads us to the Lax operator in (4.5) with flows

$$\partial_{t_m} L_n = [(L_n^m)_+ \, , \, L_n] \tag{4.16}$$

$$\partial_{t_m} r_i = \left(\left(\partial - \sum_{k=1}^{n} r_k \partial^{-1} q_k \right)^m \right)_+ r_i \tag{4.17}$$

$$\partial_{t_m} q_i = -\left(\left(-\partial + \sum_{k=1}^{n} q_k \partial^{-1} r_k \right)^m \right)_+ q_i \tag{4.18}$$

We will call the hierarchy defined by the Lax operator L_n from (4.5) and flows (4.16)-(4.18) the n-generalized two-boson KP hierarchy.

4.2 Equivalence Between n-Generalized Two-Boson KP Hierarchy and sl(n + 1) GNLS Hierarchy

We prove now the equivalence between $sl(n+1)$ GNLS hierarchy defined on basis of the linear problem (4.1) and the n-generalized two-boson KP hierarchy introduced in the previous subsection. Our result is contained in the following

Proposition. *Flows (4.16)-(4.18) of n-generalized two-boson KP hierarchy coincide with the flows produced by the recursion operator $\mathcal{R}$ (2.29) of the $sl(n+1)$ GNLS hierarchy*

We will generalize the proof given in [16] for the $n = 1$. We proceed by induction. Let us first introduce some notation. We parametrize the m-th power of L_n (4.6) as

$$L_n^m = \sum_{j \leq m} P_j(m) \partial^j \tag{4.19}$$

By defining

$$B_m \equiv (L_n^m)_+ = \sum_{j=0}^{m} P_j(m) \partial^j \tag{4.20}$$

the flow equations (4.17)-(4.18) can be rewritten as

$$\partial_{t_m} r_i = B_m r_i \quad ; \quad \partial_{t_m} q_i = -B_m^* q_i \tag{4.21}$$

The case $m = 1$ is obvious. Let us therefore make an appropriate induction assumption about m, namely that (4.21) and (2.29) agree. Consider now:

$$\begin{aligned} B_{m+1} &= \left[\left(\partial - \sum_{k=1}^{n} r_k \partial^{-1} q_k\right) \sum_{j=-1}^{m} P_j(m)\partial^j\right]_+ \qquad (4.22) \\ &= \partial B_m + P_{-1}(m) + \sum_{k=1}^{n}\sum_{j=1}^{m}\sum_{i=1}^{j}(-1)^i r_k \left(q_k P_j(m)\right)^{(i-1)} \partial^{j-i} \end{aligned}$$

where $f^{(i)} \equiv \partial_x^i f$. To calculate the constant term $P_{-1}(m)$ we note that $P_{-1}(m) = \mathrm{Res} L_n^m$ which is equal to an Hamiltonian density $\mathcal{H}_m$ and therefore

$$\begin{aligned} P_{-1}(m) = \mathcal{H}_m &= \partial^{-1}\partial_{t_m}\left(-\sum_{k=1}^{n} r_k q_k\right) = -\sum_{k=1}^{n}\partial^{-1}\left(q_k B_m r_k - r_k B_m^* q_k\right) \qquad (4.23) \\ &= -\sum_{k=1}^{n}\sum_{j=0}^{m}\partial^{-1}\left(q_k P_j(m)(r_k)^{(j)} - (-1)^j r_k (P_j(m) q_k)^{(j)}\right) \\ &= -\sum_{k=1}^{n}\sum_{j=1}^{m}\sum_{i=1}^{j}(-1)^i \left(q_k P_j(m)\right)^{(i-1)} (r_k)^{(j-i)} \end{aligned}$$

where we have used an identity

$$-\left(AB^{(j)} - (-1)^j A^{(j)} B\right) = \partial \sum_{i=1}^{j}(-1)^i (A)^{(i-1)} (B)^{(j-i)} \qquad (4.24)$$

valid for $j \geq 1$ and arbitrary A and B. With this information we can now apply B_{m+1} on r_i:

$$\begin{aligned} B_{m+1} r_i = \partial(B_m r_i) &- \sum_{k=1}^{n} r_i \partial^{-1}\left(q_k B_m r_k - r_k B_m^* q_k\right) \qquad (4.25) \\ &- \sum_{k=1}^{n} r_k \partial^{-1}\left(q_k B_m r_i - r_i B_m^* q_k\right) \end{aligned}$$

Writing now the recursion relation of the first section (2.29) with induction assumption taken into account:

$$\partial_{m+1}\begin{pmatrix} r_i \\ q_l \end{pmatrix} = \mathcal{R}_{(i,l),(j,p)} \begin{pmatrix} B_m r_j \\ -B_m^* q_p \end{pmatrix} \qquad (4.26)$$

We find from (4.26) and (2.29)

$$\partial_{m+1} r_i = -\left\{\partial(B_m r_i) - \sum_{k=1}^{n} r_i \partial^{-1}\left(q_k B_m r_k - r_k B_m^* q_k\right) - \sum_{k=1}^{n} r_k \partial^{-1}\left(q_k B_m r_i - r_i B_m^* q_k\right)\right\} \quad (4.27)$$

which agrees with (4.25) (up to a total minus sign). Similarly to (4.22) we find

$$B_{m+1}^* = -\partial B_m^* + P_{-1}(m) + \sum_{k=1}^{n}\sum_{j=1}^{m}\sum_{i=1}^{j}(-1)^{j+i} q_k (r_k)^{(i-1)} \partial^{(j-i)} P_j(m) \qquad (4.28)$$

Applying this on q_l we get

$$B^*_{m+1} q_l = -\partial(B^*_m q_l) + P_{-1}(m) q_l + \sum_{k=1}^{n} \sum_{j=1}^{m} \sum_{i=1}^{j} (-1)^{j+i} q_k (r_k)^{(i-1)} (P_j(m) q_l)^{(j-i)} \tag{4.29}$$

$$= -\partial(B^*_m q_l) - \sum_{k=1}^{n} q_l \partial^{-1} (q_k B_m r_k - r_k B^*_m q_k) - \sum_{k=1}^{n} q_k \partial^{-1} (r_k B^*_m q_l - q_l B_m r_k)$$

following again from identity (4.24). Again we find agreement (up to a total minus sign) with relation (2.29) defining recursion operator for the $sl(n+1)$ GNLS hierarchy.

5 Constrained KP Hierarchies from GNLS Hierarchy

In this section we derive a class of CKP models from GNLS hierarchy using the similarity transformations. Let us go back to the linear problem defined by A^0 from (2.39) or (1.4). In matrix notation it is given by (2.38). Performing again the phase transformation (4.3) and eliminating $\bar{\psi}_1, \ldots, \bar{\psi}_n$ from the matrix eigenvalue problem we obtain a scalar eigenvalue equation:

$$-\left[\partial + \sum_{i=1}^{n} (-1)^i r_i \prod_{k=i}^{1} \partial^{-1} q_k\right] \bar{\psi}_{n+1} = \lambda \bar{\psi}_{n+1} \tag{5.1}$$

This linear problem defines another example of constrained KP hierarchy involving the pseudo-differential operators given by:

$$\mathcal{L}_n = \partial + \sum_{i=1}^{n} (-1)^i r_i \prod_{k=i}^{1} \partial^{-1} q_k \tag{5.2}$$

$$= \partial + \sum_{i=1}^{n} a_i (\partial - S_i)^{-1} \cdots (\partial - S_1)^{-1} \tag{5.3}$$

The coefficients in (5.2) and (5.3) are related through

$$r_i = a_i e^{\int S_i} \quad ; \quad q_i = -e^{\int (S_{i-1} - S_i)} \quad ; \quad S_0 = 0 \; ; \quad i = 1, \ldots, n \tag{5.4}$$

The pseudo-differential operators of the type shown in (5.3) appear naturally in connection with the Toda lattice hierarchy [5]. We will refer to the corresponding hierarchy as the multi-boson KP-Toda hierarchy. In this section we will explain its status as a constrained KP hierarchy and establish its connection with the Toda lattice.

In [18] it was shown that the first bracket of the multi-boson KP-Toda hierarchy is a consistent reduction of the first Poisson structure of the full KP hierarchy. Here we will construct successive Miura maps taking the n-generalized two-boson KP system into a sequence of constrained KP hierarchies ending with the multi-boson KP-Toda hierarchy. This will establish the latter as a multi-hamiltonian reduction of the KP hierarchy. Our derivation will reveal the presence of discrete Schlesinger-Bäcklund symmetry of the multi-boson KP-Toda hierarchy. We will also identify the n-generalized two-boson KP hierarchy as an abelian structure used in the proof given in [18].

5.1 Similarity Transformations

We start with the Lax operator of the n-generalized two-boson KP hierarchy (4.6) and define a new variable $S_1^{(1)} \equiv S_1 + \partial \ln a_1$. Next we perform the following similarity transformation:

$$
\begin{aligned}
L_n^{(1)} &\equiv \left(\partial - S_1^{(1)}\right) L_n \left(\partial - S_1^{(1)}\right)^{-1} \\
&= \partial + a_1^{(1)} \left(\partial - S_1^{(1)}\right)^{-1} + \sum_{i=2}^{n} a_i^{(1)} (\partial - S_i)^{-1} \left(\partial - S_1^{(1)}\right)^{-1} \qquad (5.5)
\end{aligned}
$$

where we have introduced the redefined coefficients:

$$
a_1^{(1)} \equiv S_1^{(1)\prime} + \sum_{i=1}^{n} a_i \qquad (5.6)
$$

$$
a_i^{(1)} \equiv a_i' + a_i \left(S_i - S_1^{(1)}\right) \qquad i = 2, \ldots, n \qquad (5.7)
$$

If $a_i = 0$ for $i = 3, \ldots, n$ we would have already obtained in this way the Lax operator of the four-boson KP-Toda hierarchy from (5.3). Otherwise we have to continue to apply successively the similarity transformations. In such case, the next step is:

$$
\begin{aligned}
L_n^{(2)} &\equiv \left(\partial - S_2^{(2)}\right) L_n^{(1)} \left(\partial - S_2^{(2)}\right)^{-1} = \partial + a_1^{(2)} \left(\partial - S_2^{(2)}\right)^{-1} + a_2^{(2)} \left(\partial - S_1^{(1)}\right)^{-1} \left(\partial - S_2^{(2)}\right)^{-1} \\
&+ \sum_{i=3}^{n} a_i^{(2)} (\partial - S_i)^{-1} \left(\partial - S_1^{(1)}\right)^{-1} \left(\partial - S_2^{(2)}\right)^{-1} \qquad (5.8)
\end{aligned}
$$

$$
S_2^{(2)} \equiv S_2 + \partial \ln a_2^{(1)} \qquad (5.9)
$$

$$
a_1^{(2)} \equiv a_1^{(1)} + S_2^{(2)\prime} \qquad (5.10)
$$

$$
a_2^{(2)} \equiv a_2^{(1)} + \left(\partial + S_1^{(1)} - S_2^{(2)}\right) a_1^{(1)} + \sum_{i=3}^{n} a_i^{(1)} \qquad (5.11)
$$

$$
a_i^{(2)} \equiv \left(\partial + S_i - S_2^{(2)}\right) a_i^{(1)} \qquad 3 \geq i \geq n \qquad (5.12)
$$

This defines a string of successive similarity transformations. The next step will involve similarity transformation $\left(\partial - S_3^{(3)}\right) L_n^{(2)} \left(\partial - S_3^{(3)}\right)^{-1}$ with $S_3^{(3)} \equiv S_3 + \partial \ln a_3^{(2)}$ and so on. After k steps we arrive at:

$$
\begin{aligned}
L_n^{(k)} &\equiv \partial + a_1^{(k)} \left(\partial - S_k^{(k)}\right)^{-1} + a_2^{(k)} \left(\partial - S_{k-1}^{(k-1)}\right)^{-1} \left(\partial - S_k^{(k)}\right)^{-1} + \ldots \qquad (5.13) \\
&+ a_k^{(k)} \left(\partial - S_1^{(1)}\right)^{-1} \cdots \left(\partial - S_k^{(k)}\right)^{-1} + \sum_{i=k+1}^{n} a_i^{(k)} (\partial - S_i)^{-1} \left(\partial - S_1^{(1)}\right)^{-1} \cdots \left(\partial - S_k^{(k)}\right)^{-1}
\end{aligned}
$$

$$
S_k^{(k)} \equiv S_k + \partial \ln a_k^{(k-1)} \qquad (5.14)
$$

$$
a_1^{(k)} \equiv a_1^{(k-1)} + S_k^{(k)\prime} \qquad (5.15)
$$

$$
a_l^{(k)} \equiv a_l^{(k-1)} + \left(\partial + S_{k-l+1}^{(k-l+1)} - S_k^{(k)}\right) a_{l-1}^{(k-1)} \qquad l = 2, \ldots, k-1 \qquad (5.16)
$$

$$
a_k^{(k)} \equiv a_k^{(k-1)} + \left(\partial + S_1^{(1)} - S_k^{(k)}\right) a_{k-1}^{(k-1)} + \sum_{p=k+1}^{n} a_p^{(k-1)} \qquad (5.17)
$$

$$
a_p^{(k)} \equiv \left(\partial + S_p - S_k^{(k)}\right) a_p^{(k-1)} \qquad k+1 \geq p \geq n \qquad (5.18)
$$

If n is equal to $k+1$ we have obtained in equation (5.13) the Lax operator $L_n^{(n-1)}$ of the form given in (5.3) and therefore member of the n-boson KP-Toda hierarchy. Since the similarity transformations do not change Hamiltonians, the new hierarchy of the Lax operators from (5.3) will share the infinite set of involutive Hamiltonians with the n-generalized two-boson KP hierarchy. The corresponding Poisson bracket structures are obtained by applying Miura maps defined by (5.14)-(5.18).

At this point we would like to remark on the following ambiguity connected with our formalism of successive similarity transformations. Note, that $L_n^{(n-1)}$ obtained from (5.13) by setting $k=n-1$ can be further transformed by an extra similarity transformation without changing its form. This is achieved by:

$$\begin{aligned} L_n^{(n)} &\equiv \left(\partial - S_n^{(n)}\right) L_n^{(n-1)} \left(\partial - S_n^{(n)}\right)^{-1} \\ &= \partial + \sum_{l=1}^{n} a_l^{(n)} \left(\partial - S_{n-l+1}^{(n-l+1)}\right)^{-1} \cdots \left(\partial - S_n^{(n)}\right)^{-1} \end{aligned} \tag{5.19}$$

where

$$S_n^{(n)} \equiv S_n + \partial \ln a_n^{(n-1)} \tag{5.20}$$

$$a_1^{(n)} \equiv a_1^{(n-1)} + S_n^{(n)\prime} \tag{5.21}$$

$$a_l^{(n)} \equiv a_l^{(n-1)} + \left(\partial + S_{n-l+1}^{(n-l+1)} - S_n^{(n)}\right) a_{l-1}^{(n-1)} \qquad l = 2, \ldots, n \tag{5.22}$$

In (5.20)-(5.21) we recognize the Toda lattice structure. Hence the ambiguity encountered in associating the multi-boson KP-Toda hierarchy Lax operator to the underlying n-generalized two-boson KP hierarchy is an origin of the discrete symmetry of the multi-boson KP-Toda hierarchy [11]. The discrete symmetry is in this context the similarity map $L^{(n-1)} \to L^{(n)}$.

5.2 Eigenfunctions and Flow Equations for the Constrained KP Hierarchies

Let us start with a technical observation that the similarity transformation, which takes $L_n^{(k-1)}$ to $L_n^{(k)}$:

$$L_n^{(k)} = \left(\partial - S_k^{(k)}\right) L_n^{(k-1)} \left(\partial - S_k^{(k)}\right)^{-1} \tag{5.23}$$

can be equivalently written as

$$L_n^{(k)} = \Phi_k^{(k-1)} \partial \Phi_k^{(k-1)^{-1}} L_n^{(k-1)} \Phi_k^{(k-1)} \partial^{-1} \Phi_k^{(k-1)^{-1}} \tag{5.24}$$

where we have defined:

$$\Phi_k^{(k-1)} \equiv e^{\int S_k^{(k)}} = a_k^{(k-1)} e^{\int S_k} \tag{5.25}$$

This allows us to find a compact expression for a string of successive similarity transformations leading from L_n to $L_n^{(k)}$:

$$L_n^{(k)} = \prod_{l=k-1}^{0} \left(\Phi_{l+1}^{(l)} \partial \Phi_{l+1}^{(l)^{-1}}\right) L_n \prod_{l=0}^{k-1} \left(\Phi_{l+1}^{(l)} \partial^{-1} \Phi_{l+1}^{(l)^{-1}}\right) \tag{5.26}$$

It is convenient at this point to introduce a definition of the eigenfunction for the Lax operator L.

Definition. *A function Φ is called eigenfunction for the Lax operator L satisfying Sato's flow equation (4.8) if its flows are given by expression:*

$$\partial_{t_m}\Phi = \left(L^m\right)_+ \Phi \tag{5.27}$$

for the infinite many times t_m

In particular as we have seen in Section 3 the Lax L_n (4.6) of the n-generalized two-boson KP hierarchy possesses n eigenfunctions $\Phi_n^{(0)} \equiv r_i = a_i \exp\left(\int S_i\right)$.

We will now establish a main result of this subsection for the Lax $L_n^{(k)}$ defined in (5.13).

Proposition. *The Lax operators $L_n^{(k)}$ (5.13) satisfy Sato's hierarchy equations (4.8) and possess $n-k$ eigenfunctions given by:*

$$\Phi_p^{(k)} = a_p^{(k)} e^{\int S_p} \qquad p = k+1,\ldots,n \tag{5.28}$$

Especially we find that $\Phi_{k+1}^{(k)} = \exp\left(\int S_{k+1}^{(k+1)}\right)$. We will base our induction proof on the result of [19] that for given two eigenfunctions Φ_1, Φ_2 of the Lax operator L satisfying Sato's hierarchy equation (4.8) the Lax $\hat{L} \equiv \Phi_1 \partial \Phi_1^{-1} L \Phi_1 \partial^{-1} \Phi_1^{-1}$ also satisfies the hierarchy equation (4.8) and the function $\Phi_1\left(\Phi_1^{-1}\Phi_2\right)'$ is an eigenfunction of $\hat{L}$.

Start with $k=1$. Define $\Phi_i^{(1)} \equiv r_1\left(r_1^{-1} r_i\right)' = \left(\partial - (\partial \ln r_1)\right) r_i$ for $i = 2,\ldots,n$. From the result of [19] stated in the previous paragraph and equation (5.24) for $k=1$ we find that $\Phi_i^{(1)}$ is an eigenfunction of hierarchy of $L_n^{(1)}$. Furthermore substituting r_i's by a_i and S_i according to (5.4) we find the desired result $\Phi_i^{(1)} = a_i^{(1)} \exp\left(\int S_i\right)$.

Let us now assume that $\Phi_p^{(k)} = a_p^{(k)} e^{\int S_p}$ for $p = k+1,\ldots,n$ are indeed eigenfunctions belonging to the flow hierarchy of $L_n^{(k)}$. Define now

$$\Phi_p^{(k+1)} \equiv \left(\partial - (\partial \ln \Phi_{k+1}^{(k)}\right) \Phi_p^{(k)} = \left(\partial - (\partial \ln a_{k+1}^{(k)}) - S_{i+1}\right) a_p^{(k)} e^{\int S_p} \tag{5.29}$$

$$= \left[\left(\partial - (\partial \ln a_{k+1}^{(k)}) - S_{i+1} + S_p\right) a_p^{(k)}\right] e^{\int S_p} = a_p^{(k+1)} e^{\int S_p} \tag{5.30}$$

where in (5.30) we have used (5.18). Since

$$L_n^{(k+1)} = \Phi_{k+1}^{(k)} \partial {\Phi_{k+1}^{(k)}}^{-1} L_n^{(k)} \Phi_{k+1}^{(k)} \partial^{-1} {\Phi_{k+1}^{(k)}}^{-1} \tag{5.31}$$

it follows from induction assumption that $L_n^{(k+1)}$ satisfies (4.8). Furthermore since expression in (5.29) is equal to $\Phi_p^{(k+1)} \equiv \Phi_{k+1}^{(k)}\left({\Phi_{k+1}^{(k)}}^{-1} \Phi_p^{(k)}\right)'$ we have found eigenfunctions belonging to $L_n^{(k+1)}$ and they are given by (5.28). This concludes the induction proof.

As a corollary we find that the Lax operators $\mathcal{L}_n = L_n^{(n)}$ of the multi-boson KP-Toda hierarchy satisfy Sato's hierarchy equations (4.8).

Successive gauge transformations of the type given in (5.24) resulting in a Lax structures with decreasing number of eigenvalues were also considered in [20].

5.3 On the Discrete Schlesinger-Bäcklund Transformation of the Generalized Toda-AKNS model

In this subsection we will study the discrete Bäcklund transformations obtained above in the Lax formulation within the corresponding matrix formulation of GNLS hierarchy.

To set the scene let us first connect a "two-boson" Toda system with AKNS matrix formulation. The "two-boson" Toda spectral system is:

$$(\partial - a_0(n-1))\,\Psi_{n-1} = \Psi_n \tag{5.32}$$

$$\Psi_{n+1} + a_0(n)\Psi_n + a_1(n)\Psi_{n-1} = \lambda\Psi_n \tag{5.33}$$

which becomes in matrix formulation:

$$\partial\begin{pmatrix}\Psi_{n-1}\\ \Psi_n\end{pmatrix} = \begin{pmatrix}a_0(n-1) & 1\\ -a_1(n) & \lambda\end{pmatrix}\begin{pmatrix}\Psi_{n-1}\\ \Psi_n\end{pmatrix} = A_n(t,\lambda)\begin{pmatrix}\Psi_{n-1}\\ \Psi_n\end{pmatrix} \tag{5.34}$$

Under the lattice shift $n \to n+1$, Ψ_n goes

$$\Psi_{n+1} = \lambda\Psi_n - a_0(n)\Psi_n - a_1(n)\Psi_{n-1} = (\lambda - a_0(n))\,\Psi_n - a_1(n)\Psi_{n-1} \tag{5.35}$$

Recalling that $a_0(n) = a_0(n-1) + \partial \ln a_1(n)$ we can describe the transition $n \to n+1$ in form of the matrix equation involving only variables entering (5.34):

$$\begin{pmatrix}\Psi_n\\ \Psi_{n+1}\end{pmatrix} = \begin{pmatrix}0 & 1\\ -a_1(n) & (\lambda - a_0(n-1) - \partial\ln a_1(n))\end{pmatrix}\begin{pmatrix}\Psi_{n-1}\\ \Psi_n\end{pmatrix} \equiv T_n(t,\lambda)\begin{pmatrix}\Psi_{n-1}\\ \Psi_n\end{pmatrix} \tag{5.36}$$

Compatibility of (5.34) and (5.36) yields:

$$A_{n+1} = T_n A_n T_n^{-1} + T_n \partial T_n^{-1} \tag{5.37}$$

We will now establish the relation to the AKNS hierarchy. Introduce the following new variables:

$$\begin{pmatrix}\phi_n^{(2)}\\ \phi_n^{(1)}\end{pmatrix} = \begin{pmatrix}e^{-\int a_0(n-1)}\Psi_{n-1}\\ \Psi_n\end{pmatrix} \tag{5.38}$$

and

$$q_n \equiv -e^{-\int a_0(n-1)} \quad ; \quad r_n \equiv a_1(n)e^{\int a_0(n-1)} \tag{5.39}$$

Now we can rewrite (5.32)-(5.33) in matrix notation as

$$\begin{pmatrix}\partial & q_n\\ r_n & \partial - \lambda\end{pmatrix}\begin{pmatrix}\phi_n^{(2)}\\ \phi_n^{(1)}\end{pmatrix} = 0 \tag{5.40}$$

If we let $\phi_n^{(i)} \to e^{-\int^x \lambda/2}\phi_n^{(i)}$ we arrive at exactly AKNS equation:

$$\begin{pmatrix}\partial + \frac{1}{2}\lambda & q_n\\ r_n & \partial - \frac{1}{2}\lambda\end{pmatrix}\begin{pmatrix}e^{-\int^x \lambda/2}\phi_n^{(2)}\\ e^{-\int^x \lambda/2}\phi_n^{(1)}\end{pmatrix} = 0 \tag{5.41}$$

Using the Toda chain equations

$$\partial a_0(n) = a_1(n+1) - a_1(n) \tag{5.42}$$
$$\partial a_1(n) = a_1(n)\left(a_0(n) - a_0(n-1)\right) \tag{5.43}$$

and (5.35) or (5.36) we find that the shift $n \to n-1$ of the Toda lattice results in

$$\begin{pmatrix} \lambda - \partial \ln q_n & q_n \\ -1/q_n & 0 \end{pmatrix} \begin{pmatrix} \phi_n^{(2)} \\ \phi_n^{(1)} \end{pmatrix} = \begin{pmatrix} \phi_{n-1}^{(2)} \\ \phi_{n-1}^{(1)} \end{pmatrix} \tag{5.44}$$

together with

$$q_n \to q_{n-1} = -r_n q_n^2 + \partial^2 q_n - \frac{(\partial q_n)^2}{q_n} \quad ; \quad r_n \to r_{n-1} = -\frac{1}{q_n} \tag{5.45}$$

while the shift $n \to n+1$ of the Toda lattice results in (equivalently to (5.36))

$$\begin{pmatrix} 0 & r_n^{-1} \\ -r_n & (\lambda - \partial \ln r_n) \end{pmatrix} \begin{pmatrix} \phi_n^{(2)} \\ \phi_n^{(1)} \end{pmatrix} = \begin{pmatrix} \phi_{n+1}^{(2)} \\ \phi_{n+1}^{(1)} \end{pmatrix} \tag{5.46}$$

together with

$$q_n \to q_{n+1} = -\frac{1}{r_n} \quad ; \quad r_n \to r_{n+1} = -r_n^2 q_n + \partial^2 r_n - \frac{(\partial r_n)^2}{r_n} \tag{5.47}$$

These are so-called Schlesinger transformations discussed in [21, 22, 23]. Here we obtained them from a lattice shift of the Toda lattice underlying the AKNS construction. The similarity transformation on the level of corresponding Lax operator captures, as shown in a previous subsection, a Toda lattice structure within the continuous constrained KP hierarchy and is fully equivalent to the discrete Schlesinger-Bäcklund transformation obtained here because (5.20) and (5.22) correspond to the Toda chain equations (5.42) and (5.43).

For completeness let us comment on the Schlesinger-Bäcklund transformations in the case of the general Toda hierarchy. We start this time with the spectral equation:

$$\begin{aligned} \partial \Psi_n &= \Psi_{n+1} + a_0(n)\Psi_n \\ \lambda \Psi_n &= \Psi_{n+1} + a_0(n)\Psi_n + \sum_{k=1}^{M} a_k(n)\Psi_{n-k} \end{aligned} \tag{5.48}$$

The spectral equation (5.48) can be rewritten as a matrix equation, which in components is given by:

$$\begin{pmatrix} \partial - a_0(n-M) & -1 & 0 & \ldots & 0 \\ 0 & \partial - a_0(n-M+1) & -1 & 0 & \ldots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \partial - a_0(n-1) & -1 \\ a_M(n) & a_{M-1}(n) & \ldots & a_1(n) & \partial - \lambda \end{pmatrix} \begin{pmatrix} \Psi_{n-M} \\ \Psi_{n-M+1} \\ \vdots \\ \Psi_{n-1} \\ \Psi_n \end{pmatrix} = 0 \tag{5.49}$$

Under the transition $n \to n+1$ on the lattice we get:

$$\begin{pmatrix} \Psi_{n-M+1} \\ \Psi_{n-M+2} \\ \vdots \\ \Psi_n \\ \Psi_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & \ddots & 1 \\ -a_M(n) & -a_{M-1}(n) & \cdots & -a_1(n) & (\lambda - a_0(n)) \end{pmatrix} \begin{pmatrix} \Psi_{n-M} \\ \Psi_{n-M+1} \\ \vdots \\ \Psi_{n-1} \\ \Psi_n \end{pmatrix} \tag{5.50}$$

Introduce now AKNS notation:

$$r_i(n) = a_i(n) e^{\int a_0(n-i)} \quad ; \quad q_i = -e^{\int (a_0(n-i) - a_0(n-i-1))} \quad i = 1, \ldots, M \tag{5.51}$$

$$\phi_n^{(1)} \equiv \Psi_n \quad ; \quad \phi_n^{(i)} \equiv e^{-\int a_0(n-i+1)} \Psi_{n-i+1} \quad ; \quad i = 2, \ldots, M+1 \tag{5.52}$$

where for consistency we set $a_0(n-M-1) = 0$. The relation (5.50) reads in the AKNS variables as:

$$\begin{pmatrix} \phi_{n+1}^{(M+1)} \\ \phi_{n+1}^{(M)} \\ \vdots \\ \phi_{n+1}^{(2)} \\ \phi_{n+1}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & \ddots & r_1^{-1}(n) \\ -r_M(n) & -r_{M-1}(n) & \cdots & -r_1(n) & (\lambda - \partial \ln r_1(n)) \end{pmatrix} \begin{pmatrix} \phi_n^{(M+1)} \\ \phi_n^{(M)} \\ \vdots \\ \phi_n^{(2)} \\ \phi_n^{(1)} \end{pmatrix} \tag{5.53}$$

This transformation generalizes the Schlesinger-Bäcklund transformation from the AKNS system to a general AKNS system based on arbitrary Toda lattice.

A Lie Algebra Preliminaries

In order to be self contained we recall some basic results on the theory of Lie algebras. We first establish the commutation relations in the Chevalley basis,

$$\{H_a\,,\,H_b\} = 0$$

$$\{H_a\,,\,E_\alpha\} = K_{\alpha a} E_\alpha$$

$$\{E_\alpha\,,\,E_\beta\} = \begin{cases} \epsilon(\alpha,\beta) E_{\alpha+\beta} & \alpha+\beta \quad \text{is a root} \\ \sum_{a=1}^{n} n_a H_a & \alpha+\beta = 0 \\ 0 & \text{otherwise} \end{cases} \tag{A.1}$$

where $K_{\alpha a} = \frac{2\alpha \cdot \alpha_a}{\alpha_a^2} = \sum n_b K_{ba}$, for K_{ab} the Cartan matrix. A root α can be expanded in terms of simple roots as $\alpha = \sum n_a \alpha_a$. The integers l_a are defined from the expansion $\frac{\alpha}{\alpha^2} = \frac{\sum l_a \alpha_a}{\alpha_a^2}$. $\epsilon(\alpha,\beta)$ constants are related among themselves by the Jacobi identities and the antisymmetry of the bracket. In particular they satisfy

$$\epsilon(\alpha,\beta) = -\epsilon(\beta,\alpha) = \epsilon(-\beta,-\alpha) \tag{A.2}$$

and

$$\epsilon(\alpha,\beta) = \epsilon(\beta,\gamma) = \epsilon(\gamma,\alpha) \tag{A.3}$$

for $\alpha+\beta+\gamma=0$.

We choose a special element E in the Lie algebra defined as in terms of fundamental weights μ_a, where $2\mu_a.\alpha_b/\alpha_b^2 = \delta_{ab}$ as

$$E = \frac{2\mu_a.H}{\alpha_a^2} \tag{A.4}$$

The element E decomposes the Lie algebra $\mathcal{G} = \mathcal{M} + \mathcal{K} = Ker(adE) + Im(adE)$ where $\mathcal{K}$ is composed of all generators of $\mathcal{G}$ commuting with E and is spanned by $\{H_a, E_\alpha\}$ with α not containing the simple root α_a while $\mathcal{M}$ is its orthogonal complement. Hermitean symmetric spaces are associated to those Lie algebras $\mathcal{G}$ with roots such that they either do not contain α_a or contain it only once, i.e.

$$\frac{2\mu_a.\alpha}{\alpha_a^2} = \pm 1, 0 \tag{A.5}$$

for all roots of $\mathcal{G}$. This fact implies that

$$[E,[E,E_\alpha]] = E_\alpha \quad \text{or} \quad 0 \tag{A.6}$$

From the Jacobi identities it can be shown that (see e.g. [13]) $\mathcal{G}/\mathcal{K}$ is a symmetric space,i.e.

$$[\mathcal{K},\mathcal{K}] \in \mathcal{K}, \quad [\mathcal{M},\mathcal{K}] \in \mathcal{M}, \quad [\mathcal{M},\mathcal{M}] \in \mathcal{K}, \tag{A.7}$$

The Lie algebras satisfying (A.5) are $su(n), so(2n), sp(n), E_6, andE_7$ and generate the following Hermitian symmetric spaces $\frac{su(n+1)}{su(n)\times u(1)}$, $\frac{so(2n)}{u(n)}$, $\frac{sp(n)}{u(n)}$, $\frac{E_6}{so(10)\times u(1)}$ and $\frac{E_7}{E_6\times u(1)}$.

The curvature tensor associated to these symmetric spaces is defined as

$$R^\delta_{\alpha\beta-\gamma}E_\delta = [E_\alpha,[E_\beta,E_{-\gamma}]] \tag{A.8}$$

with the hermiticity property

$$R^{\delta\,*}_{\alpha\beta-\gamma} = R^{-\delta}_{-\alpha\,-\beta\,\gamma} \tag{A.9}$$

B sl(3) ZS-AKNS Matrix Model Solution for (1.3)

Recursion equations for Z-S approach for the case of $A^{(0)}$ as in (1.4) for $n=2$. In this case (2.36) gives:

$$Y_m = -\partial^{-1}\partial_{m-1}q_1 E_{\alpha_1} = \begin{pmatrix} 0 & -\partial^{-1}\partial_{m-1}q_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{B.1}$$

while (2.37) yields

$$O_m^{\mathcal{M}}(1) = O_{\alpha_2}E_{\alpha_2} + O_{-\alpha_2}E_{-\alpha_2} + O_{\alpha_1+\alpha_2}E_{\alpha_1+\alpha_2} + O_{-\alpha_1-\alpha_2}E_{-\alpha_1-\alpha_2} \tag{B.2}$$

with the coefficients:

$$O_{\alpha_2} = \partial_{m-1} q_2 \quad ; \quad O_{-\alpha_2} = -\partial_{m-1} r_2 + r_1 \partial^{-1} \partial_{m-1} q_1 \tag{B.3}$$

$$O_{\alpha_1+\alpha_2} = q_2 \partial^{-1} \partial_{m-1} q_1 \quad ; \quad O_{-\alpha_1-\alpha_2} = -\partial_{m-1} r_1 \tag{B.4}$$

plugging this into algebraic parts of (2.34)-(2.35) we obtain expression for

$$O_m^{\mathcal{K}}(1) = O_{\alpha_1} E_{\alpha_1} + O_{-\alpha_1} E_{-\alpha_1} + O_{h_1} H_1 + O_{h_2} H_2 \tag{B.5}$$

with

$$O_{\alpha_1} = \frac{1}{q_2} \left(\partial q_2 \partial^{-1} \partial_{m-1} q_1 + q_1 \partial_{m-1} q_2 \right) \tag{B.6}$$

$$O_{-\alpha_1} = \partial^{-1} \partial_{m-1} r_1 q_2 \tag{B.7}$$

$$O_{h_1} = \partial^{-1} \left(q_1 \partial^{-1} \partial_{m-1} (q_2 r_1) - r_1 q_2 \partial^{-1} \partial_{m-1} q_1 \right) \tag{B.8}$$

$$O_{h_2} = -\partial^{-1} \partial_{m-1} q_2 r_2 \tag{B.9}$$

After inserting back into the dynamical parts of (2.34)-(2.35) the above results lead to the recursion operator of the same form as the one for the four-boson KP-Toda hierarchy.

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