IC/95/91
INTERNAL REPORT
(Limited Distribution)

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

STABILITY ANALYSIS FOR A GENERAL AGE-DEPENDENT VACCINATION MODEL

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ABSTRACT

An SIR epidemic model of a general age-dependent vaccination model is investigated when the fertility, mortality and removal rates depend on age. We give threshold criteria for the existence of equilibriums and perform stability analysis. Furthermore a critical vaccination coverage that is sufficient to eradicate the disease is determined.

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May 1995

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1. Introduction

Several recent papers have dealt with age-dependent vaccination models. For example, Hethcote [10], Katzman and Dietz [12], Anderson and May [1], Busenberg, Cooke and Iannelli [3] and El-Doma [5,6,7,8]. In our previous paper [8] we studied the existence and uniqueness of solution to the model equations and determined the large time behaviour to the solution of the model equations. It is known that the large time behaviour of the total population size is completely determined by a characteristic equation which has a unique real root. The size of the total population will be increasing if the root is positive, decreasing if the root is negative and constant if the root is zero.

In this paper we determine the steady states for the model equations and perform stability analysis. We show that there is a parameter which determine whether the disease will become endemic or die out. And then we show that this endemic state is locally asymptotically stable. For the trivial equilibrium, we show that, depending on the size of the parameter, this state is locally asymptotically stable or unstable. Furthermore the size of the parameter could be used as a means of control to determine a vaccination coverage that is sufficient to eradicate the disease with minimum vaccination coverage.

The organization of this paper is as follows: in Section 2 we describe the model and obtain the model equations; in Section 3 we reduce the model equations to several subsystems; in Section 4 we determine the steady states and the limiting equation; in Section 5 we perform stability analysis.

2. A General Age-Dependent Vaccination Model

In this section, we consider an age-structured population of variable size exposed to a communicable disease. We assume the following:

1 – $s(a,t)$, $i(a,t)$ and $r(a,t)$, respectively, denote the age density for susceptibles, infectives and immune individuals of age a at time t. Then

$\int_{a_2}^{a_1} s(a,t)da$ = total number of susceptibles at time t of ages between a_1 and a_2.

$\int_{a_2}^{a_1} i(a,t)da$ = total number of infectives at time t of ages between a_1 and a_2.

$\int_{a_2}^{a_1} r(a,t)da$ = total number of immune individuals of ages between a_1 and a_2.

And that the total population consists entirely of susceptibles, infectives and immune individuals.

2 – The transmission of the disease occurs according to the following law:
$k(a)s(a,t)\int_0^\infty i(a,t)da$ where $k(a)$ is a bounded non-negative continuous function of a.

3 - The fertility rate $\beta(a)$ is non-negative, continuous, with compact support $[0,T]$, $(T \geq 0)$. And the birth rates are given by:

$$s(0,t) = \int_0^\infty \beta(a)[s(a,t) + i(a,t) + r(a,t)]da$$

$i(0,t) = 0$, $r(0,t) = 0$, i.e. all newborns are susceptibles and the maternally transmitted immunity is zero.

4 - The death rate $\mu(a)$ is the same for susceptibles, infectives and immunes, and $\mu(a)$ is non-negative, bounded, continuous and eventually non-decreasing.

5 - The cure rate $\gamma(a)$ is a bounded non-negative continuous function of a.

6 - The vaccination rate $\nu(a)$ is a bounded non-negative continuous function of a.

7 - The initial age-distributions: $s(a,0) = s_0(a)$, $i(a,0) = i_0(a)$, $r(a,0) = r_0(a)$ are assumed to be continuous and integrable functions of a in $[0,\infty)$.

These assumptions lead to the following system of nonlinear integro-differential equations:

$$\begin{cases} \frac{\partial s(a,t)}{\partial a} + \frac{\partial s(a,t)}{\partial t} + (\mu(a) + \nu(a))s(a,t) = -k(a)s(a,t)\int_0^\infty i(a,t)da\ , \\ \qquad a > 0,\ t > 0 \\ \frac{\partial i(a,t)}{\partial a} + \frac{\partial i(a,t)}{\partial t} + (\mu(a) + \gamma(a))i(a,t) = k(a)s(a,t)\int_0^\infty i(a,t)da\ , \\ \qquad a > 0, t > 0 \\ \frac{\partial r(a,t)}{\partial a} + \frac{\partial r(a,t)}{\partial t} + \mu(a)r(a,t) = \nu(a)s(a,t) + \gamma(a)i(a,t),\ a > 0,\ t > 0 \\ s(0,t) = \int_0^\infty \beta(a)[s(a,t) + i(a,t) + r(a,t)]da,\quad t \geq 0, \\ i(0,t) = r(0,t) = 0,\ t \geq 0 \\ s(a,0) = s_0(a),\ i(a,0) = i_0(a),\ r(a,0) = r_0(a),\quad a \geq 0\ . \end{cases} \tag{2.1}$$

Here we note that the SI version of this model together with numerical solution is given in [4].

3. Formal Reduction of the Model

In this section, we develop some preliminary formal analysis of problem (2.1). We define $p(a,t)$ by:

$$p(a,t) = s(a,t) + i(a,t) + r(a,t)\ .$$

Then $p(a,t)$ satisfies the following McKendrick-Von Foerster equation:

$$\begin{cases} \frac{\partial p(a,t)}{\partial a} + \frac{\partial p(a,t)}{\partial t} + \mu(a)p(a,t) = 0,\ a > 0,\ t > 0 \\ p(0,t) = \int_0^\infty \beta(a)p(a,t)da \overset{\text{def}^n}{=} B(t),\ t \geq 0 \\ p(a,0) = p_0(a) = s_0(a) + i_0(a) + r_0(a),\ a \geq 0\ . \end{cases} \tag{3.1}$$

$s(a,t)$ satisfies the following:

$$\begin{cases} \frac{\partial s(a,t)}{\partial a} + \frac{\partial s(a,t)}{\partial t} + [\mu(a) + \nu(a)]s(a,t) = -k(a)s(a,t)\int_0^\infty i(a,t)da, \\ \qquad a > 0,\ t > 0 \\ s(0,t) = B(t),\ t \geq 0 \\ s(a,0) = s_0(a),\ a \geq 0\ . \end{cases} \tag{3.2}$$

And $i(a,t)$ satisfies the following:

$$\begin{cases} \frac{\partial i(a,t)}{\partial a} + \frac{\partial i(a,t)}{\partial t} + [\mu(a) + \gamma(a)]i(a,t) = k(a)s(a,t)\int_0^\infty i(a,t)da, \\ \qquad a > 0,\ t > 0 \\ i(0,t) = 0,\ t \geq 0 \\ i(a,0) = i_0(a),\ a \geq 0\ . \end{cases} \tag{3.3}$$

And accordingly $r(a,t)$ satisfies:

$$r(a,t) = p(a,t) - s(a,t) - i(a,t)\ . \tag{3.4}$$

So, it is clear that (3.1), (3.2), (3.3) and (3.4) are equivalent to the original problem (2.1). The existence and uniqueness of solution to this problem is proved in [8]. Furthermore, since problem (3.1) is McKendrick-Von Foerster equation, $p(a,t)$ has a unique solution that exists for all time, (see Bellman and Cooke [2], Hoppensteadt [11] and Feller [9]).

The unique solution of problem (3.1) is given by:

$$p(a,t) = \begin{cases} p_0(a-t)\pi(a)/\pi(a-t), & a > t \\ B(t-a)\pi(a), & a < t \end{cases} \tag{3.5}$$

where $\pi(a)$ is given by:

$$\pi(a) = e^{-\int_0^a \mu(s)ds} \tag{3.6}$$

and $B(t)$ has the following asymptotic behaviour

$$B(t) = [c + \theta(t)]e^{p^* t} \tag{3.7}$$

where p^* is the unique real number which satisfies the following characteristic equation:

$$\int_0^\infty \beta(a)\pi(a)e^{-p_a^*}da = 1 \tag{3.8}$$

and $\theta(t)$ is a function such that $\theta(t) \to 0$ as $t \to \infty$ and c is a constant.

Note that since $B(t)$ is known, we can solve for $s(a,t)$ from (3.2) and then use this solution to determine $i(a,t)$ from (3.3).

4. Limiting Equation and Steady States

In this section, we discuss the steady state solution of problem (2.1). We note that the large time behaviour of the total population is determined by the characteristic equation (3.8). We restrict our attention here to the case $p^* = 0$. For the other cases see El-Doma [8].

Theorem (4.1)

Suppose $p^* = 0$. Then if $c\int_0^\infty\int_0^\infty \pi(a+s)e^{-\int_a^{a+s}\gamma(\sigma)d\sigma}k(a)e^{-\int_0^a \nu(\sigma)d\sigma}dads > 1$ there exists a unique $I_\infty = \lim_{t\to\infty} I(t) > 0$ satisfying:

$$1 = c\int_0^\infty\int_0^\infty \pi(a+s)e^{-\int_a^{a+s}\gamma(\sigma)d\sigma}k(a)e^{-\int_0^a \nu(\sigma)d\sigma}e^{-I_\infty\int_0^a k(\sigma)d\sigma}dads \tag{4.1}$$

And in this case $s_\infty(a) = c\pi(a)e^{-\int_0^a(\nu(\sigma)+I_\infty k(\sigma))d\sigma}$,

$$i_\infty(a) = c\pi(a)\int_0^a e^{-\int_\sigma^a \gamma(s)ds}k(\sigma)e^{-\int_0^\sigma \nu(s)ds}e^{-I_\infty\int_0^\sigma k(s)ds}d\sigma ,$$

$$r_\infty(a) = c\pi(a)[1 - e^{-\int_0^a \nu(s)ds}e^{-I_\infty\int_0^a k(\sigma)d\sigma} - I_\infty\int_0^a e^{-\int_\sigma^a \gamma(s)ds}k(\sigma)e^{-\int_0^\sigma \nu(s)ds}e^{-I_\infty\int_0^\sigma k(s)ds}d\sigma].$$

Otherwise; $I_\infty = 0$, $s_\infty(a) = c\pi(a)e^{-\int_0^a \nu(\sigma)d\sigma}$, $i_\infty(a) = 0$ and $r_\infty(a) = c\pi(a)[1 - e^{-\int_0^a \nu(\sigma)d\sigma}]$.

Proof:

By integrating (3.3) along characteristics $t - a =$ constant we obtain the following:

$$i(a,t) = \begin{cases} i_0(a-t)e^{-\int_0^t[\mu(a-t+s)+\gamma(a-t+s)]ds} \\ \quad + \int_0^\infty e^{-\int_\sigma^t(\mu(a-t+s)+\gamma(a-t+s))ds} \\ \qquad \times k(a-t+\sigma)s(a-t+\sigma,\sigma)I(\sigma)d\sigma, \ a > t \\ \\ \int_0^a e^{-\int_\sigma^a(\mu(s)+\gamma(s))ds}k(\sigma)s(\sigma,t-a+\sigma)I(t-a+\sigma)d\sigma, \ a < t \end{cases} \tag{4.2}$$

where $I(t) = \int_0^\infty i(a,t)da$.

By integrating (4.2) we obtain the following:

$$\begin{aligned} I(t) = &\int_t^\infty i_0(a-t)e^{-\int_0^t(\mu(a-t+s)+\gamma(a-t+s))ds}da \\ &+ \int_t^\infty\int_0^t e^{-\int_\sigma^t(\mu(a-t+s)+\gamma(a-t+s))ds}k(a-t+\sigma)s(a-t+\sigma,\sigma)I(\sigma)d\sigma da \\ &+ \int_0^t\int_0^a e^{-\int_\sigma^a(\mu(s)+\gamma(s))ds}k(\sigma)\ s(\sigma,t-a+\sigma)I(t-a+\sigma)d\sigma da . \end{aligned}$$

We note that

$$\int_t^\infty i_0(a-t)e^{-\int_0^t(\mu(a-t+s)+\gamma(a-t+s))ds}da = \int_0^\infty i_0(a)e^{-\int_0^t(\mu(a+s)+\gamma(a+s))ds}da \to 0$$

as $t \to \infty$ by the dominated convergence theorem and assumption (4) and (7).

Now, by change of the variables of the integration we get:

$$I(t) = \int_0^\infty i_0(a)e^{-\int_0^t(\mu(a+s)+\gamma(a+s))ds}da$$

$$+ \int_0^t\int_0^\infty e^{-\int_a^{a+\sigma}(\mu(s)+\gamma(s))ds}k(a)s(a,t-\sigma)I(t-\sigma)dad\sigma . \tag{4.3}$$

In a similar fashion if we integrated (3.2) along characteristics $t - a =$ constant we obtain the following:

$$s(a,t) = \begin{cases} s_0(a-t)e^{-\int_0^t(\mu(a-t+s)+\nu(a-t+s)+k(a-t+s)I(s))ds}, & a > t \\ B(t-a)e^{-\int_0^a(\mu(s)+\nu(s)+k(s)I(t-a+s))ds}, & a < t \end{cases} \tag{4.4}$$

So, letting $p^* = 0$ and substituting (4.4) in (4.3) we obtain:

$$I_\infty[1 - c\int_0^\infty\int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}e^{-I_\infty\int_0^a k(s)ds}dad\sigma] . \tag{4.5}$$

That is if $I_\infty \neq 0$, then we obtain the following equation for I_∞:

$$c\int_0^\infty\int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}e^{-I_\infty\int_0^a k(s)ds}dad\sigma = 1 . \tag{4.6}$$

Since the left-hand side of (4.6) is a monotone decreasing function of I_∞ (4.6) have a unique solution $I_\infty > 0$ iff

$$c\int_0^\infty\int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}dad\sigma > 1 .$$

Moreover in this case I_∞ is given by (4.6). And by (4.4) and (3.7) $s_\infty(a)$ is given by:

$$s_\infty(a) = c\pi(a)e^{-\int_0^a \nu(\sigma)d\sigma}e^{-I_\infty\int_0^a k(\sigma)d\sigma} . \tag{4.7}$$

By using (4.7) and (4.2) we get that $i_\infty(a)$ satisfies:

$$i_\infty(a) = c\pi(a)I_\infty\int_0^a e^{-\int_\sigma^a \gamma(s)ds}k(\sigma)e^{-\int_0^\sigma \nu(s)ds}e^{-I_\infty\int_0^\sigma k(s)ds}d\sigma . \tag{4.8}$$

And therefore $r_\infty(a)$ is given by:

$$r_\infty(a) = c\pi(a)\left[1 - e^{-\int_0^a \nu(s)ds} e^{-I_\infty \int_0^a k(\sigma)d\sigma} - I_\infty \int_0^a e^{-\int_\sigma^a \gamma(s)ds} k(\sigma) e^{-\int_0^\sigma \nu(s)ds} e^{-I_\infty \int_0^\sigma k(s)ds} d\sigma\right] . \tag{4.9}$$

If $I_\infty = 0$, then from (4.4) $s_\infty(a)$ satisfies:

$$s_\infty(a) = c\pi(a)e^{-\int_0^a \nu(s)ds} . \tag{4.10}$$

From (4.2) we see that $i_\infty(a) = 0$. And therefore $r_\infty(a)$ is given by:

$$r_\infty(a) = c\pi(a)[1 - e^{-\int_0^a \nu(s)ds}] . \tag{4.11}$$

It is worth noting that if $c\int_0^\infty \int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}dad\sigma \le 1$ then $I_\infty = 0$ is the only solution, on the other hand if

$$c\int_0^\infty \int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}dad\sigma > 1 , \quad \text{then } I_\infty = 0 \text{ and } I_\infty > 0$$

are possible solutions.

Also the above theorem assert the existence of an endemic disease when $c\int_0^\infty \int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}dad\sigma > 1$, so, it is clear that if the vaccination rate $\nu(a)$ is high enough it is not possible to attain the inequality. For this model the minimum vaccination coverage ν_c is defined to satisfy:

$$c\int_0^\infty \int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu_c(s)ds}dad\sigma = 1 \tag{4.12}$$

(4.12) could be used as a device for determining the effectiveness of a certain vaccination strategy. For example there are vaccination strategies that are followed for the eradication of important communicable diseases such as Measles and Rubella (see Hethcote [10], Katzman and Dietz [12]).

Here, we note that from (4.3) and (4.4), $I(t)$ has the following limiting equation:

$$I(t) = c\int_0^\infty \int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}e^{-\int_0^a k(s)I(t-a-\sigma+s)ds}I(t-\sigma)dad\sigma . \tag{4.13}$$

Also, we note that here we are assuming that the total population $p(a,t)$ has already reached its steady state $p_\infty(a) = c\pi(a)$.

5. Stability Results

In this section, we determine the stability of the steady states described in Theorem (4.1). Here, we note that we are assuming that the total population has already reached its steady state $p_\infty(a) = c\pi(a)$.

We define $\omega(t)$ to be the perturbation and is given by:

$$\omega(t) = I(t) - I_\infty$$

where $I(t)$ is the limiting equation given by (4.13). The linearization of ω yields the following equation:

$$\omega(t) = c\int_0^\infty \left\{\int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}e^{-I_\infty\int_0^a k(s)ds}da\right.$$
$$\left. -I_\infty \int_0^\sigma \int_0^\infty \pi(a+\sigma)e^{-\int_{a+s}^{a+\sigma}\gamma(c)dc}k(a+s)e^{-\int_0^{a+s}\nu(c)dc}k(a)e^{-I_\infty\int_0^{a+s}k(c)dc}dads\right\}\omega(t-\sigma)d\sigma \tag{5.1}$$

which has the following characteristic equation:

$$1 = c\int_0^\infty e^{-\lambda\sigma}\left\{\int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}e^{-\int_0^a \nu(s)ds}k(a)e^{-I_\infty\int_0^a k(s)ds}\right.$$
$$\left. -I_\infty \int_0^\sigma \int_0^\infty \pi(a+\sigma)e^{-\int_{a+s}^{a+\sigma}\gamma(c)dc}k(a+s)e^{-\int_0^{a+s}\nu(c)dc}k(a)e^{-I_\infty\int_0^{a+s}k(c)dc}dads\right\}d\sigma . \tag{5.2}$$

Setting $\lambda = x + iy$ we get the following equations:

$$1 = c\int_0^\infty \int_0^\infty e^{-x\sigma}\cos\sigma y\pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}e^{-I_\infty\int_0^a k(s)ds}dad\sigma$$
$$-cI_\infty \int_0^\infty \int_0^\sigma \int_0^\infty e^{-x\sigma}\cos\sigma y\pi(a+\sigma)e^{-\int_{a+s}^{a+\sigma}\gamma(c)dc}k(a+s)e^{-\int_0^{a+s}\nu(c)dc}k(a)e^{-I_\infty\int_0^{a+s}k(c)dc}dadsd\sigma \tag{5.3}$$

$$O = \int_0^\infty \int_0^\infty e^{-x\sigma}\sin\sigma y\pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}e^{-I_\infty\int_0^a k(s)ds}dad\sigma \tag{5.4}$$
$$-cI_\infty \int_0^\infty \int_0^\sigma \int_0^\infty e^{-x\sigma}\sin\sigma y\pi(a+\sigma)e^{-\int_{a+s}^{a+\sigma}\gamma(c)dc}k(a+s)e^{-\int_0^{a+s}\nu(s)dc}k(a)e^{-I_\infty\int_0^{a+s}k(c)dc}dadsd\sigma .$$

The following result describes the local asymptotic stability of the trivial equilibrium $I_\infty = 0$.

Theorem (5.1).

The trivial equilibrium $I_\infty = 0$ is locally asymptotically stable if

$$c\int_0^\infty \int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}dad\sigma < 1 ,$$

and unstable if

$$c\int_0^\infty \int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}dad\sigma > 1 .$$

Proof:

We start by noting that for the trivial equilibrium the characteristic equation satisfies:

$$1 = c\int_0^\infty \int_0^\infty e^{-x\sigma}\cos\sigma y\pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}dad\sigma \tag{5.5}$$

$$0 = c\int_0^\infty \int_0^\infty e^{-x\sigma} \sin \sigma y \pi(a+\sigma) e^{-\int_a^{a+\sigma} \gamma(s)ds} k(a) e^{-\int_0^a \nu(s)ds} da d\sigma \,. \tag{5.6}$$

So, if $c\int_0^\infty \int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}dad\sigma < 1$, (5.5) impies that $x < 0$. And if $c\int_0^\infty \int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}dad\sigma > 1$, then if we choose $y = 0$ in (5.5), (5.6) then the right hand side of (5.5) is a monotone decreasing function of x and goes to zero as $x \to \infty$, while its value at $x = 0$ is greater than one, hence there exists a unique $x > 0$ such that (5.5) and (5.6) are satisfied. So, the trivial equilibrium is unstable if $c\int_0^\infty \int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}dad\sigma > 1$.

We note that if $c\int_0^\infty \int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}dad\sigma = 1$, then $\lambda = 0$ is a solution, and there are no solutions which are pure imaginary.

Since $s(a,t)$ and $r(a,t)$ are determined by $I(t)$, we see that this leads to the following two results stated in Theorems (5.2) and (5.3).

Theorem (5.2)

The equilibrium $s_\infty(a) = c\pi(a)e^{-\int_0^a \nu(s)}$ is locally asymptotically stable if

$$c\int_0^\infty \int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}dad\sigma < 1 \,,$$

and unstable if

$$c\int_0^\infty \int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}dad\sigma > 1 \,.$$

Theorem (5.3)

The equilibrium $r_\infty(a) = c\pi(a)[1 - e^{-\int_0^a \nu(s)ds}]$ is stable if

$$c\int_0^\infty \int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}dad\sigma < 1 \,,$$

and unstable if

$$c\int_0^\infty \int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}dad\sigma > 1 \,.$$

In the next result, we describe the stability of the positive endemic equilibrium $I_\infty > 0$.

Theorem (5.4)

Suppose that $c\int_0^\infty \int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}dad\sigma > 1$, then $I_\infty > 0$ is locally asymptotically stable.

Proof:

Let $J_1(\sigma, I_\infty)$, $J_2(\sigma, I_\infty)$ be defined by:

$$J_1(\sigma, I_\infty) = c\int_0^\infty \pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(s)ds}k(a)e^{-\int_0^a \nu(s)ds}e^{-I_\infty \int_0^a k(s)ds}da$$

$$J_2(\sigma, I_\infty) = cI_\infty \int_0^\sigma \int_0^\infty \pi(a+\sigma)e^{-\int_{a+s}^{a+\sigma}\gamma(c)dc}k(a+s)e^{-\int_0^{a+s}\nu(c)dc}k(a)e^{-I_\infty \int_0^{a+s} k(c)dc}dads \,.$$

We note that $J_1(0, I_\infty) - J_2(0, I_\infty) = c\int_0^\infty \pi(a)k(a)e^{-\int_0^a \nu(s)ds}e^{-I_\infty \int_0^a k(s)ds}da > 0$ for $k(a)$ positive in a set of nonzero measure. Also

$$\frac{\partial J_1(\sigma, I_\infty)}{\partial \sigma} - \frac{\partial J_2(\sigma, I_\infty)}{\partial \sigma} =$$

$$-c\int_0^\infty (\mu(a+\sigma) + \gamma(a+\sigma))\pi(a+\sigma)e^{-\int_a^{a+\sigma}\gamma(c)dc}k(a)e^{-\int_0^a \nu(c)dc}e^{-I_\infty \int_0^a k(c)dc}da$$
$$+cI_\infty \int_0^\sigma \int_0^\infty (\mu(a+\sigma)+\gamma(a+\sigma))\pi(a+\sigma)e^{-\int_{a+s}^{a+\sigma}\gamma(c)dc}k(a+s)e^{-\int_0^{a+s}\nu(c)dc}e^{-I_\infty \int_0^{a+s} k(c)dc}dads$$
$$-cI_\infty \int_0^\infty \pi(a+\sigma)k(a+\sigma)e^{-\int_0^{a+\sigma}\nu(c)dc}k(a)e^{-I_\infty \int_0^{a+\sigma} k(c)dc}da \,. \tag{5.7}$$

Here we note that:

$$cI_\infty \int_0^\sigma \int_0^\infty (\mu(a+s) + \gamma(a+s))\pi(a+\sigma)e^{-\int_{a+s}^{a+\sigma}\gamma(c)dc}k(a+s)$$
$$\times e^{-\int_0^{a+s}\nu(c)dc}k(a)e^{-I_\infty \int_0^{a+s} k(c)dc}dads$$
$$\leq cI_\infty \int_0^\sigma \int_0^\infty (\mu(a+\sigma) + \gamma(a+\sigma))\pi(a+\sigma)e^{-\int_{a+s}^{a+\sigma}\gamma(c)dc}k(a+s)$$
$$\times e^{-\int_0^a \nu(c)dc}e^{-I_\infty \int_0^{a+s} k(c)dc}dads$$
$$= c\int_0^\infty (\mu(a+\sigma)+\gamma(a+\sigma))\pi(a+\sigma)e^{-\int_0^a \nu(c)dc}k(a)[e^{-I_\infty \int_0^a k(c)dc}e^{-\int_a^{a+\sigma}\gamma(c)dc} - e^{-I_\infty \int_0^{a+\sigma} k(c)dc}]da$$
$$-c\int_0^\sigma \int_0^\infty (\mu(a+\sigma)+\gamma(a+\sigma))\pi(a+\sigma)e^{-\int_0^a \nu(c)dc}k(a)e^{-I_\infty \int_0^{a+s} k(c)dc}e^{-\int_{a+s}^{a+\sigma}\gamma(c)dc}\gamma(a+s)dads \,. \tag{5.8}$$

By using (5.7) and (5.8) we find that:

$$\frac{\partial J_1(\sigma, I_\infty)}{\partial \sigma} - \frac{\partial J_2(\sigma, I_\infty)}{\partial \sigma} \leq 0 \,.$$

Now, we look at

$$I_\infty \int_0^\sigma \int_0^\infty \pi(a+\sigma)e^{-\int_{a+s}^{a+\sigma}\gamma(c)dc}k(a+s)e^{-\int_0^{a+s}\nu(c)dc}e^{-I_\infty \int_0^{a+s} k(c)dc}dads$$
$$\leq \; MI_\infty \int_0^\sigma \int_0^\infty \pi(a+\sigma)k(a+s)e^{-I_\infty \int_0^{a+s} k(c)dc}dads, \quad M = \sup_{a\in[0,\infty)} k(a)$$
$$\leq \; M\sigma\pi(\sigma) \to 0 \text{ as } \sigma \to \infty \text{ by assumption (4).}$$

So, $J_1(\sigma, I_\infty) - J_2(\sigma, I_\infty)$ decreases to zero as $\sigma \to \infty$. But if $x \geq 0$, $y \neq 0$ then

$$\int_0^\infty e^{-x\sigma} \sin y\sigma [J_1(\sigma, I_\infty) - J_2(\sigma, I_\infty)]d\sigma \neq 0$$

since $\sin \sigma y$ is periodic and odd and $J_1(0, I_\infty) - J_2(0, I_\infty) > 0$ and $[J_1(\sigma, I_\infty) - J_2(\sigma, I_\infty)]$ is decreasing. Finally, if $y = 0$ then (5.3) shows that

$$1 = c \int_0^\infty \int_0^\infty e^{-x\sigma} \pi(a+\sigma) e^{-\int_a^{a+\sigma} \gamma(s)ds} k(a) e^{-\int_0^a \nu(s)ds} e^{-I_\infty \int_0^a k(s)ds} da d\sigma$$

$$-cI_\infty \int_0^\infty \int_0^\sigma \int_0^\infty e^{-x\sigma} \pi(a+\sigma) e^{-\int_{a+s}^{a+\sigma} \gamma(c)dc} k(a+s) e^{-\int_0^{a+s} \nu(c)dc} k(a) e^{-I_\infty \int_0^{a+s} k(c)dc} dadsd\sigma .$$

Now, (4.6) implies that $c \int_0^\infty \int_0^\infty e^{-x\sigma} \pi(a+\sigma) e^{-\int_a^{a+\sigma} \gamma(s)ds} k(a) e^{-\int_0^a \nu(s)ds} e^{-I_\infty \int_0^a k(s)ds} da d\sigma < 1$ for $x \geq 0$ and so

$$\int_0^\infty \int_0^\sigma \int_0^\infty e^{-x\sigma} \pi(a+\sigma) e^{-\int_{a+s}^{a+\sigma} \gamma(c)dc} k(a+s) e^{-\int_0^{a+s} \nu(c)dc} k(a) e^{-I_\infty \int_0^{a+s} k(c)dc} dadsd\sigma < 0,$$

which is impossible so, $y \neq 0$, which gives the local asymptotic stability.

Since $s_\infty(a)$, $i_\infty(a)$ and $r_\infty(a)$ given by (4.7), (4.8) and (4.9) are determined by I_∞, the following result is a direct consequence of Theorem (5.4).

Theorem (5.5).

Suppose that $c \int_0^\infty \int_0^\infty \pi(a+\sigma) e^{-\int_a^{a+\sigma} \gamma(s)ds} k(a) e^{-\int_0^a \nu(s)ds} da d\sigma > 1$, then $s_\infty(a)$, $i_\infty(a)$ and $r_\infty(a)$ given respectively by (4.7), (4.8) and (4.9) are locally asymptotically stable.

Acknowledgments

The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste. He would also like to thank Professor Narasimhan, Director of the Mathematics Section, ICTP for a Visiting Fellowship.

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