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ASYMPTOTIC DESCRIPTION OF TWO METASTABLE PROCESSES OF SOLIDIFICATION FOR THE CASE OF LARGE RELAXATION TIME

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ABSTRACT

The non-isothermal Cahn-Hilliard equations in the n-dimensional case ($n = 2, 3$) are considered. The interaction length is proportional to a small parameter, and the relaxation time is proportional to a constant. The asymptotic solutions describing two metastable processes are constructed and justified. The soliton type solution describes the first stage of separation in alloy, when a set of "superheated liquid" appears inside the "solid" part. The Van der Waals type solution describes the free interface dynamics for large time. The smoothness of temperature is established for large time and the Mullins-Sekerka problem describing the free interface is derived.

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1. Introduction

The aim of this paper is to consider two metastable processes of solidification. The simplest processes of phase separation and solidification in a binary alloy are described by the non-isothermal *Cahn-Hilliard equations* [2, 7-9]:

$$\frac{\partial}{\partial t}\left(\theta + \frac{l}{2}\varphi\right) = k\Delta\theta + f(x,t),$$
$$-\tau\frac{\partial\varphi}{\partial t} = \xi^2\Delta\left(\xi^2\Delta\varphi + \frac{1}{2a}(\varphi - \varphi^3) + \theta\right). \tag{1}$$

Here $x \in \Omega$, $\Omega \subset R^n$ is a bounded domain with smooth (C^∞) boundary $\partial\Omega$, $n = 2, 3$; Δ is the Laplace operator; θ is the temperature; $k > 0$ is the thermal conductivity coefficient; $f(x,t)$ is a smooth function; $l > 0$ is the latent heat of melting; $\xi > 0$ is the surface tension; $\tau > 0$ is the time of relaxation; $a > 0$ is the interaction length; φ is the normalized concentration. The value $\varphi = 0$ corresponds to the uniform mixture, whereas the values $\varphi = \pm 1$ correspond to the pure phases, i.e. to the species which are distinguished from the alloy.

The system (1) is interesting from the physical viewpoint in the case

$$\xi \ll 1, \qquad a \ll 1. \tag{2}$$

Let us introduce a small parameter $\varepsilon > 0$ and let

$$a = \varepsilon, \qquad \frac{\xi^2}{a} = \text{const}, \qquad \tau = \text{const}, \qquad \varepsilon \to 0. \tag{2'}$$

For simplicity, we also assume that k is a constant and $l = 2$. Then, preserving the notation and performing simple transformations, we can rewrite system (1) in the form of the *Cahn-Hilliard system with a small parameter*

$$\frac{\partial}{\partial t}(\theta + \varphi) = \Delta\theta + f(x,t),$$
$$-\kappa\frac{\partial\varphi}{\partial t} = \Delta\left(\varepsilon^2\Delta\varphi + \varphi - \varphi^3 + \varepsilon\kappa_1\theta\right), \tag{3}$$

where $\kappa > 0$, κ_1 are constants, $x \in \Omega$, $\varepsilon \to 0$.

The initial data

$$\theta|_{t=0} = \theta^0(x,\varepsilon), \qquad \varphi|_{t=0} = \varphi^0(x,\varepsilon), \tag{4}$$

where θ^0, φ^0 are certain smooth (for $\varepsilon > 0$) functions, and the boundary conditions

$$\left.\frac{\partial\theta}{\partial N}\right|_\Sigma = 0, \quad \left.\frac{\partial\varphi}{\partial N}\right|_\Sigma = 0, \quad \left.\frac{\partial}{\partial N}\Delta\varphi\right|_\Sigma = 0 \tag{5}$$

complete the problem. Here N is the external normal to $\partial\Omega$, $\Sigma = [0,T] \times \partial\Omega$, $T > 0$.

The detailed mathematical analysis of the solution for arbitrary initial data θ^0, φ^0 is impossible at present. So we shall consider only two specific for the metastable stage processes, choosing the special initial data respectively. It is well known that there exist so-called stable, unstable and metastable stages of processes in alloys [31]. Under the assumptions (2′) they correspond to the cases: $\bar\varphi \geq 1$ or $\bar\varphi \leq -1$, and $-1/\sqrt{3} \leq \bar\varphi \leq 1/\sqrt{3}$, and $-1 < \bar\varphi < -1/\sqrt{3}$ or $1/\sqrt{3} < \bar\varphi < 1$ respectively. Here and below $\bar f(x) = w-\lim_{\varepsilon\to 0} f(x,\varepsilon)$ denotes the weak limit in the $\mathcal{D}'$ sense. The numbers ± 1 correspond to the zero points of the equilibrium chemical potential $F'(\varphi) = \varphi^3 - \varphi$. The numbers $\pm 1/\sqrt{3}$ correspond to local maxima/minima of F'.

The physical setting of our problem is the following: to simplify problem we assume that the initial concentration $\varphi^0(x) \in (1/\sqrt{3}, 1)$ for all $x \in \Omega$. The set of such points with concentration will be called "solid". Since the interval $(1/\sqrt{3}, 1)$ belongs to the domain of attraction to the point $\varphi^+_{\text{eq}} = 1$, the concentration ought to increase and to tend to φ^+_{eq}. But it is impossible to obtain the situation when $\varphi(x,t) = \varphi^+_{\text{eq}}$ at each point of Ω, since the global mass $m(\varphi)$,

$$m(\varphi) = \int_\Omega \varphi dx,$$

conserves in time and $m(\varphi^0) < |\Omega|$. Thus one must assume the appearance of subdomains $\Omega^\pm_t$ such that $\bar\varphi \in (1/\sqrt{3}, 1]$ as $x \in \Omega^+_t$ and $\bar\varphi \in [-1, -1/\sqrt{3})$ as $x \in \Omega^-_t$. After that the next stage of the solidification starts, when the subdomains $\Omega^\pm_t$ transform. The soliton type asymptotic solution, constructed in the present paper, describes the first stage when the "liquid" part Ω^-_t appears inside the "solid" part, but the volume of Ω^-_t is still small enough.

The Van der Waals type asymptotic solution describes the motion of $\Omega^\pm_t$ when $|\Omega^\pm_t| \geq$ const uniformly in ε. Our construction allows us as well to establish that the temperature remains a smooth function (in the leading term with respect to ε) during these processes. Therefore, the solidification process, described by Cahn–Hilliard equations (3), differs, in principle, from the dynamics of "solid"/"liquid", described by both the phase field model and the Cahn–Hilliard model (1) for small time of relaxation. Actually, according to the phase field model and (1) for $\tau \sim \varepsilon$, the temperature has a weak discontinuity on the free interface, whereas, according to the model (3), the temperature is almost the same on the "solid" and "liquid" domains (for both $|\Omega^-_t| \ll 1$ and $|\Omega^-_t| \sim$ const). So we can say that the appearing set Ω^-_t is the domain of "superheated liquid". We shall consider only the case $\overline{\varphi^0} \in (1/\sqrt{3}, 1]$. Nevertheless, it is clear that our construction also allows us to describe the analogous processes in the case $\overline{\varphi^0} \in [-1, -1/\sqrt{3})$.

The Cahn–Hilliard equation (the second equation in (1) with $\theta =$ const) was proposed by J. W. Cahn and J. E. Hilliard [7–9] as a simple model for the process of phase separation of a binary alloy at a fixed temperature. The surveys of physical aspects of this model are given also by V. P. Skripov & A. V. Skripov [41], J. D. Gunton & M. Droz [18], Y. S. Lipatov & V. V. Shilov [22], A. Novick-Cohen & L. A. Segel [31], T. Nose [32]. This equation arises as well in the study of mathematical biology and ecology, see D. S. Cohen & J. D. Murray [13] and M. Hazewinkel & J. Kaashoek & B. Leynse [19]. The numerical results for the Cahn–Hilliard equation were got by C. M. Elliott & D. A. French [17] and T. M. Rogers & K. R. Elder & R. C. Desai [40]. The equilibrium theory for the Cahn–Hilliard equation was investigated by J. Carr & M. E. Gurtin & M. Slemrod [10] in the one-dimensional case such that $\xi^2 \sim a \ll 1$. The existence of extremely slow evolving solutions of this equation in the same case was established by J. Carr & R. L. Pego [11] and N. Alikakos & P. W. Bates & G. Fusco [1] (see also references cited therein). The initial value problem for the Cahn–Hilliard equation is investigated in detail in the case when τ, ξ, a are constants, $n \leq 3$ and $\theta =$ const. For the description of the results the reader is referred to C. M. Elliott & S. Zheng [16], S. Zheng [46], P. W. Bates & P. C. Fife

[3], R. Temam [45], where the existence and uniqueness theorems, as well as the existence of an attractor are proved.

Nevertheless, it is clear that these results do not describe the limit problem of (1) as $a, \xi \to 0$. So, there we have the problem about the correspondence between the initial problem (1) with $a, \xi \ll 1$ and the limit problem as $a, \xi \to 0$. A similar relation between the *phase field system* with a small parameter $\varepsilon \ll 1$ and the limit (as ε tends to zero) *modified Stefan problem* was established by E. Radkevich [39], V. Danilov & G. Omel'yanov & E. Radkevich [14, 15, 33, 34], P. Plotnikov & V. Starovoitov [38], M. Soner [42], see also the paper by B. Stoth [43] and the papers [35, 36] about analogous problems. Moreover, it is impossible to reduce the problem with a small parameter $\varepsilon \to 0$ to the already studied problem with $\varepsilon =$ const. Actually, by changing the scale $t' = t/\varepsilon^2$, $x' = x/\varepsilon$, system (1) can be transformed as follows

$$\frac{\partial}{\partial t'}(\theta' + \varphi') = \Delta_{x'}\theta' + f',$$
$$-\kappa\frac{\partial \varphi'}{\partial t'} = \Delta_{x'}\left(\Delta_{x'}\varphi' + \varphi' - (\varphi')^3 + \varepsilon\kappa_1\theta'\right), \tag{6}$$
$$\theta'|_{t'=0} = \theta^{0'}(x', \varepsilon), \qquad \varphi'|_{t'=0} = \varphi^{0'}(x', \varepsilon).$$

Here $F'(x', t', \varepsilon) = F(\varepsilon x, \varepsilon^2 t, \varepsilon)$, $f' = \varepsilon^2 f(\varepsilon x, \varepsilon^2 t)$.

Now we note that the conditions

$$\|\theta^{0'}; L^2(\Omega')\| \leq \text{const}, \qquad \|\varphi^{0'}; L^2(\Omega')\| \leq \text{const},$$

which are natural for $\varepsilon =$ const, imply the senseless assumption

$$|\Omega| \propto \varepsilon^n \qquad \text{as} \qquad \varepsilon \to 0,$$

since

$$\int_{\Omega'} (\varphi^{0'}(x', \varepsilon))^2 dx' = \frac{1}{\varepsilon^n}\int_\Omega (\varphi^0(x, \varepsilon))^2 dx = \text{const}$$

and φ^0 varies from -1 to 1. Here and below $\|f; X\|$ denotes the norm of f in the space X.

Thus we have to admit that

$$\|\theta^{0'}; L^2(\Omega')\| \leq \frac{c}{\varepsilon^{n/2}}, \qquad \|\varphi^{0'}; L^2(\Omega')\| \leq \frac{c}{\varepsilon^{n/2}},$$

which corresponds to $|\Omega| \propto$ const. Here $c =$ const.

Further, the solvability of (6) for $t' \leq T_1'$, where T_1' is a constant, means only the solvability of the initial problem with the small parameter for times $t \leq \varepsilon^2 T_1' \to 0$ as $\varepsilon \to 0$. Let the estimate

$$\|\varphi'; L^\infty(0, T_1'; L^2(\Omega'))\| + \|\nabla_{x'}\varphi'; L^2(Q')\| + \|\nabla_{x'}\theta'; L^2(Q')\| \leq \frac{c_1}{\varepsilon^{n/2}}$$

hold for $t' \leq T_1'$. Here $Q' = [0, T_1'] \times \Omega'$. Then, using the construction common for parabolic equations, we see that the energy equality

$$\frac{\kappa}{2}\frac{d}{dt'}\int_{\Omega'} \varphi'^2 dx' + \int_{\Omega'} (\Delta_{x'}\varphi')^2 dx' + 3\int_{\Omega'} \varphi'^2 |\nabla_{x'}\varphi'|^2 dx'$$

$$= \int_{\Omega'} |\nabla_{x'}\varphi'|^2 dx' + \varepsilon\kappa_1 \int_{\Omega'} (\nabla_{x'}\varphi', \nabla_{x'}\theta') dx'$$

implies the following estimate

$$\frac{d}{dt'}\|\varphi'; L^2(\Omega')\|^2 + \varepsilon^2\lambda_0^2 b\,\|\varphi'; L^2(\Omega')\|^2 \le \frac{c_1^2}{\varepsilon^n}$$

for all $t' \le T_1'$. Here $\lambda_0 > 0$ is the minimal eigenvalue of the Laplace operator $-\Delta_x$ on Ω; (f, g) denotes the scalar product of vectors f, g in R^n; $b = \text{const}$.

Therefore,

$$\|\varphi'; L^2(\Omega')\|^2(t') \le \frac{c^2}{\varepsilon^n} e^{-\varepsilon^2\lambda_0^2 bt'} + \frac{c_1^2}{\varepsilon^{n+2} b\lambda_0^2}\left(1 - e^{-\varepsilon^2\lambda_0^2 bt'}\right).$$

Hence,

$$\|\varphi'; L^2(\Omega')\|\,(t') \le \frac{c_2}{\varepsilon^{n/2+1}}.$$

The last inequality yields the useless estimate for the solution with respect to initial variables

$$\|\varphi; L^2(\Omega)\|\,(t) \le \frac{c_2}{\varepsilon} \qquad \text{as} \qquad \varepsilon \to 0.$$

So, problems with a small parameter have to be examined from the viewpoint of both *constructing an asymptotic solution and proving the existence and uniqueness theorems.*

The multidimensional Cahn–Hilliard equation with a small parameter was considered by B. Stoth [44] in the spherical symmetry case, and by R. L. Pego [37].

The non-isothermal multidimensional equations for the process of phase separation were obtained by H. Alt & I. Pawlow [2], where the existence of a weak solution satisfying a generalization of (1) is established in the case $\tau, \xi, a = \text{const}$. The Van der Waals type asymptotic solution of the multidimensional Cahn–Hilliard system (1) was constructed and established by G. Omel'yanov & V. Danilov & E. Radkevich [35, 36] in the fast relaxation case

$$a = \varepsilon, \qquad \xi^2/a = \text{const}, \qquad \tau/a = \text{const}, \qquad \varepsilon \to 0.$$

In this paper the asymptotic solutions of problem (3)–(5) are constructed and justified under the assumption that the classical smooth solution of (3)–(5) exists. The asymptotic solutions are constructed on the basis of a *modified two-scale method* (V. Maslov & V. Tsupin [23], V. Maslov & G. Omel'yanov [24, 25], V. Maslov & V. Danilov & K. Volosov [28]) for obtaining solutions with localized "fast" variation. A similar method was used by E. Radkevich [39] and V. Danilov & G. Omel'yanov & E. Radkevich [14, 15, 33–36] for obtaining the asymptotic solutions of both the phase field system and the Cahn–Hilliard system in the fast relaxation case.

It is very important to note that the boundary layer construction of the asymptotic solution cannot be used in problems with free boundary. The "boundary layer construction" means the construction of a pair of asymptotic solutions $Y_\pm(\eta, x, t, \varepsilon)$ like boundary layers on the left ($\eta < 0$) and right ($\eta > 0$) sides of the free boundary $\Gamma_t = \{x, \eta = S(x,t)/\varepsilon = 0\}$ along the direction normal to Γ_t. As in elliptic problems, $Y_\pm$ vanish (or tend to some constants) at the points sufficiently far from Γ_t, and $Y_+ = Y_-$ on Γ_t. Obviously, in general, the function

$$Y\left(\frac{S}{\varepsilon}, x, t, \varepsilon\right) = \begin{cases} Y_+, & \Omega_t^+ \cup \Gamma_t = \{x, S(x,t) \ge 0\}, \\ Y_-, & \Omega_t^- \cup \Gamma_t = \{x, S(x,t) \le 0\} \end{cases} \tag{7}$$

is continuous only on $\Omega = \Omega_t^+ \cup \Omega_t^- \cup \Gamma_t$, and cannot be a classical solution of our problem for $\varepsilon > 0$, so Y is a weak formal asymptotic solution. In fact, the function (7) does not describe the behavior of the solution close to Γ_t, since Y is not the solution on Γ_t. But the main purpose of the whole construction is to describe both the solution close to Γ_t and the evolution of Γ_t. Nevertheless, it is possible to define the classical solution on Ω in the case when the analytical extensions $\widetilde{Y_\pm}$ of $Y_\pm$ exist, and $\widetilde{Y_+}|_{\eta<0} = Y_-$, $\widetilde{Y_-}|_{\eta>0} = Y_+$. The additional conditions for the existence of such analytical extensions, the so-called *Hugoniot type conditions*, are exactly the conditions on the free boundary (Gibbs–Thomson law and so on) (see V. Maslov & G. Omel'yanov [24] for analogous problem). This defect of the boundary layer construction was eliminated by using the so-called method of matched asymptotic expansions, see G. Caginalp & P. C. Fife [5]. Nevertheless, a version [37] of this method results in unbounded "lower" terms of the asymptotic expansion. Obviously, this follows from the fact that the necessary condition for the existence of a bounded solution of the model equations is of no practical use, since the normalization condition was posed (see [37], p. 272). The possibility of constructing an asymptotical expansion up to an arbitrary precision is a very important condition in the theory of perturbations. This is the first necessary step to justify the asymptotic. As a rule, the unboundedness of lower terms means that the leading term is uncorrected. For example, the unboundedness of the lower term of the asymptotic expansion for weak nonlinear problems indicates that the first term includes some additional terms, which were not posed at the initial instant of time. It leads to the so-called three-waves and three-trains processes describing the non-linear interaction between the waves and a generation of new waves for positive time, see V. P. Maslov & G. A. Omel'yanov [26, 27].

2. Soliton type asymptotic solution

Let us formulate the main result of this section. By $\theta_0 = \theta_0(x,t)$, $\varphi_0 = \varphi_0(x,t) \in (1/\sqrt{3}, 1)$, $\psi = \psi(x)$, we denote the solutions of the following model problems

$$\kappa\frac{\partial\varphi_0}{\partial t} = \Delta(\varphi_0^3 - \varphi_0), \qquad x \in \Omega, \quad t > 0,$$
$$\varphi_0\Big|_{t=0} = \overline{\varphi^0}(x), \qquad \frac{\partial\varphi_0}{\partial N}\Big|_\Sigma = 0, \tag{8}$$

$$\frac{\partial\theta_0}{\partial t} = \Delta\theta_0 + f(x,t) - \frac{\partial\varphi_0}{\partial t}, \qquad x \in \Omega, \quad t > 0,$$
$$\theta_0\Big|_{t=0} = \overline{\theta^0}(x), \qquad \frac{\partial\theta_0}{\partial N}\Big|_\Sigma = 0, \tag{9}$$

$$\kappa D_\nu = \frac{1}{3}\mathcal{K}_t\big(1 + G(\check{\varphi}_0)\big) + G_0\frac{\partial\check{\varphi}_0}{\partial\nu}, \qquad \psi|_{\Gamma_0} = 0. \tag{10}$$

Here $D_\nu = 1/|\nabla\psi|$ is the normal velocity of motion of the surface $\Gamma_t = \{x \in \Omega, \psi(x) = t\}$; $\mathcal{K}_t = \text{div}(\nu)$ is the mean curvature of Γ_t, $\nu = \nabla\psi/|\nabla\psi|$ is the vector normal to Γ_t; $\check{F} = F(x, \psi(x))$ for all continuous functions $F(x,t)$; $\partial/\partial\nu = (\nu, \nabla)$;

$$G(\check{\varphi}_0) = \frac{IQ^2}{2(Q - I)}, \qquad G_0 = \frac{G(\check{\varphi}_0)}{\check{\varphi}_0}, \qquad \frac{\partial\check{\varphi}_0}{\partial\nu} = (\nu, \nabla\varphi_0(x, \psi)),$$

$$Q = \sqrt{3(\check\varphi_0)^2 - 1}, \quad I = \sqrt{2}\check\varphi_0 \ln J, \quad J = (\sqrt{2}\check\varphi_0 + Q)/\sqrt{b}, \quad b = 1 - \check\varphi_0^2.$$

Theorem 1. *Let $\Gamma_0 = \{x \in \Omega, \psi(x) = 0\}$ be a sufficiently smooth closed surface of codimension 1. Let sufficiently smooth solutions of problem (8)-(10) exist, and $\varphi_0 \in (1/\sqrt{3}, 1)$, and let* $\operatorname{dist}(\Gamma_t, \partial\Omega) \geq \text{const}$ *for all $t \in [0, T]$. Then, for any $M \geq 0$, there exists a formal asymptotic (up to $\mathcal{O}(\varepsilon^{M+1})$) solution of problem (3)-(5). The leading term of this asymptotic solution has the form*

$$\theta(x, t, \varepsilon) = \theta_0(x, t) + \mathcal{O}(\varepsilon),$$
$$\varphi(x, t, \varepsilon) = \varphi_0(x, t) + \chi(\eta, x) + \mathcal{O}(\varepsilon). \tag{11}$$

Here

$$\chi = -8Q^2\left\{e^{\xi} + 8be^{-\xi} + 8\check\varphi_0\right\}^{-1}, \tag{12}$$
$$\xi = \beta(\eta + \psi_1(x)), \quad \eta = (t - \psi(x))/\varepsilon, \quad \beta = Q/|\nabla\psi|,$$

$\psi_1 \in C^\infty$ is the solution of the inhomogeneous linearization of equation (10).

Remark 1. From (11), (12) it easily follows that our asymptotic solution is a so-called self-similar solution. It implies the special choice of the initial data (4) in an ε-neighbourhood of Γ_0, i.e., φ^0 must exhibit the special behavior (not only in the principle term, but also in the lower terms of the expansion of $\varphi^0(x, \varepsilon)$ with respect to ε). The initial temperature θ^0 can be an arbitrary smooth function in the principle term, nevertheless its lower terms must exhibit a special behavior close to Γ_0.

Remark 2. It is not too difficult to prove that equation (10) is a quasilinear parabolic equation, in which x_ν (along the vector $\nu = \nabla\psi/|\nabla\psi|\big|_{\Gamma_t}$) is a time like variable, and x_t (tangential to Γ_t) are space like variables. So, the additional condition in (10) is actually the initial condition [14, 15]. The classical solvability and uniqueness of solutions of quasilinear parabolic problems with smooth coefficients are the result of the realization of some matching conditions between the initial and boundary data [21]. So, we assume that these matching conditions are realized.

At first let us consider the statement of Theorem 1. Formulas (11), (12) and the solutions of problems (8)-(10) describe the motion of the soliton χ on the smooth "background" φ_0 (a soliton type solution was obtained also in [17] by numerical simulations for the one-dimensional Cahn-Hilliard equation). Obviously, the surface Γ_t is the set of maximum magnitude of $|\chi|$

$$A = \max_{x\in\Omega} |\chi| = -\chi|_{\Gamma_t} = Q^2\{\frac{9}{8} + \check\varphi_0 - \check\varphi_0^2\}^{-1}.$$

It is easy to prove that this solution exists if and only if $\check\varphi_0 \in (1/\sqrt{3}, 1)$. The amplitude A is a monotonically increasing function, $A'_{\check\varphi_0} > 0$, and trivial calculations show that $A \to 0$ and $G \to 0$ as $\check\varphi_0 \to 1/\sqrt{3}$. It is also clear that there exists a value $\varphi^* \in (1/\sqrt{3}, 1)$ such that $A < \check\varphi_0$ as $\check\varphi_0 \in (1/\sqrt{3}, \varphi^*)$, and $A > \check\varphi_0$ as $\check\varphi_0 \in (\varphi^*, 1)$. Thus, moving into the domain with $\check\varphi_0 \in (\varphi^*, 1)$, the soliton solution describes how the set $\Omega^-_{t,\varepsilon} = \{x \in \Omega, \check\varphi_0 + \chi < 0\}$ with negative concentration arises. Let us consider the behavior of the solution as φ_0 tends to 1. Setting $\varphi_0 = 1 - \tilde\varphi(x, t)\exp(-1/\delta)$, $\delta \ll 1$, we get the following relations

$$A = \frac{16}{9} + \mathcal{O}\big(e^{-1/\delta}\big), \qquad G = -1 + \mathcal{O}(\delta).$$

Hence, $D_\nu \sim \delta$ and the velocity of the soliton motion decreases as $\delta \to 0$. On the other hand, the volume of the set $\Omega^-_{t,\varepsilon}$ increases, since $b \sim \exp(-1/\delta)$ and $|\Omega^-_{t,\varepsilon}| \sim |\Omega^-_{t,\varepsilon/\delta}| \sim \varepsilon/\delta$ for $\delta \ll 1$. Thus, this solution describes the appearance of a sufficiently large domain of "superheated liquid", since the concentration $\varphi \sim -7/9$ on $\Omega^-_{t,\varepsilon/\delta}$ and the temperature θ_0 is almost independent of φ_0 at these points. Nevertheless, this asymptotic solution is correct only if $|\Omega^-_{t,\varepsilon/\delta}| \to 0$ as $\varepsilon \to 0$.

We shall also see that the first corrections of the asymptotic expansions for the temperature and concentration have the form of smoothed shock waves. So,

$$w - \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(\theta - \theta_0) = A_{1,\theta}H(t - \psi), \qquad w - \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(\varphi - \varphi_0 - \chi) = A_{1,\varphi}H(t - \psi),$$

where H is the Heaviside function,

$$A_{1,\theta} = 2\frac{I}{\check\varphi_0}D_\nu, \qquad A_{1,\varphi} = 2\frac{\kappa I}{\check\varphi_0 Q^2}D_\nu$$

are the amplitudes of jumps on Γ_t. It is easy to calculate that $A_{1,\theta}$, $A_{1,\varphi}$ are bounded as $\check\varphi_0 \to 1$.

Let us prove Theorem 1 and consider the general method for constructing the asymptotic solution of this problem up to an arbitrary precision.

First, we introduce the following classes of functions

$$\begin{aligned}
\mathcal{H} &= \Big\{f(\eta, x, t) \in C^\infty(R^1 \times Q),\ \exists f^\pm \in C^\infty(\bar Q), \\
&\qquad \lim_{\eta\to\pm\infty} \eta^m D^{r\alpha\gamma}(f(\eta, x, t) - f^\pm(x, t)) = 0\Big\}, \\
\mathcal{S} &= \Big\{f(\eta, x, t) \in \mathcal{H},\ f^+ = f^- = 0\Big\}, \\
\mathcal{P} &= \Big\{f(\eta, x', t) \in C^\infty(R^1_+ \times \Sigma),\ \lim_{\eta\to\infty} \eta^m D^{r\alpha'\gamma} f(\eta, x', t) = 0\Big\},
\end{aligned}$$

where

$$D^{r\alpha\gamma} = \frac{\partial^r}{\partial\eta^r}\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\gamma}{\partial t^\gamma} \quad \forall\, m \geq 0, \quad r \geq 0, \quad \gamma \geq 0, \quad |\alpha| \geq 0, \quad |\alpha'| \geq 0.$$

Here and below $Q = [0, T] \times \Omega$.

Lemma 1. 1. *Let $S(x, t) \in C^\infty(\bar Q)$ be such that $\partial S/\partial t|_{\Gamma_t} \neq 0$, where $\Gamma_t = \{(x, t) \in Q, S(x, t) = 0\}$. Then, for any function $f(\eta, x, t) \in \mathcal{H}$, we get*

$$f\Big(\frac{S(x, t)}{\varepsilon}, x, t\Big) = f\Big(\beta\frac{t - \psi(x)}{\varepsilon}, x, t\Big) + \mathcal{O}(\varepsilon),$$

where $t = \psi(x)$ is the equation of the surface $S(x, t) = 0$ and $\beta = \partial S/\partial t|_{\Gamma_t}$.

2. *Let $\mu(\eta, x, t)$, $\zeta(\eta, x, t) \in \mathcal{H}$ be such that $\mu^\pm = \pm 1$, $\zeta^+ = 1$, $\zeta^- = 0$. Then, for any function $f \in \mathcal{H}$, we have the representations*

$$f = \frac{1}{2}(f^+ + f^-) + \frac{1}{2}(f^+ - f^-)\mu(\eta, x, t) + \omega_1(\eta, x, t),$$

$$f = f^- + (f^+ - f^-)\zeta(\eta, x, t) + \omega_2(\eta, x, t),$$

where ω_i are functions from $\mathcal{S}$.

3. *The relations*

$$(t-\psi)^k f\Big(\frac{t-\psi}{\varepsilon}, x, t\Big) = \mathcal{O}(\varepsilon^k), \qquad k \geq 0,$$

$$g(x,t)\, f\Big(\frac{t-\psi(x)}{\varepsilon}, x, t\Big) = g(x,\psi(x))\, f\Big(\frac{t-\psi(x)}{\varepsilon}, x, \psi(x)\Big) + \mathcal{O}(\varepsilon)$$

hold for any functions $f(\eta,x,t) \in \mathcal{S}$, $g(x,t) \in C^\infty(\bar{Q})$.

The proof obviously follows from the definition (see also [24]).

Remark. The representation $S = t - \psi(x)$ does not mean that the solution must move with velocity of order $\mathcal{O}(1)$. Actually, since the function $\psi(x)$ can increase rapidly along the direction normal to the surface $\Gamma_0 = \{x \in \Omega, \psi(x) = 0\}$, the motion of the solution can be arbitrary slowly.

Let us begin to construct the self-similar asymptotic solution of problem (3)–(5). First of all, we note that the leading term of the asymptotic expansion for θ must be a smooth function, since the leading term of φ is a soliton. This implies that the asymptotic solution has the following form

$$\theta(x,t,\varepsilon) = \vartheta^M(x,t,\varepsilon) + \varepsilon \mathcal{V}^M\Big(\frac{S(x,t,\varepsilon)}{\varepsilon}, \frac{x_N}{\varepsilon}, x, t, \varepsilon\Big) + \mathcal{O}(\varepsilon^{M+1}),$$

$$\varphi(x,t,\varepsilon) = \Phi^M(x,t,\varepsilon) + \mathcal{W}^M\Big(\frac{S(x,t,\varepsilon)}{\varepsilon}, \frac{x_N}{\varepsilon}, x, t, \varepsilon\Big) + \mathcal{O}(\varepsilon^{M+1}),$$

$$\vartheta^M(x,t,\varepsilon) = \sum_{j=0}^{M} \varepsilon^j \theta_j(x,t), \qquad \Phi^M(x,t,\varepsilon) = \sum_{j=0}^{M} \varepsilon^j \varphi_j(x,t), \tag{13}$$

$$\begin{aligned} \mathcal{V}^M(\eta,\tau,x,t,\varepsilon) &= \sum_{j=1}^{M} \varepsilon^{j-1}\{U_j(\eta,x,t) + Y_j(\tau,x',t)\}, \\ \mathcal{W}^M(\eta,\tau,x,t,\varepsilon) &= \chi(\eta,x,t) + \sum_{j=1}^{M} \varepsilon^j\{W_j(\eta,x,t) + Z_j(\tau,x',t)\}. \end{aligned}$$

Here x_N is the distance from $\partial\Omega$ along the internal normal, $x' \in \partial\Omega$,

$$S,\ \theta_i,\ \varphi_j \in C^\infty(\bar{Q}), \qquad Y_j(\tau,x',t),\ Z_j(\tau,x',t) \in \mathcal{P},$$

$$\chi(\eta,x,t) \in \mathcal{S}, \qquad U_j(\eta,x,t),\ W_j(\eta,x,t) \in \mathcal{H},$$

and

$$U_j^- = 0, \quad W_j^- = 0, \quad \frac{\partial S}{\partial t}\Big|_{\Gamma_t} \neq 0, \quad \Gamma_t = \{x \in \Omega,\ S(x,t) = 0\}. \tag{14}$$

By Lemma 1, without loss of generality, we can assume that

$$S = t - \psi(x), \qquad \chi = \chi(\eta, x).$$

Substituting (12) into equations (3), we get the relations

$$\begin{aligned} \frac{\partial}{\partial t}(\vartheta^M + \Phi^M) - \Delta\vartheta^M - f = \frac{1}{\varepsilon}\Big\{\hat{\mathcal{L}}_2\mathcal{V}^M - \frac{\partial}{\partial\eta}\mathcal{W}^M\Big\} \\ -\Big\{\Big(\frac{\partial}{\partial\eta} + \mathcal{L}_1\Big)\mathcal{V}^M + \frac{\partial}{\partial t}\mathcal{W}^M\Big\} + \varepsilon\Big(\Delta - \frac{\partial}{\partial t}\Big)\mathcal{V}^M, \end{aligned} \tag{15}$$

$$\begin{aligned} &\frac{1}{\varepsilon^2}\hat{\mathcal{L}}_2\big(\hat{\mathcal{L}}_2\mathcal{W}^M + \mathcal{W}^M - (\Phi^M + \mathcal{W}^M)^3\big) \\ &+\frac{1}{\varepsilon}\Big\{\Big(\kappa\frac{\partial}{\partial\eta} - \hat{\mathcal{L}}_2\hat{\mathcal{L}}_1\Big)\mathcal{W}^M - \hat{\mathcal{L}}_1\big(\hat{\mathcal{L}}_2\mathcal{W}^M + \mathcal{W}^M - (\Phi^M + \mathcal{W}^M)^3\big)\Big\} \\ &+\Big\{(\hat{\mathcal{L}}_2\Delta + \hat{\mathcal{L}}_1^2)\mathcal{W}^M + \kappa_1\hat{\mathcal{L}}_2\mathcal{V}^M + \kappa\frac{\partial}{\partial t}(\Phi^M + \mathcal{W}^M) \\ &+\Delta\big(\hat{\mathcal{L}}_2\mathcal{W}^M + \Phi^M + \mathcal{W}^M - (\Phi^M + \mathcal{W}^M)^3\big)\Big\} \\ &-\varepsilon\Big\{\hat{\mathcal{L}}_1(\Delta\mathcal{W}^M + \kappa_1\mathcal{W}^M) + \Delta(\hat{\mathcal{L}}_1\mathcal{W}^M - \kappa_1\vartheta^M)\Big\} \\ &+\varepsilon^2\Delta\left\{\Delta(\Phi^M + \mathcal{W}^M) + \kappa_1\mathcal{V}^M\right\} = 0. \end{aligned} \tag{16}$$

Here

$$\hat{\mathcal{L}}_2 = |\nabla\psi|^2\frac{\partial^2}{\partial\eta^2} + |\nabla x_N|^2\frac{\partial^2}{\partial\tau^2}, \qquad \hat{\mathcal{L}}_1 = \frac{\partial}{\partial\eta}\hat{\Pi} - \frac{\partial}{\partial\tau}\hat{\Pi}_b,$$

$$\hat{\Pi} = 2(\nabla\psi, \nabla) + \Delta\psi, \qquad \hat{\Pi}_b = 2(\nabla x_N, \nabla) + \Delta x_N.$$

First, let us obtain the regular terms of expansion (13). Passing to the limit as $\eta \to \pm\infty$, $\tau \to \infty$, from (15), (16) we get

$$\begin{aligned} &\frac{\partial}{\partial t}(\vartheta^M + \Phi^M) - \Delta\vartheta^M - f = -\frac{\partial}{\partial t}\mathcal{W}^{M\pm} - \varepsilon\left(\frac{\partial}{\partial t} - \Delta\right)\mathcal{V}^{M\pm}, \\ &\kappa\frac{\partial}{\partial t}(\Phi^M + \mathcal{W}^{M\pm}) + \Delta\big(\Phi^M + \mathcal{W}^{M\pm} - (\Phi^M + \mathcal{W}^{M\pm})^3\big) \\ &+\varepsilon\kappa_1\Delta\vartheta^M + \varepsilon^2\Delta\big(\Delta(\Phi^M + \mathcal{W}^{M\pm}) + \kappa_1\mathcal{V}^{M\pm}\big). \end{aligned} \tag{17}$$

Introducing the notation

$$\theta_j^\pm = \theta_j + U_j^\pm, \quad \Phi_j^\pm = \varphi_j + W_j^\pm, \qquad j = 1, 2, \ldots, M, \tag{18}$$

and setting the terms of the order $\mathcal{O}(\varepsilon^j)$ equal to zero, we get equations (8), (9) for the leading terms, as well the following equations for the lower terms of the asymptotic expansion

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - \Delta\right)\theta_k^\pm = f_{k,\theta}^\pm(x,t), \quad x \in \Omega_t^\pm, \quad t > 0, \\ &\kappa\frac{\partial}{\partial t}\Phi_k^\pm - \Delta\big((3\varphi_0^2 - 1)\Phi_k^\pm\big) = f_{k,\varphi}^\pm(x,t). \end{aligned} \tag{19}$$

Here $\Omega_t^{\pm}$ are subdomains of Ω such that

$$\Omega_t^+ = \{x \in \Omega, \psi(x) < t\}, \quad \Omega_t^- = \{x \in \Omega, \psi(x) > t\}, \quad \Omega = \Omega_t^+ \cup \Omega_t^- \cup \Gamma_t,$$

$f_{k,\theta}^{\pm}(x,t)$, $f_{k,\varphi}^{\pm}(x,t)$ are functions of the previous terms of the expansion and their derivatives. In particular,

$$f_{1,\theta}^{\pm} = -\frac{\partial}{\partial t}\Phi_1^{\pm}, \qquad f_{1,\varphi}^{\pm} = \kappa_1 \Delta\theta_0.$$

Now we note that the supports of fast variations of boundary-layer functions and of functions rapidly varying on a neighbourhood of Γ_t do not intersect (up to terms $\mathcal{O}(\varepsilon^{\infty})$), since dist$(\partial\Omega, \Gamma_t) \geq$ const. Hence the solution in a neighbourhood of Γ_t and in a neighbourhood of $\partial\Omega$ is constructed differently.

Let us consider a neighbourhood of Γ_t. Passing to the limit as $\tau \to \infty$, we obtain that the terms in (15), (16) belong to the space $\mathcal{S}$, since relations (17) hold. So we can use the method developed by V. Danilov, G. Omel'yanov and E. Radkevich [14, 15, 35, 36, 39]: step by step we decompose the coefficients at ε^j in (15), (16), $j = -2, -1, 0, \ldots$, into the Taylor expansion at the point $t = \psi(x)$ and use the relation $\eta = (t - \psi)/\varepsilon$. Further, passing to the functions of independent variables η, x, we obtain the asymptotic solution on the surface Γ_t. Finally, we define a sufficiently smooth extension of these functions on $[0,T] \times \Omega$, so that the lower terms of the asymptotic expansion exist and belong to the space $\mathcal{H}$.

Let us denote $\check{F} = F(\eta, x, t)|_{t=\psi(x)}$ and consider the terms $\mathcal{O}(\varepsilon^{-2})$ in (16):

$$\frac{\partial^2}{\partial\eta^2}\Big\{|\nabla\psi|^2\frac{\partial^2\check{\chi}}{\partial\eta^2} + \check{\chi} - (\check{\varphi}_0 + \check{\chi})^3\Big\} = 0.$$

After the integration, we get

$$\frac{\partial}{\partial\eta}\Big\{|\nabla\psi|^2\frac{\partial^2\check{\chi}}{\partial\eta^2} + \check{\chi} - (\check{\varphi}_0 + \check{\chi})^3\Big\} = \text{const}.$$

Obviously, const $= 0$ since $\check{\chi} \in \mathcal{S}$. Therefore

$$|\nabla\psi|^2\frac{\partial^2\check{\chi}}{\partial\eta^2} + \check{\chi} - (\check{\varphi}_0 + \check{\chi})^3 = c, \qquad \check{\chi} \to 0 \quad \text{as} \quad \eta \to \pm\infty. \tag{20}$$

Choosing $c = -\check{\varphi}_0^3$, we get that the solution on Γ_t has the form (12), where the "constant" of integration $\psi_1 = \psi_1(x)$ is an arbitrary function from $C^{\infty}(\bar{\Omega})$.

Let us extend $\check{\chi}$ by the identity to $\chi = \check{\chi}(\eta, x)$ for all $(x,t) \in Q$. Now, setting the terms $\mathcal{O}(\varepsilon^{-2+k})$ equal to zero, from (15), (16) we get the following equations

$$\frac{\partial^2 \check{U}_k}{\partial\eta^2} = \check{F}_k^{\theta}, \qquad \check{U}_k \to 0 \quad \text{as} \quad \eta \to -\infty, \tag{21}$$

$$\frac{\partial^2}{\partial\eta^2}\,\hat{L}\,\check{W}_k = \check{F}_k^{\varphi}, \qquad \check{W}_k \to 0 \quad \text{as} \quad \eta \to -\infty. \tag{22}$$

Here

$$\hat{L} = |\nabla\psi|^2\frac{\partial^2}{\partial\eta^2} + 1 - 3(\check{\varphi}_0 + \chi)^2,$$

$\check{F}_k^{\varphi}$, $\check{F}_k^{\theta}$ are functions of $\theta_0, \varphi_0, \ldots, U_{k-1}, W_{k-1}$ and of their derivatives at the point $t = \psi(x)$, $k = 1, 2, \ldots, M$. In particular,

$$\begin{aligned}\check{F}_1^{\varphi} &= \frac{\partial^2}{\partial\eta^2}\Big\{\hat{\Pi}\frac{\partial\chi}{\partial\eta} + 3(\check{\varphi}_0 + \chi)^2\varphi_1\Big\} \\ &+3\frac{\partial\varphi_0}{\partial t}\Big\{2 + \eta\frac{\partial}{\partial\eta}\Big\}\frac{\partial}{\partial\eta}(\varphi_0 + \chi)^2 - \kappa|\nabla\psi|^{-2}\frac{\partial\chi}{\partial\eta}\Big|_{t=\psi},\end{aligned} \tag{23}$$

$$\check{F}_1^{\theta} = |\nabla\psi|^{-2}\frac{\partial\chi}{\partial\eta}. \tag{24}$$

It is not too difficult to prove that the following statement holds (see also [14, 24])

Lemma 2. *The solutions $\check{U}_k \in \mathcal{H}$, $\check{W}_k \in \mathcal{H}$ of* (21), (22) *exist if and only if*

$$\check{F}_k^{\theta} \in \mathcal{S}, \qquad \check{F}_k^{\varphi} \in \mathcal{S}, \tag{25}$$

$$\int_{-\infty}^{\infty} \check{F}_k^{\theta} d\eta = 0, \qquad \int_{-\infty}^{\infty} \check{F}_k^{\varphi} d\eta = 0, \tag{26}$$

$$\int_{-\infty}^{\infty} \check{f}_k^{\varphi}\frac{\partial\chi}{\partial\eta} d\eta = 0, \tag{27}$$

where

$$\check{f}_k^{\varphi} = \int_{-\infty}^{\eta}\int_{-\infty}^{\eta'} \check{F}_k^{\varphi}(\eta'', x)\, d\eta'' d\eta'.$$

Using (23), (24), it is easy to see that the conditions (25), (26) hold automatically for $k = 1$.

Further, since

$$\check{f}_1^{\varphi} = \hat{\Pi}\frac{\partial\chi}{\partial\eta} + 3(\varphi_1 + \xi\varphi_{0t})(2\varphi_0\chi + \chi^2)\Big|_{t=\psi} + \sqrt{2}\,\kappa\,\beta\ln(J^2R),$$

$$R = \{e^{\xi} + 4\check{\varphi}_0 - 2\sqrt{2}Q\}\{e^{\xi} + 4\check{\varphi}_0 + 2\sqrt{2}Q\}^{-1},$$

simple calculations yield the statement

Lemma 3. *For $k = 1$ condition* (27) *is equivalent to equation* (10).

Now, since (25)–(27) are satisfied for $k = 1$, we can obtain the functions $\check{U}_1$, $\check{W}_1$:

$$\check{U}_1 = \check{U}_1^+(x)\zeta(\eta, x), \qquad \check{W}_1 = \check{W}_1^+(x)\zeta(\eta, x) + \omega_1(\eta, x).$$

Here

$$\zeta = \frac{\sqrt{2}}{a}\ln(J^2R) \in \mathcal{H}, \quad \zeta^+ = 1, \quad \zeta^- = 0,$$

$$\check{U}_1^+ = -aD_{\nu}, \qquad \check{W}_1^+ = \check{U}_1^+\kappa/Q^2,$$

$$\omega_1(\eta, x) = \omega_{1,1}(\eta, x) + \psi_2(x)\chi_{\eta}(\eta, x) \in \mathcal{S},$$

ψ_2 is the "constant" of integration, $a = 2I/\check{\varphi}_0$.

Let us define the extensions U_1, W_1

$$U_1 = u_1(x,t)\zeta(\eta,x), \qquad W_1 = w_1(x,t)\zeta(\eta,x) + \omega_1(\eta,x). \tag{28}$$

Here

$$u_1 = \theta_{1e}^+ - \theta_{1e}^-, \qquad w_1 = \Phi_{1e}^+ - \Phi_{1e}^-, \tag{28'}$$

$\theta_{1e}^\pm$, $\Phi_{1e}^\pm$ are sufficiently smooth extensions of $\theta_1^\pm$, $\Phi_1^\pm$ in $\Omega_t^\pm \cup \Gamma_{t,\delta}^\mp$, such that the heat equations (19) are satisfied for $k = 1$. Here $0 < \delta \ll 1$ is an arbitrary number.

Thus we have the following conditions for jumps of $\theta_1^\pm$, $\Phi_1^\pm$ on Γ_t

$$[\theta_1^\pm]\big|_{\Gamma_t} = aD_\nu, \qquad [\Phi_1^\pm]\big|_{\Gamma_t} = \frac{\kappa a}{Q^2}D_\nu \tag{29}$$

Here we use the notation $[f^\pm]|_{\Gamma_t} = f^-|_{\Gamma_t+0} - f^+|_{\Gamma_t-0}$ and note that the vector ν is directed from Ω_t^+ to Ω_t^-.

Let us consider equations (21), (22) in the case $k = 2$. The right-hand sides of these equations belong to $\mathcal{S}$, since equation (8) and the first equation (19) hold for $k = 1$. Further, after some calculations, we get the following statement

Lemma 4. *For $k = 2$ conditions (26) are equivalent to the equalities*

$$\left[\frac{\partial\theta_1^\pm}{\partial\nu}\right]\Big|_{\Gamma_t} = -aD_\nu\Big\{(1+\frac{\kappa}{Q^2})D_\nu + K_t - \frac{4}{abQ}\frac{\partial\bar\varphi_0}{\partial\nu}\Big\},$$

$$\left[\frac{\partial}{\partial\nu}(3\varphi_0^2-1)\Phi_1^\pm\right]\Big|_{\Gamma_t} = -12|\nabla\psi|\frac{\partial\varphi_0}{\partial t}\Big|_{\Gamma_t}\Big\{QK_t - D_\nu \mathrm{div}(Q\nabla\psi) + \frac{a}{2}\frac{\partial\bar\varphi_0}{\partial\nu}\Big\} \tag{30}$$

$$+D_\nu^2\frac{\kappa a}{2Q^2}(\kappa + Q^2\Delta\psi).$$

Finally, for $k = 2$, after some trivial but cumbersome calculations condition (27) can be transformed to the following linear inhomogeneous equation for the phase correction ψ_1:

$$\kappa\frac{\partial\psi_1}{\partial\nu} = G_1\hat{\mathcal{K}}'\psi_1 + f^{\psi_1}(x). \tag{31}$$

Here $\hat{\mathcal{K}}'$ is the variation of the operator $\mathcal{K}$ from (10), the right-hand side $f^{\psi_1}(x)$ depends on the functions ψ, θ_0, φ_0, $\theta_1^\pm$, $\Phi_1^\pm$.

The following constructions are performed similarly:

1. Calculating $\hat{U}_k^\pm$, $\hat{W}_k^\pm$, we get conditions for jumps of the functions $\theta_k^\pm$, $\Phi_k^\pm$ on Γ_t;
2. By using formulas similar to (28), we define the extensions U_k, W_k on Q. Since $\theta_k^\pm$, $\Phi_k^\pm$ is the solution of (19), we see that conditions (25) hold;
3. Conditions (26) imply conditions for the normal derivatives of $\theta_{k-1}^\pm$, $\Phi_{k-1}^\pm$ on Γ_t;
4. Condition (27) yields an equation (similar to (31)) for the "constant" of integration ψ_k.

In fact, since

$$f(\beta(\eta+\psi_1)) + \varepsilon\beta f'_\eta\psi_2 + \varepsilon^2\beta f'_\eta\psi_3 + \ldots = f(\beta(\eta+\psi_1+\varepsilon\psi_2+\varepsilon^2\psi_3+\ldots)) + \mathcal{O}(\varepsilon^2),$$

the functions ψ_k, $k \geq 1$, are the lower corrections to the principle phase ψ. So, these functions describe the front of the soliton wave more precisely.

We must also pose the initial conditions for the heat equations (8), (9), (19). Since θ_0, φ_0 are smooth functions on $\Omega\times[0,T]$ and $\theta_k^\pm$, $\Phi_k^\pm$ are smooth functions on $\Omega_t^\pm\times[0,T]$, let us define the initial data as follows

$$\overline{\theta^0} = w - \lim_{\varepsilon\to0}\theta^0(x,\varepsilon), \quad \overline{\varphi^0} = w - \lim_{\varepsilon\to0}\varphi^0(x,\varepsilon), \qquad x\in\Omega,$$

$$\overline{\theta_k^0}^{\,\pm} = w - \lim_{\varepsilon\to0}\frac{1}{\varepsilon^k}\Big(\theta^0(x,\varepsilon) - \theta_0 - \sum_{j=1}^{k-1}\varepsilon^j(\theta_j + U_j + Y_j)\big|_{t=0}\Big), \quad x\in\Omega_0^\pm,$$

$$\overline{\varphi_k^0}^{\,\pm} = w - \lim_{\varepsilon\to0}\frac{1}{\varepsilon^k}\Big(\varphi^0(x,\varepsilon) - \varphi_0 - \chi - \sum_{j=1}^{k-1}\varepsilon^j(\varphi_j + W_j + Z_j)\big|_{t=0}\Big), \quad x\in\Omega_0^\pm.$$

These formulas yield the initial data (8), (9) and the initial data for equations (19):

$$\theta_k^\pm = \overline{\theta_k^0}^{\,\pm}(x), \qquad \varphi_k^\pm = \overline{\varphi_k^0}^{\,\pm}(x), \qquad x\in\Omega_0^\pm. \tag{32}$$

Obviously, the behavior of smooth (for the $\varepsilon > 0$) functions $\theta^0(x,\varepsilon)$, $\varphi^0(x,\varepsilon)$ may be arbitrary outside an ε-neighbourhood of Γ_0, but θ^0, φ^0 must be of a special form in this neighbourhood.

Now let us consider the boundary conditions on the external boundary $\partial\Omega$ and calculate the boundary-layer functions. The soliton part of the asymptotic solution satisfies both boundary conditions (5) up to $\mathcal{O}(\varepsilon^\infty)$. So, for φ on Σ, a discrepancy in the second boundary condition arises only from the regular part of the solution, since $\mathrm{dist}(\Gamma_t,\partial\Omega) \geq$ const and, in general,

$$\frac{\partial}{\partial N}\Delta\varphi_0\Big|_\Sigma \neq 0.$$

Let us put $Z_j = 0$ for $j = 1,2$ since

$$\frac{\partial}{\partial N}\Delta\varepsilon^3 Z_3\left(\frac{x_N}{\varepsilon}, x'\right)\Big|_{x_N=0} = \mathcal{O}(1).$$

Then, using the construction common for the boundary layer asymptotic solutions, we obtain the equation for Z_3:

$$\frac{\partial^2}{\partial\tau^2}Z_3 - qZ_3 = 0, \qquad Z_3\to0, \quad \tau\to\infty, \tag{32}$$

where $q = 3\varphi_0^2|_\Sigma - 1 > 0$. Obviously,

$$Z_3 = c_3(x',t)\exp(-\sqrt{q}\,\tau).$$

Now we can see that the boundary condition

$$\frac{\partial^3}{\partial\tau^3}Z_3\Big|_{\tau=0} = \frac{\partial}{\partial N}\Delta\varphi_0\Big|_\Sigma \tag{33}$$

leads to the formula

$$c_3 = -q^{-3/2}\frac{\partial}{\partial N}\Delta\varphi_0\Big|_\Sigma.$$

Further, we note that the appearance of the boundary-layer function $\varepsilon^3 Z_3$ necessarily implies a correction of the Neumann condition in the term $\mathcal{O}(\varepsilon^2)$. Let, for definiteness, $\partial\Omega\cap\partial\Omega_t^- = \partial\Omega$. Then the Neumann condition for Φ_2^- has the form

$$\frac{\partial\Phi_2^-}{\partial N}\Big|_\Sigma = \frac{\partial Z_3}{\partial\tau}\Big|_{\tau=0} = \frac{1}{q}\frac{\partial}{\partial N}\Delta\varphi_0\Big|_\Sigma. \tag{34}$$

The appearance of the boundary-layer functions Z_k implies the boundary-layer terms Y_j in the θ asymptotic expansion. Since the boundary $\partial\Omega$ is fixed, $Y_j = 0$, $j = 1,\ldots,4$, we get the equation for Y_5:

$$\frac{\partial^2 Y_5}{\partial\tau^2} = \frac{\partial}{\partial t}Z_3(\tau,x',t), \qquad Y_5\to 0, \quad \tau\to\infty. \tag{35}$$

Therefore

$$Y_5 = -q^{-5/2}\frac{\partial^2}{\partial N\partial t}\Delta\varphi_0\big|_\Sigma \exp(-\sqrt{q}\,\tau).$$

Conversely, the appearance of Y_5 leads to a correction of the Neumann condition for the temperature in the term $\mathcal{O}(\varepsilon^4)$. Thus, the Neumann condition for θ_4^- has the form

$$\frac{\partial\theta_4^-}{\partial N}\Big|_\Sigma = \frac{\partial Y_5}{\partial\tau}\Big|_{\tau=0} = \frac{1}{q^2}\frac{\partial^2}{\partial N\partial t}\Delta\varphi_0\Big|_\Sigma. \tag{36}$$

Finally, the asymptotic expansion in a small neighbourhood of $\partial\Omega$ has the following form

$$\begin{aligned}\theta &= \theta_0(x,t) + \sum_{j=1}^{4}\varepsilon^j\theta_j^-(x,t) + \sum_{j=5}^{M}\varepsilon^j(\theta_j^-(x,t) + Y_j(\tau,x',t)) + \mathcal{O}(\varepsilon^{M+1}),\\ \varphi &= \varphi_0(x,t) + \sum_{j=1}^{2}\varepsilon^j\Phi_j^-(x,t) + \sum_{j=3}^{M}\varepsilon^j(\Phi_j^-(x,t) + Z_j(\tau,x',t)) + \mathcal{O}(\varepsilon^{M+1}).\end{aligned}$$

Here Z_3, $Y_5\in\mathcal{P}$ are described above, $Z_k\in\mathcal{P}$, $k\geq 4$, and $Y_j\in\mathcal{P}$, $j\geq 6$, are calculated from linear inhomogeneous problems like (32), (33), (35). In turn, we obtain the boundary conditions for problems (19):

1. Conditions (8), (9) for θ_0, φ_0;
2. Conditions
$$\frac{\partial\theta_j^-}{\partial N}\Big|_\Sigma = 0, \qquad \frac{\partial\Phi_l^-}{\partial N}\Big|_\Sigma = 0$$
for θ_j^+, $j = 1,\ldots,3$, Φ_1^-;
3. Conditions (34), (36) for Φ_2^-, θ_4^-;
4. Conditions
$$\frac{\partial\theta_j^-}{\partial N}\Big|_\Sigma = \frac{\partial Y_{j+1}}{\partial\tau}\Big|_{\tau=0}, \qquad \frac{\partial\Phi_k^-}{\partial N}\Big|_\Sigma = \frac{\partial Z_{k+1}}{\partial\tau}\Big|_{\tau=0}$$
for θ_j^-, $j\geq 5$ and Φ_k^-, $k\geq 3$.

Theorem 1 is proved. Moreover, analyzing our construction, we obtain the statement.

Theorem 2. *Let the assumptions of Theorem 1 hold. Then for any integer $M\geq 0$ there exist the functions*

$$\theta_M^{as} = \theta_0 + \sum_{j=1}^{M}\varepsilon^j(\theta_j + U_j + Y_j) + \varepsilon^{M+1}(U_{M+1} + Y_{M+1}),$$
$$\varphi_M^{as} = \varphi_0 + \chi + \sum_{j=1}^{M}\varepsilon^j(\varphi_j + W_j + Z_j) + \varepsilon^{M+1}(W_{M+1} + Z_{M+1}) \tag{37}$$

such that

$$\frac{\partial}{\partial t}(\theta_M^{as} + \varphi_M^{as}) - \Delta\theta_M^{as} - f(x,t) = \varepsilon^M\mathcal{F}_M^\theta,$$
$$\kappa\frac{\partial\varphi_M^{as}}{\partial t} + \Delta\left(\varepsilon^2\Delta\varphi_M^{as} + \varphi_M^{as} - (\varphi_M^{as})^3 + \varepsilon\kappa_1\theta_M^{as}\right) = \varepsilon^M\mathcal{F}_M^\varphi, \tag{38}$$

$$\frac{\partial\theta_M^{as}}{\partial N}\Big|_\Sigma = \varepsilon^{M+1}F_M^\theta, \qquad \frac{\partial\varphi_M^{as}}{\partial N}\Big|_\Sigma = 0, \qquad \frac{\partial}{\partial N}\Delta\varphi_M^{as}\Big|_\Sigma = \varepsilon^{M-1}F_M^\varphi, \tag{39}$$

$$\theta_M^{as}|_{t=0} = \theta^0(x,0) + \varepsilon(\overline{\theta_1^{0\pm}} + \tilde{\theta}), \quad \varphi_M^{as}|_{t=0} = \varphi^0(x,0) + \chi\Big|_{t=0} + \varepsilon(\overline{\varphi_1^{0\pm}} + \tilde{\varphi}). \tag{40}$$

Here $\tilde{\theta}$, $\tilde{\varphi}$, $\mathcal{F}_M^{\varphi,\theta}$, $F_M^{\varphi,\theta}$ are (smooth for $\varepsilon > 0$) functions such that

$$\|\tilde{\theta}; L^2(\Omega)\| + \|\tilde{\varphi}; L^2(\Omega)\| \leq c_0\sqrt{\varepsilon}, \tag{40'}$$

$$\begin{aligned}\|\mathcal{F}_M^\theta; C(\bar{Q})\| + \|\mathcal{F}_M^\varphi; C(\bar{Q})\| &\leq c_1,\\ \|F_M^\theta; C(\Sigma)\| + \|F_M^\varphi; C(\Sigma)\| &\leq c_2,\\ \|\mathcal{F}_M^\theta; L^2(\Omega)\| + \|\mathcal{F}_M^\varphi; L^2(\Omega)\| &\leq c_3\sqrt{\varepsilon},\end{aligned} \tag{41}$$

where the constants c_j are independent of ε.

Remark. Estimate (40′) is better, if θ^0, φ^0 are of special form in a neighbourhood of Γ_0.

3. Justification of the soliton type asymptotic solution

In this section we shall obtain estimates for the differences between the exact θ, φ and asymptotic θ_M^{as}, φ_M^{as} solutions of problem (3)-(5). Let us introduce the notation $\sigma = \theta - \theta_M^{as}$, $\omega = \varphi - \varphi_M^{as}$ and let the initial data θ^0, φ^0 exhibit a special behavior. Then, from (3)-(5) and (38)-(40), we get the following problem for the remainders σ, ω:

$$\frac{\partial}{\partial t}(\sigma + \omega) - \Delta\sigma = -\varepsilon^M\mathcal{F}_M^\theta, \tag{42}$$

$$\kappa\frac{\partial\omega}{\partial t} + \Delta\left(\varepsilon^2\Delta\omega + \omega(1 - 3\varphi_M^2 - 3\varphi_M\omega - \omega^2) + \varepsilon\kappa_1\sigma\right) = -\varepsilon^M\mathcal{F}_M^\varphi, \tag{43}$$

$$\left.\frac{\partial\sigma}{\partial N}\right|_\Sigma = -\varepsilon^{M+1}F^\theta_M, \quad \left.\frac{\partial\omega}{\partial N}\right|_\Sigma = 0, \quad \left.\frac{\partial}{\partial N}\Delta\omega\right|_\Sigma = -\varepsilon^{M-1}F^\varphi_M,$$
$$\sigma|_{t=0} = -\varepsilon^{M+1/2}f^\theta_M, \qquad \omega|_{t=0} = -\varepsilon^{M+1/2}f^\varphi_M. \tag{44}$$

Here $\mathcal{F}^{\theta,\varphi}_M$, $F^{\theta,\varphi}_M$ are smooth functions satisfying (41), $f^{\theta,\varphi}_M$ are functions such that

$$\|f^\theta_M; L^2(\Omega)\| + \|f^\varphi_M; L^2(\Omega)\| \le c\sqrt{\varepsilon} \tag{45}$$

with constant c independent of ε. To simplify the notation, we omit the superscript denoting asymptotic solutions.

The main result of this section is

Theorem 3. *Let there exist a sufficiently smooth solution of problem* (3)–(5) *on the time interval* $[0,T]$, *where the quantity* $T>0$ *is independent of* ε. *Let also the assumptions of Theorem* 1 *be satisfied,* $M\ge 2$ *and there exist a constant* $\gamma>0$ *such that* $\varphi_0 - 1/\sqrt{3}\ge\gamma$ *uniformly in* $x\in\Omega$, $t\in[0,T]$. *Then the estimates hold*

$$\|\omega; L^\infty((0,T); L^2(\Omega))\| + \|\sigma; L^\infty((0,T); L^2(\Omega))\| \le c\,\varepsilon^{M+1}, \tag{46}$$
$$\|\nabla\omega; L^2(Q)\| + \|\nabla\sigma; L^2(Q)\| \le c\,\varepsilon^{M+1/2}, \quad \|\Delta\omega; L^2(Q)\| \le c\,\varepsilon^{M-1/2}$$

with constant c *independent of* ε.

First, we obtain an auxiliary result.

Theorem 4. *Under the assumptions of Theorem 3, the following estimates hold*

$$\|\omega; L^\infty((0,T); L^2(\Omega))\| + \|\sigma; L^\infty((0,T); L^2(\Omega))\| \le c\,\varepsilon^{M+1/2}, \tag{47}$$
$$\|\nabla\sigma; L^2(Q)\| + \|\nabla\omega; L^2(Q)\| \le c\,\varepsilon^{M+1/2}, \quad \|\Delta\omega; L^2(Q)\| \le c\,\varepsilon^{M-1/2}.$$

Proof. Multiplying equations (42), (43) by σ, ω respectively and integrating on Ω, we get the relations

$$\frac{1}{2}\frac{d}{dt}\|\sigma\|^2 + \int_\Omega \omega_t\sigma dx + \|\nabla\sigma\|^2 = -\varepsilon^M\int_\Omega \sigma\mathcal{F}^\theta_M dx - \varepsilon^{M+1}\int_{\partial\Omega}\sigma F^\theta_M dx', \tag{48}$$

$$\frac{\kappa}{2}\frac{d}{dt}\|\omega\|^2 + \varepsilon^2\|\Delta\omega\|^2 + 3\|\omega\nabla\omega\|^2 = \int_\Omega(\nabla\omega, \nabla\omega(1-3\varphi_M^2))dx$$
$$-3\int_\Omega(\nabla\omega, \nabla\varphi_M\omega^2)dx + \varepsilon\kappa_1\int_\Omega(\nabla\omega,\nabla\sigma)dx \tag{49}$$
$$-\varepsilon^M\int_\Omega\omega\mathcal{F}^\varphi_M dx + \varepsilon^{M+1}\int_{\partial\Omega}\omega F^\varphi_M dx' + \varepsilon^{M+2}\kappa_1\int_{\partial\Omega}\omega F^\theta_M dx'.$$

Here and below $\|f\|$ denotes the $L^2(\Omega)$ norm of f.

Further, multiplying (42) by ω, integrating on Ω and summing with (48), we obtain

$$\frac{1}{2}\frac{d}{dt}\left\{\|\omega\|^2 + \|\sigma\|^2 + 2\int_\Omega\omega\sigma dx\right\} + \|\nabla\sigma\|^2 + \int_\Omega(\nabla\omega,\nabla\sigma)dx$$
$$= -\varepsilon^M\int_\Omega(\omega+\sigma)\mathcal{F}^\theta_M dx' - \varepsilon^{M+1}\int_{\partial\Omega}(\omega+\sigma)F^\theta_M dx'. \tag{50}$$

Let us fix a constant $K>1/\gamma$. Multiplying (49) by K and summing with (50), we get the equality

$$\frac{1}{2}\frac{d}{dt}\left\{(1+\kappa K)\|\omega\|^2 + \|\sigma\|^2 + 2\int_\Omega\omega\sigma dx\right\}$$
$$+\|\nabla\sigma\|^2 + \varepsilon^2K\|\Delta\omega\|^2 + 3K\|\omega\nabla\omega\|^2$$
$$= K\int_\Omega(\nabla\omega,\nabla\omega(1-3\varphi_M^2))dx - 3K\int_\Omega(\nabla\omega,\nabla\varphi_M\omega^2)dx$$
$$+(\varepsilon\kappa_1K-1)\int_\Omega(\nabla\omega,\nabla\sigma)dx - \varepsilon^M\int_\Omega\left\{(\omega+\sigma)\mathcal{F}^\theta_M + \omega K\mathcal{F}^\varphi_M\right\}dx \tag{51}$$
$$-\varepsilon^{M+1}\int_{\partial\Omega}\left\{(\omega+\sigma)F^\theta_M - \omega K(F^\varphi_M + \varepsilon\kappa_1F^\theta_M)\right\}dx'.$$

We shall analyze the terms in the right-hand side of (51).

Lemma 5. *Let* φ_M *be the asymptotical expansion* (37). *Then*

$$\int_\Omega(\nabla\omega,\nabla\omega(1-3\varphi_M^2))dx = -\int_\Omega(3\varphi_0^2-1)|\nabla\omega|^2dx - \frac{1}{\varepsilon^2}\int_\Omega\omega^2\Psi_1dx + I, \tag{52}$$
$$I = \int_\Omega|\nabla\omega|^2(g_1\chi+\varepsilon g_2)dx + \frac{1}{\varepsilon^2}\int_\Omega\omega^2(\Psi_1(\Psi_2+\varepsilon\Psi_3)+\varepsilon^2\Psi_4)dx,$$

where

$$\Psi_1 = A_1\frac{\cosh^3\rho + A\cosh\rho + (1+2A)/\alpha}{(\cosh\rho+\alpha)^4}\in\mathcal{S}, \tag{53}$$
$$\rho = \xi - \frac{1}{2}\ln 8b, \quad \alpha = \dot\varphi_0\sqrt{2/b}, \quad A = Q^2\sqrt{2/b}, \quad A_1 = \dot\varphi_0A^2/6b,$$

g_i, Ψ_k *are smooth functions such that*

$$|g_i|\le\text{const}, \quad \Psi_2\in\mathcal{S}, \quad \Psi_3\in\mathcal{H}, \quad |\Psi_4|\le\text{const}.$$

Proof of Lemma 5. Using the expansion (37) and rewriting φ_M in the form $\varphi_M = \varphi_0 + \chi + \varepsilon\varphi^*_M$, we get

$$\int_\Omega(\nabla\omega,\nabla\omega(1-3\varphi_M^2))dx = -\int_\Omega(3\varphi_0^2-1)|\nabla\omega|^2dx$$
$$+\int_\Omega|\nabla\omega|^2(g_1\chi+\varepsilon g_2)dx + \frac{3}{2}\int_\Omega\omega^2\Delta\varphi_M^2dx, \tag{54}$$

where $g_1 = -3(\chi+2\varphi_0)$, $g_2 = (2\varphi_0+2\chi+\varepsilon\varphi^*_M)\varphi^*_M$. Obviously, g_i are bounded functions. Further, simple calculations yield the relation

$$\frac{1}{2}\Delta\varphi_M^2 = \frac{1}{\varepsilon^2}|\nabla\psi|^2\left\{\chi_\eta^2 + (\varphi_0+\chi)\chi_{\eta\eta}\right\} + \frac{1}{\varepsilon}\tilde\Psi + \Psi_4. \tag{55}$$

Here

$$\tilde\Psi = -2\chi_\eta(|\nabla\psi|^2W_{1\eta} + (\nabla\psi,\nabla\varphi_0))$$
$$+(\varphi_0+\chi)\left(|\nabla\psi|^2W_{1\eta\eta} - \hat\Pi\chi_\eta\right) + (\varphi_1+W_1)|\nabla\psi|^2\chi_{\eta\eta},$$

Ψ_4 is bounded (in the C-sense) and $\hat{\Pi}$ is the operator described in (16). Let us rewrite the function χ in the form

$$\chi = -A\{\cosh\rho + \alpha\}^{-1},$$

where ρ, $A > 0$, $\alpha > 0$ are described in (53). Now it is easy to calculate that

$$3|\nabla\psi|^2\{\chi_\eta^2 + (\varphi_0 + \chi)\chi_{\eta\eta}\} = \Psi_1(1 - \Psi_2),$$

where Ψ_1 has the form (53) and

$$\Psi_2 = \Big\{\frac{A+1}{\alpha}\cosh^2\rho + (\alpha^2+2)\cosh\rho + 2\alpha\Big\}\Big\{\cosh^3\rho + A\cosh\rho + (2A+1)/\alpha\Big\}^{-1}.$$

Obviously, $\Psi_2 \in \mathcal{S}$. Finally, let us note that the term $\tilde{\Psi}$ can be written in the form $\tilde{\Psi} = \Psi_1\Psi_3$, where Ψ_3 is a function from $\mathcal{H}$, since W_1 satisfies (22) and χ_η, $\chi_{\eta\eta}$ vanish like $1/\cosh\rho$ as $\eta \to \pm\infty$. This equality, (54), and (55) complete the proof of Lemma 4.

Now, using (42), (44) and integrating (51) with respect to t, we get

$$\begin{aligned}
&\frac{1}{2}\Big\{(1+\kappa K)\|\omega\|^2 + \|\sigma\|^2\Big\}(t) + \int_0^t \Big\{\|\nabla\sigma\|^2 + \varepsilon^2 K\|\Delta\omega\|^2 \\
&+3K\|\omega\nabla\omega\|^2 + K\int_\Omega(3\varphi_0^2 - 1)|\nabla\omega|^2dx + \frac{K}{\varepsilon^2}\int_\Omega \omega^2\Psi_1 dx\Big\}dt' \\
&= \frac{1}{2}\varepsilon^{2M+1}\Big\{(1+\kappa K)\|f_M^\varphi\|^2 + \|f_M^\theta\|^2 + 2\int_\Omega f_M^\varphi f_M^\theta dx\Big\} - \int_\Omega \omega\sigma dx \\
&+\int_0^t\Big\{KI - 3K\int_\Omega(\nabla\omega, \nabla\varphi_M\omega^2)dx + (\varepsilon\kappa_1 K - 1)\int_\Omega(\nabla\omega,\nabla\sigma)dx \\
&-\varepsilon^M\int_\Omega\Big[(\omega+\sigma)\mathcal{F}_M^\theta + \omega K\mathcal{F}_M^\varphi\Big]dx \\
&\qquad -\varepsilon^{M+1}\int_{\partial\Omega}\Big[(\omega+\sigma)F_M^\theta - \omega K(F_M^\varphi + \varepsilon\kappa_1 F_M^\theta)\Big]dx'\Big\}dt'.
\end{aligned} \tag{56}$$

It is easy to see that

$$2\Big|\int_\Omega \omega\sigma dx\Big| \le \alpha_1\|\omega\|^2 + \frac{1}{\alpha_1}\|\sigma\|^2,$$

$$2\Big|\int_\Omega(\nabla\omega,\nabla\sigma)dx\Big| \le \alpha_2\,\|\nabla\omega\|^2 + \frac{1}{\alpha_2}\,\|\nabla\sigma\|^2,$$

$$\alpha_1 = \frac{1}{2}\Big(\kappa K + \sqrt{(\kappa K)^2+4}\,\Big), \quad \alpha_2 = \frac{1}{2}\Big(\gamma K - 1 + \sqrt{(\gamma K - 1)^2 + 4}\,\Big).$$

Let us choose the constant K large enough, so that $\gamma K - \alpha_2 \ge 1/2$. It is possible, since $\gamma K - \alpha_2$ varies from 0 to 1. Further, by the embedding theorem (see, for example, [21]) and (41), we get

$$\begin{aligned}
&\varepsilon^{M+1}\Big|\int_{\partial\Omega}\Big[(\omega+\sigma)F_M^\theta - \omega K(F_M^\varphi + \varepsilon\kappa_1 F_M^\theta)\Big]dx'\Big| \\
&\le c\,\varepsilon^{M+1}(\|\omega; L^2(\partial\Omega)\| + \|\sigma; L^2(\partial\Omega)\|) \le c\,\varepsilon^{2M+2} + \frac{1}{4}\,\|\omega\|_1^2 + \frac{1}{4}\,\|\sigma\|_1^2.
\end{aligned}$$

Here and below c denotes a universal constant, $\|f\|_k$ is the $H^k(\Omega)$ norm of f, and H^k denotes the Sobolev space.

Therefore, choosing ε small enough, from (56) we obtain the following inequality

$$\begin{aligned}
&\frac{\alpha_1 - 1}{2\alpha_1}\Big\{\|\omega\|^2 + \|\sigma\|^2\Big\}(t) + \int_0^t\Big\{\frac{1}{4}\,\|\nabla\omega\|^2 + \frac{1}{4}\,\|\nabla\sigma\|^2 \\
&+\varepsilon^2 K\,\|\Delta\omega\|^2 + 3K\,\|\omega\nabla\omega\|^2 + \frac{K}{\varepsilon^2}\int_\Omega \omega^2\Psi_1 dx\Big\}dt' \le c\,\varepsilon^{2M+1} \\
&+\int_0^t\Big\{K\,|I| + c(\|\omega\|^2 + \|\sigma\|^2) + 3K\Big|\int_\Omega(\nabla\omega,\nabla\varphi_M\omega^2)dx\Big|\Big\}dt'.
\end{aligned} \tag{57}$$

To estimate the integral I we shall need the following

Lemma 6 ([25], [33]). *For any function $\varphi(t) \in C^\infty([0,T])$ and for any nonnegative functions f, v,*

$$f(t,x) \in L^\infty(0,T; L^2(R^1)\cap L^1(R^1)), \quad v(t,x) \in L^\infty(0,T; S(R^1)),$$

where S is the Schwartz space, there exists a constant $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0,\varepsilon_0]$,

$$\int_{-\infty}^{\infty} f(t,x)v\Big(t, \frac{x-\varphi(t)}{\varepsilon}\Big)dx \le \delta\|f; L^1(R_x^1)\| + c(\delta)\,\varepsilon^{3/2}r(\varepsilon)\|f; L^2(R_x^1)\|,$$

where $\delta \ge k\varepsilon^{1/2-\mu}$ is an arbitrary constant, $\mu \in (0,1/2)$; $k > 0$ is a constant. Here $0 < c(\delta) \le \mathrm{const}/\delta^2$ and $r(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Remark. A similar estimate was established by M. Berger and L. Fraenkel [4] for the boundary layer problem. The support (up to $\mathcal{O}(\varepsilon^\infty)$) of the boundary layer part of the asymptotic solution lies in the ε^δ-neighbourhood of the boundary $\partial\Omega$, $\delta \in (0,1)$ is a constant, and the remainder is equal to zero on $\partial\Omega$. So, by Lemma 6, we have some additional condition, which means that $f = 0$ on a surface inside the domain $\Omega_\alpha = \{x, |v(t,(x-\varphi)/\varepsilon)| \ge \alpha\}$ for sufficiently small α. This condition, very natural for elliptic problems, was used by M. Berger and L. Fraenkel extensively. Moreover, the proof [4] is incorrect without this condition. Obviously, such an additional condition does not appear for the free boundary problems.

Let us estimate the principal terms of I.

Lemma 7. *Let ε be small enough. Then*

$$\int_\Omega |\nabla\omega|^2|g_1|\chi dx \le \delta_1\,\|\nabla\omega\|^2 + c\,\varepsilon^{9/2}\|\Delta\omega\|^2, \tag{58}$$

$$\frac{1}{\varepsilon^2}\int_\Omega \omega^2\Psi_1|\Psi_2|dx \le \frac{\delta_2}{\varepsilon^2}\int_\Omega \omega^2\Psi_1 dx + c\,\varepsilon^2\|\omega\|_1^2, \tag{59}$$

where $\delta_i > 0$ are arbitrary constants.

Proof. Denote by $\mathcal{N}_\mu$ a μ neighbourhood of the interface $\Gamma_{t'}$, where $\mu \geq 0$ is a constant independent of ε. Since $\chi = \mathcal{O}(\varepsilon^\infty)$ outside $\mathcal{N}_\mu$, we have

$$\int_\Omega |\nabla\omega|^2 |g_1| \chi dx \leq c_0 \int_{\mathcal{N}_\mu} |\nabla\omega|^2 \chi dx + \varepsilon^k c_0 \int_\Omega |\nabla\omega|^2 dx$$

for any positive k. Here $c_0 = \max_{x\in\Omega} |g_1|$.

Choosing μ sufficiently small, we pass to the variables $y = (y_1, \dots, y_n)$ in $\mathcal{N}_\mu$, where y_1 is the coordinate normal to $\Gamma_{t'} = \{x \in \Omega,\ t' = \psi(x)\}$. Then in $\mathcal{N}_\mu = \{\ y,\ |y_1| \leq \mu,\ Y_i^- \leq y_i \leq Y_i^+,\ i = 2, \dots, n\}$ we have

$$\int_{\mathcal{N}_\mu} |\nabla\omega|^2 \chi dx = \prod_{i=2}^n \int_{Y_i^-}^{Y_i^+} \int_{-\mu}^{\mu} |\widetilde{\nabla\omega}|^2 v(\frac{y_1}{\varepsilon}, y, t') J\, dy_1 dy_i,$$

where J is the Jacobian of this change of variables,

$$|\widetilde{\nabla\omega}| = |\nabla_x \omega|\big|_{x=x(y,t')}, \qquad v(\eta, y, t') = \chi(\beta(\eta + \psi_1))\big|_{x=x(y,t')}.$$

By Lemma 6 and the embedding theorem for $n = 1$, we get

$$\int_{\mathcal{N}_\mu} |\nabla\omega|^2 \chi dx$$

$$\leq \prod_{i=2}^n \int_{Y_i^-}^{Y_i^+} \left\{ \delta_1 \int_{-\mu}^{\mu} |\widetilde{\nabla\omega}|^2 J\, dy_1 + c\varepsilon^{3/2} r(\varepsilon) \left(\int_{-\mu}^{\mu} |\widetilde{\nabla\omega}|^4 J^2 dy_1 \right)^{1/2} \right\} dy_i$$

$$\leq \delta_1 \|\nabla\omega\|^2 + c\, \varepsilon^{3/2 - k_1 - k_2} r(\varepsilon) \left\{ \varepsilon^{4k_1/3} \|\nabla\omega\|^2 + \varepsilon^{4k_2} (\|\nabla\omega\|^2 + \|\Delta\omega\|^2) \right\}$$

$$\leq \delta_1 \|\nabla\omega\|^2 + c\, \varepsilon^{1/2} r(\varepsilon) \left\{ \|\nabla\omega\|^2 + \varepsilon^4 \|\Delta\omega\|^2 \right\}.$$

Here we choose $k_2 = 1 + k_1/3$ and use that $J > 0$ is a bounded smooth function.

This implies estimate (58). Similarly,

$$\frac{1}{\varepsilon^2} \int_{\mathcal{N}_\mu} \omega^2 \Psi_1 |\Psi_2| dx \leq \frac{\delta_2}{\varepsilon^2} \int_\Omega \omega^2 \Psi_1 dx + c\, r(\varepsilon) \varepsilon^{-1/2}$$

$$\times \prod_{i=2}^n \int_{Y_i^-}^{Y_i^+} \left\| \tilde\omega \sqrt{J\tilde\Psi_1}\ ; L^2(-\mu, \mu) \right\|^{3/2} \left\| \tilde\omega \sqrt{J\tilde\Psi_1}; H^1(-\mu, \mu) \right\|^{1/2} dy_i$$

$$\leq \frac{\delta_2}{\varepsilon^2} \left\| \omega \sqrt{\Psi_1}\ \right\|^2 + c\rho(\varepsilon) \left\{ \frac{1}{\varepsilon^2} \left\| \omega \sqrt{\Psi_1} \right\|^2 + \varepsilon^2 \|\omega\|^2 + \varepsilon^4 \|\nabla\omega\|^2 \right\}.$$

Lemma 7 is proved.

Further, choosing ε small enough, we have the trivial estimate

$$\varepsilon \int_\Omega |\nabla\omega|^2 |g_2| dx + \frac{1}{\varepsilon} \int_\Omega \omega^2 \Psi_1 |\Psi_3 + \varepsilon \Psi_4| dx$$
$$\leq c\|\omega\|^2 + \delta_3 \left\{ \|\nabla\omega\|^2 + \frac{1}{\varepsilon^2} \int_\Omega \omega^2 \Psi_1 dx \right\} \tag{60}$$

with an arbitrary constant $\delta_3 > 0$.

Let us estimate the last term in the right-hand side of (57). Using the Galliardo-Nierenberg inequality, we get that

$$\left| \int_\Omega (\nabla\omega, \nabla\varphi_M \omega^2) dx \right| \leq c \int_\Omega |\omega|\ |\nabla\omega|^2 dx + \frac{c}{\varepsilon} \int_\Omega |\nabla\omega|\ \omega^2 dx$$
$$\leq c\|\omega\|^{(8-n)/4}\ \|\omega\|_2^{(4+n)/4} + \frac{c}{\varepsilon}\ \|\nabla\omega\|\ \|\omega\|^{(8-n)/4}\ \|\omega\|_2^{n/4} \tag{61}$$
$$\leq \delta_4 \{ \|\nabla\omega\|^2 + \varepsilon^2 \|\omega\|_2^2 \} + c\, \varepsilon^{-2(4+n)/(4-n)}\ \|\omega\|^{2(8-n)/(4-n)}.$$

Choosing reasonable constants δ_i and using (58)-(61), we can transform (57) as follows

$$U(t) + c \int_0^t \Big\{ \|\nabla\omega\|^2 + \|\nabla\sigma\|^2 + \varepsilon^2 \|\Delta\omega\|^2 + \|\omega\nabla\omega\|^2$$
$$+ \frac{1}{\varepsilon^2} \left\| \sqrt{\Psi_1}\ \omega \right\|^2 \Big\} dt' \leq c\, \varepsilon^{2M+1} + c \int_0^t \left\{ U(t') + \varepsilon^{-r} (U(t'))^{1+\lambda} \right\} dt'. \tag{62}$$

Here

$$U(t) = \{\|\omega\|^2 + \|\sigma\|^2\}(t), \qquad \lambda = \frac{4}{4-n}, \qquad r = 2\frac{4+n}{4-n}.$$

Let us fix a number $T_1 \in (0, T]$, $T < \infty$, and let $t \in [0, T_1]$. Then, according to the Gronuoll lemma, (62) yields

$$U(t) \leq c \left(\varepsilon^{2M+1} + \varepsilon^{-r} \int_0^{T_1} U^{1+\lambda} dt' \right).$$

Let $z = \max_{t\in[0,T_1]} U(t)$. Then

$$z \leq c\, (\varepsilon^{2M+1} + \varepsilon^{-r} T_1 z^{1+\lambda}). \tag{63}$$

To analyze the last relation, we need the following lemma proved in [29].

Lemma 8. *Let positive numbers p, q, λ satisfy the estimate*

$$q < \frac{\lambda}{1+\lambda} (p(1+\lambda))^{-1/\lambda}. \tag{64}$$

Then the solutions of the inequality

$$0 < z < q + p\, z^{1+\lambda}$$

fill the domain consisting of an interval adjoining $z = 0$ and a half-line, separated from each other by the point $z_* = q(1+\lambda)/\lambda$.

In our case $p = cT_1\varepsilon^{-r}$. Therefore, since ε is small enough, inequality (71) holds for any $M \geq 2$. Thus we obtain the estimate

$$\max_{t\in[0,T_1]} U(t) \leq \varepsilon^{2M+1}. \tag{65}$$

It is easy to see that (65) and (62) yield the estimates (47). This completes the proof of Theorem 4.

Proof of Theorem 3. Let us choose the number $M' = M + 1$, where $M \geq 2$. Then, by Theorem 4, we get

$$\theta = \theta^{as}_{M'} + \varepsilon^{M+3/2}\sigma_{M'}, \qquad \varphi = \varphi^{as}_{M'} + \varepsilon^{M+3/2}\omega_{M'},$$

where $\sigma_{M'}$, $\omega_{M'}$ are functions from $L^\infty(0,T;L^2(\Omega))$ uniformly bounded in ε. Nevertheless,

$$\theta^{as}_{M'} = \theta^{as}_M + \varepsilon^{M+1}\mathcal{Y}^\theta_{M+1}, \qquad \varphi^{as}_{M'} = \varphi^{as}_M + \varepsilon^{M+1}\mathcal{Y}^\varphi_{M+1},$$

$$\mathcal{Y}^\theta_{M+1} = \theta_{M+1}(x,t) + U_{M+1}(\eta,x,t) + Y_{M+1}(\tau,x',t),$$

$$\mathcal{Y}^\varphi_{M+1} = \varphi_{M+1}(x,t) + W_{M+1}(\eta,x,t) + Z_{M+1}(\tau,x',t),$$

where $\eta = (t - \psi(x))/\varepsilon$, $\tau = x_N/\varepsilon$. It is easy to calculate that

$$\|\mathcal{Y}^\theta_{M+1}; L^\infty(0,T;L^2(\Omega))\| + \|\mathcal{Y}^\varphi_{M+1}; L^\infty(0,T;L^2(\Omega))\| \leq c.$$

Therefore,

$$\theta = \theta^{as}_M + \varepsilon^{M+1}\sigma^*_M, \qquad \varphi = \varphi^{as}_M + \varepsilon^{M+1}\omega^*_M,$$

where

$$\sigma^*_M = \mathcal{Y}^\theta_{M+1} + \sqrt{\varepsilon}\,\sigma_{M'}, \qquad \omega^*_M = \mathcal{Y}^\varphi_{M+1} + \sqrt{\varepsilon}\,\omega_{M'}$$

are functions such that the estimate

$$\|\mathcal{Y}^\theta_{M+1}; L^\infty(0,T;L^2(\Omega))\| + \|\mathcal{Y}^\varphi_{M+1}; L^\infty(0,T;L^2(\Omega))\| + \|\sigma_{M'}; L^\infty(0,T;L^2(\Omega))\| + \|\omega_{M'}; L^\infty(0,T;L^2(\Omega))\| \leq \text{const}$$

holds uniformly in ε. This estimate and Theorem 4 complete the proof of Theorem 3.

4. Asymptotic behavior of the solution for large time

The purpose of this section is to consider the process for large time, i.e., when the first stage of separation is completed and some domains of "solid" and "liquid" are formed. Respectively, our aim is to study the dynamics of the interfaces between these domains. We shall assume that the distances between different interfaces are greater than a constant (uniformly with respect to time and ε). It is clear that in this case we can consider only one couple of such domains in Ω without loss of generality. We shall also assume that the right hand side f in (3) is a smooth function, slowly varying with respect to t and preserving the zero mean value:

$$\int_\Omega f\,dx = 0.$$

Changing the time scale $t = \tau/\varepsilon$, we rewrite the system (3) as follows

$$\varepsilon\frac{\partial}{\partial\tau}(\theta + \varphi) = \Delta\theta + f(x,\tau),$$
$$-\kappa\varepsilon\frac{\partial\varphi}{\partial\tau} = \Delta\left(\varepsilon^2\Delta\varphi + \varphi - \varphi^3 + \varepsilon\kappa_1\theta\right), \tag{66}$$

Let us pose the boundary conditions (5) on the external boundary and the initial conditions

$$\theta|_{\tau=0} = \theta^0(x,\varepsilon), \qquad \varphi|_{\tau=0} = \varphi^0(x,\varepsilon), \tag{67}$$

where φ^0 is a certain smooth (for $\varepsilon > 0$) function such that

$$\lim_{\varepsilon\to 0}\varphi^0 = 1 \quad \text{as} \quad x \in \Omega_0^+, \qquad \lim_{\varepsilon\to 0}\varphi^0 = -1 \quad \text{as} \quad x \in \Omega_0^-.$$

Here $\Omega_0^\mp$ denote subdomains of Ω such that $\Omega = \Omega_0^+ \cup \Omega_0^- \cup \Gamma_0$, Γ_0 is a sufficiently smooth closed surface of codimension 1. Let $\Gamma_0 \cap \partial\Omega = \emptyset$ and, for definiteness, $\partial\Omega_0^- \cap \partial\Omega = \partial\Omega$.

Precisely as in similar problems for both the phase field system [14, 15, 33, 34, 39] and the Cahn–Hilliard system with the fast relaxation time [35, 36], one can assume that as $\varepsilon \to 0$ the limiting temperature has a weak discontinuity on the free interface, since the limiting concentration is the Heaviside function. In fact, it is not true.

Lemma 9. *Let $\Gamma_\tau = \{x \in \Omega, \tau = \psi(x)\}$ be a smooth surface of codimension 1 such that* $\operatorname{dist}(\Gamma_\tau, \partial\Omega) \geq \text{const}$ *for all $\tau \in [0, T_0]$. Let also θ, φ be the solution of problem* (66), (67), (5) *such that $w-\lim_{\varepsilon\to 0}\varphi$ is the Heaviside function* $\mathrm{H}(\tau - \psi(x))$. *Then the temperature θ is a smooth function up to terms $O(\varepsilon)$.*

This result is in agreement with the behavior of the soliton type asymptotic solution as the "background" φ_0 tends to ± 1. Actually, the leading term of the asymptotic expansion for the temperature is a smooth function and the amplitude of jump for the first correction remains bounded as $\varphi_0 \to \pm 1$ (see section 2). So, it is very natural to assume that the leading term of temperature will be smooth during the whole time of bifurcation (from $t \sim$ const to $t \sim 1/\varepsilon$), when the concentration of "superheated liquid" tends to -1 in $\Omega^-_{t,\varepsilon/\delta}$ (respectively, when the concentration of "supercooled solid" tends to 1 in $\Omega^+_{t,\varepsilon/\delta}$). The statement of Lemma 9 means that this assumption is true.

Returning to the problem for large time and taking into account the statement of Lemma 9, we shall assume that the leading term of the asymptotic expansion for the initial temperature θ^0 is a smooth function uniformly in $\varepsilon \geq 0$.

Let us formulate the main result of this section. Denote by $\theta_0 = \theta_0(x,\tau)$ the solution of the following Neumann problem

$$\Delta\theta_0 = -f(x,\tau), \quad x \in \Omega, \quad \tau > 0; \qquad \frac{\partial\theta_0}{\partial N}\Big|_\Sigma = 0. \tag{68}$$

Let also $\Phi_1^\pm = \Phi_1^\pm(x,\tau)$, $\psi = \psi(x)$ be the solution of the following model problems

$$\Delta\Phi_1^\pm = -\frac{\kappa_1}{2}f(x,\tau), \qquad x \in \Omega_\tau^\pm, \quad \tau > 0;$$
$$\frac{\partial\Phi_1^-}{\partial N}\Big|_\Sigma = 0, \quad \left[\Phi_1^\pm\right]\Big|_{\Gamma_\tau} = 0, \quad \left[\frac{\partial\Phi_1^\pm}{\partial\nu}\right]\Big|_{\Gamma_\tau} = \kappa D_\nu, \tag{69}$$

$$\operatorname{div}\left(\frac{\nabla\psi}{|\nabla\psi|}\right) = \kappa_2\check{\Phi}_1^\pm, \qquad \psi|_{\Gamma_0} = 0. \tag{70}$$

By $\theta_1^\pm = \theta_1^\pm(x,\tau)$ we also denote the solution of the problem

$$\Delta\theta_1^\pm = \frac{\partial\theta_0}{\partial\tau}, \qquad x \in \Omega_\tau^\pm, \quad \tau > 0;$$
$$\frac{\partial\theta_1^-}{\partial N}\Big|_\Sigma = 0, \quad \left[\theta_1^\pm\right]\Big|_{\Gamma_\tau} = 0, \quad \left[\frac{\partial\theta_1^\pm}{\partial\nu}\right]\Big|_{\Gamma_\tau} = 2D_\nu. \tag{71}$$

Here we use the same notation as in Section 2, $\kappa_2 = 3\sqrt{2}$.

The uniqueness of θ_0, $\Phi_1^\mp$, $\theta_1^\pm$ follows from the normalization conditions

$$\int_{\Omega} \theta_0 dx = K_0^\theta, \quad \int_{\Omega_T^+} \Phi_1^+ dx + \int_{\Omega_T^-} \Phi_1^- dx = K_1^\varphi, \quad \int_{\Omega_T^+} \theta_1^+ dx + \int_{\Omega_T^-} \theta_1^- dx = K_1^\theta,$$

where K_1^φ, K_i^θ are the coefficients of the expansions

$$\int_\Omega \theta^0(x,\varepsilon)dx = \sum_{i\geq 0} \varepsilon^i K_i^\theta, \qquad \int_\Omega \varphi^0(x,\varepsilon)dx = \sum_{i\geq 0} \varepsilon^i K_i^\varphi.$$

Theorem 5. *Let there exist sufficiently smooth solutions of problems* (69)–(71), *and let* $\operatorname{dist}(\Gamma_T, \partial\Omega) \geq \text{const}$ *for all* $T \in [0, T_0]$. *Then, for any* $M \geq 0$, *there exists a formal asymptotic (up to* $\mathcal{O}(\varepsilon^{M+1})$*) solution of problem* (66), (67), (5). *The two leading terms of this asymptotic solution have the form*

$$\begin{aligned}
\theta(x,T,\varepsilon) &= \theta_0(x,T) + \varepsilon\Big\{\frac{1}{2}\Big(\theta_{1c}^+(x,T) + \theta_{1c}^-(x,T)\Big) \\
&\quad + \frac{1}{2}\Big(\theta_{1c}^+(x,T) - \theta_{1c}^+(x,T)\Big)\chi(\eta,x)\Big\} + \mathcal{O}(\varepsilon^2), \\
\varphi(x,T,\varepsilon) &= \chi(\eta,x) + \varepsilon\Big\{\frac{1}{2}\Big(\Phi_{1c}^+(x,T) + \Phi_{1c}^-(x,T)\Big) \\
&\quad + \frac{1}{2}\Big(\Phi_{1c}^+(x,T) - \Phi_{1c}^+(x,T)\Big)\chi(\eta,x) + \omega_1(\eta,x)\Big\} + \mathcal{O}(\varepsilon^2).
\end{aligned} \tag{72}$$

Here

$$\chi = \tanh\big(\beta(\eta + \psi_1(x))\big); \qquad \eta = (T - \psi(x))/\varepsilon, \quad \beta = (\sqrt{2}\,|\nabla\psi|)^{-1},$$

$\theta_{1c}^\pm$, $\Phi_{1c}^\pm$ *are sufficiently smooth extensions of solutions to problems* (69), (71) $\theta_1^\pm$, $\Phi_1^\mp$ *in* $\Omega_T^\pm \cup \Gamma_{T,\delta}^\mp$, *where* $\Gamma_{T,\delta}^\mp$ *is a layer of width* δ *lying in* $\Omega_T^\mp$ *and being adjacent to* Γ_T, $\delta > 0$ *is an arbitrary number;* $\omega_1 \in \mathcal{S}$; $\psi_1 \in C^\infty$ *is the solution of the inhomogeneous linearization of equation* (71).

Let us prove Theorem 5 and consider the general method for constructing the asymptotic solution of this problem up to an arbitrary precision. By Lemma 9, the asymptotic solution has the following form

$$\begin{aligned}
\theta(x,T,\varepsilon) &= \vartheta^M(x,T,\varepsilon) + \varepsilon\mathcal{V}^M\Big(\frac{S(x,T,\varepsilon)}{\varepsilon}, \frac{x_N}{\varepsilon}, x, T, \varepsilon\Big) + \mathcal{O}(\varepsilon^{M+1}), \\
\varphi(x,T,\varepsilon) &= \varepsilon\Phi^M(x,T,\varepsilon) + \mathcal{W}^M\Big(\frac{S(x,T,\varepsilon)}{\varepsilon}, \frac{x_N}{\varepsilon}, x, T, \varepsilon\Big) + \mathcal{O}(\varepsilon^{M+1}), \\
\vartheta^M(x,T,\varepsilon) &= \sum_{j=0}^{M} \varepsilon^j \theta_j(x,T), \qquad \Phi^M(x,T,\varepsilon) = \sum_{j=1}^{M} \varepsilon^{j-1}\varphi_j(x,T),
\end{aligned} \tag{73}$$

$$\mathcal{V}^M(\eta,\tau,x,T,\varepsilon) = \sum_{j=1}^{M} \varepsilon^{j-1}\{\rho_j(x,T)V_j(\eta,x,T) + U_j(\eta,x,T) + Y_j(\tau,x',T)\},$$

$$\begin{aligned}
\mathcal{W}^M(\eta,\tau,x,T,\varepsilon) &= \chi(\eta,x,T) \\
&\quad + \sum_{j=1}^{M} \varepsilon^j\{\gamma_j(x,T)G_j(\eta,x,T) + W_j(\eta,x,T) + Z_j(\tau,x',T)\}.
\end{aligned}$$

Here x_N is the distance from $\partial\Omega$ along the internal normal, $x' \in \partial\Omega$,

$$S, \rho_i, \gamma_j, \theta_i, \varphi_j \in C^\infty(\bar{Q}), \qquad Y_j(\tau,x',T),\ Z_j(\tau,x',T) \in \mathcal{P},$$

$$V_i(\eta,x,T), \chi(\eta,x,T), U_j(\eta,x,T), G_j(\eta,x,T), W_j(\eta,x,T) \in \mathcal{H},$$

and

$$\begin{gathered}
\chi^\pm = \pm 1, \quad U_j^- = 0, \quad W_j^- = 0, \quad \frac{\partial S}{\partial T}\Big|_{\Gamma_T} \neq 0, \\
\rho_i|_{\Gamma_T} = 0, \qquad \gamma_j|_{\Gamma_T} = 0, \qquad \Gamma_T = \{x \in \Omega,\ S(x,T) = 0\}.
\end{gathered} \tag{74}$$

By Lemma 1 and (74), without loss of generality, we can assume that

$$\begin{aligned}
&S = T - \psi(x), \qquad \chi = \chi(\eta,x), \\
&V_i(\eta,x,T) = \alpha_i^+ + \alpha_i^-\chi(\eta,x), \qquad G_j(\eta,x,T) = \mu_j^+ + \mu_j^-\chi(\eta,x), \\
&\alpha_i^\pm = \frac{1}{2}(V_i^+(x,T) \pm V_i^-(x,T)), \qquad \mu_j^\pm = \frac{1}{2}(G_j^+(x,T) \pm G_j^-(x,T)).
\end{aligned} \tag{75}$$

Precisely as in [35, 36] and section 2, substituting (73) into equations (66), we get the relations for the regular parts of the asymptotic solution

$$\begin{aligned}
&\Delta\vartheta^M + f = \varepsilon\Big\{\frac{\partial}{\partial T}(\mathcal{W}^{M\pm} + \vartheta^M) - \Delta\mathcal{V}^{M\pm}\Big\} + \varepsilon^2\frac{\partial}{\partial T}(\mathcal{V}^{M\pm} + \Phi^M), \\
&\Delta(\mathcal{W}^{M\pm} - (\mathcal{W}^{M\pm})^3) \\
&\quad + \varepsilon\Big\{\kappa\frac{\partial}{\partial T}\mathcal{W}^{M\pm} - \Delta\Big((3(\mathcal{W}^{M\pm})^2 - 1)\Phi^M + \kappa_1\vartheta^M\Big)\Big\} \\
&\quad + \varepsilon^2\Big\{\Delta\Big((\Delta - 3(\Phi^M)^2)\mathcal{W}^{M\pm} + \kappa_1\mathcal{V}^{M\pm}\Big) + \kappa\frac{\partial}{\partial T}\Phi^M\Big\} \\
&\quad + \varepsilon^3\Delta(\Delta\Phi^M - (\Phi^M)^3) = 0.
\end{aligned} \tag{76}$$

It is easy to see, that to pass to the limit as $\eta \to \pm\infty$, $\tau \to \infty$ is equivalent to calculate the weak limit as $\varepsilon \to 0$.

So, as ε tends to zero, we have

$$\Delta\Big((1 - (\chi^\pm)^2)\chi^\pm\Big) = 0,$$

which does not contradict the assumption (74), i.e., $\chi^\pm = \pm 1$.

Let us introduce the notation

$$\theta_j^\pm = \theta_j + \rho_j V_j^\pm + U_j^\pm, \quad \Phi_j^\pm = \varphi_j + \gamma_j G_j^\pm + W_j^\pm. \tag{77}$$

Now, setting the terms of the order $\mathcal{O}(\varepsilon^j)$ equal to zero, from (76) we get equations (68), (69), (71) for the leading terms and similar equations for the lower terms

$$\Delta\theta_k^\pm = f_{k,\theta}^\pm(x,T), \qquad \Delta\Phi_k^\pm = f_{k,\varphi}^\pm(x,T).$$

Here $f_{k,\theta}^{\pm}(x,\tau)$, $f_{k,\varphi}^{\pm}(x,\tau)$ are functions of $\theta_0, \theta_1^{\pm}, \dots, \Phi_{k-1}^{\pm}$ and their derivatives.

Let us consider a neighborhood of Γ_T. Using the same construction as in Section 2 and [35, 36], and taking the terms $\mathcal{O}(\varepsilon^{-2})$ in relations similar to (16), we get the equality

$$\frac{\partial^2}{\partial\eta^2}\left\{\frac{1}{2\beta^2}\frac{\partial^2\check{\chi}}{\partial\eta^2}+\check{\chi}-\check{\chi}^3\right\}=0, \qquad \beta=1/\sqrt{2}\,|\nabla\psi|.$$

After the integration, we obtain

$$\frac{\partial}{\partial\eta}\left\{\frac{1}{2\beta^2}\frac{\partial^2\check{\chi}}{\partial\eta^2}+\check{\chi}-\check{\chi}^3\right\}=\text{const}.$$

Obviously, const $=0$ since $\check{\chi}\in\mathcal{H}$. Therefore

$$\frac{1}{2\beta^2}\frac{\partial^2\check{\chi}}{\partial\eta^2}+\check{\chi}-\check{\chi}^3=c, \qquad \check{\chi}\to\pm1 \quad \text{as} \quad \eta\to\pm\infty. \tag{78}$$

It can be easily shown that, if and only if $c=0$, equation (78) has the solution such that $\check{\chi}\in\mathcal{H}$ and $\check{\chi}^+\neq\check{\chi}^-$ exists. Therefore the solution on Γ_T has the usual Van der Waals tanh form

$$\check{\chi}(\eta,x)=\tanh(\beta(\eta+\psi_1)),$$

where the "constant" of integration $\psi_1=\psi_1(x)$ is an arbitrary function from $C^\infty(\bar{\Omega})$. Since $\check{\chi}^{\pm}=\pm1$, we can extend $\check{\chi}$ by the identity to $\chi=\check{\chi}(\eta,x)$ for all $(x,\tau)\in\bar{Q}$.

Further, setting the lower terms equal to zero, we obtain equations (21), (22), where

$$\hat{L}=\frac{1}{2\beta^2}\frac{\partial^2}{\partial\eta^2}+1-3\chi^2,$$

$\check{F}_k^{\varphi}$, $\check{F}_k^{\theta}$ are functions of the higher terms of the asymptotic expansion at the point $\tau=\psi(x)$, $k=1,2,\dots,M$. In particular,

$$\check{F}_1^{\theta}=0, \qquad \check{F}_1^{\varphi}=\frac{\partial^2}{\partial\eta^2}\left\{\hat{\Pi}\frac{\partial\chi}{\partial\eta}+3\chi^2\varphi_1\right\}\Big|_{\tau=\psi},$$

$$\check{F}_2^{\theta}=2\beta^2\left\{2(\nabla\psi,\nabla\rho_1)\frac{\partial V_1}{\partial\eta}+\frac{\partial\chi}{\partial\eta}-\frac{\eta}{2\beta^2}\frac{\partial\rho_1}{\partial\tau}\frac{\partial^2V_1}{\partial\eta^2}\right\}\Big|_{\tau=\psi}.$$

Obviously, we get $\check{U}_1=0$. Let us consider the solvability conditions for equations (21), (22) in $\mathcal{H}$. It is easy to see that for $k=1$ conditions (25), (26) hold automatically. Further, since

$$\check{f}_1^{\varphi}=\hat{\Pi}\frac{\partial\chi}{\partial\eta}+3\check{\varphi}_1(\chi^2-1),$$

simple calculations yield the statement

Lemma 10. *Condition (27) for $k=1$ is equivalent to the equality*

$$\operatorname{div}\left(\frac{\nabla\psi}{|\nabla\psi|}\right)=3\sqrt{2}\,\check{\varphi}_1. \tag{79}$$

Now we can obtain the function $\check{W}_1=\omega_1(\eta,x)$, where

$$\omega_1(\eta,x)=\omega_{1,1}(\eta,x)+\psi_2(x)\chi_\eta(\eta,x)\in\mathcal{S},$$

$\omega_{1,1}$ is a particular solution of the inhomogenious equation (22) for $k=1$, and ψ_2 is the "constant" of integration. Therefore, the functions $W_1^{\pm}$ vanish, and the equality (79) can be rewritten in the form (70), since $\gamma_1|_{\Gamma_T}=0$.

Let us define the extensions W_1, $\gamma_1 G_1$

$$\gamma_1G_1+W_1=\gamma_1\mu_1^+(x,\tau)+\gamma_1\mu_1^-(x,\tau)\chi(\eta,x)+\omega_1(\eta,x).$$

Here

$$\gamma_1\mu_1^+=\frac{1}{2}(\Phi_{1c}^+-2\varphi_1+\Phi_{1c}^-), \qquad \gamma_1\mu_1^-=\frac{1}{2}(\Phi_{1c}^+-\Phi_{1c}^-),$$

$\Phi_{1c}^{\pm}$ are sufficiently smooth extensions of $\Phi_1^{\pm}$ in $\Omega_T^{\pm}\cup\Gamma_{T,\delta}^{\mp}$, such that the Poisson equation is satisfied. Here $0<\delta\ll1$ is an arbitrary number.

Now, let us consider equations (21), (22) in the case $k=2$. It is easy to establish that conditions (25) are satisfied and for $k=2$ conditions (26) have the form

$$\begin{aligned}&\left(2(\nabla\psi,\nabla\rho_1)+\frac{1}{2\beta^2}\frac{\partial\rho_1}{\partial\tau}\right)\Big|_{\tau=\psi}[\check{V}_1]=2,\\&\left(2(\nabla\psi,\nabla\gamma_1)+\frac{1}{2\beta^2}\frac{\partial\gamma_1}{\partial\tau}\right)\Big|_{\tau=\psi}[\check{G}_1]=2\kappa,\end{aligned} \tag{80}$$

where $[V]=V^- - V^+$. Note that

$$\nabla\rho|_{\Gamma_T}=-\rho_T\nabla\psi|_{\Gamma_T}$$

for any smooth function ρ such that $\rho|_{\Gamma_T}=0$. Hence, relations (80) imply

$$(\nabla\psi,\nabla\rho_1)|_{\Gamma_T}[\check{V}_1]=2, \quad (\nabla\psi,\nabla\gamma_1)|_{\Gamma_T}[\check{G}_1]=2\kappa.$$

Moreover, the last equalities can be rewritten in the form of Stefan conditions in (69), (71), since θ_1, $\varphi_1\in C^\infty(\bar{Q})$ and ρ_1, γ_1 are equal to zero on Γ_T. Let us also note that the realization of the continuity conditions in (69), (71) is the result of our construction.

Finally, after some trivial but cumbersome calculations condition (27) for $k=2$ can be transformed to a linear inhomogeneous equation for the phase correction ψ_1.

The following constructions are performed in the same way, see also [35, 36] and Section 2.

Now let us consider the boundary conditions on the external boundary $\partial\Omega$ and calculate the boundary-layer functions. The principle term of the asymptotic solution $\varphi=\chi$ satisfies both boundary conditions (5) up to $\mathcal{O}(\varepsilon^\infty)$. So, a discrepancy in the second boundary condition for φ on Σ arises only in terms $\mathcal{O}(\varepsilon)$. Let us put $Z_j=0$ for $j=1,2,3$, then we obtain the equation for Z_4:

$$\frac{\partial^2}{\partial\tau^2}Z_4-2Z_4=0, \qquad Z_4\to0, \quad \tau\to\infty.$$

Obviously,

$$Z_4=c_4(x',\tau)e^{-\sqrt{2}\,\tau}, \qquad c_4=-2^{-3/2}\frac{\partial}{\partial N}\Delta\Phi_1^-\Big|_{\Sigma}.$$

Further, we note that the appearance of the boundary-layer function $\varepsilon^4 Z_4$ necessarily requires some corrections in the Neumann condition in the term $\mathcal{O}(\varepsilon^3)$. So, the Neumann condition for Φ_3^- has the form

$$\frac{\partial \Phi_3^-}{\partial N}\Big|_\Sigma = \frac{\partial Z_4}{\partial \tau}\Big|_{\tau=0} = \frac{1}{2}\frac{\partial}{\partial N}\Delta\Phi_1^-\Big|_\Sigma .$$

The appearance of the boundary-layer functions Z_k yields boundary-layer terms Y_j in the θ asymptotic expansion. Since the boundary $\partial\Omega$ is fixed, $Y_j = 0$, $j = 1, \dots, 6$, and we get the equation for Y_7:

$$\frac{\partial^2 Y_7}{\partial \tau^2} = \frac{\partial}{\partial \tau} Z_4(\tau, x', T), \qquad Y_7 \to 0, \quad \tau \to \infty.$$

Therefore

$$Y_7 = -2^{-5/2}\frac{\partial^2}{\partial N \partial T}\Delta\Phi_1^-\big|_\Sigma \exp(-\sqrt{2}\,\tau).$$

Conversely, the appearance of Y_7 leads to a correction of the Neumann condition for temperature in the term $\mathcal{O}(\varepsilon^6)$. Thus, the Neumann condition for θ_6^- has the form

$$\frac{\partial \theta_6^-}{\partial N}\Big|_\Sigma = \frac{\partial Y_7}{\partial \tau}\Big|_{\tau=0} = \frac{1}{4}\frac{\partial^2}{\partial N \partial T}\Delta\Phi_1^-\Big|_\Sigma .$$

The following constructions are performed similarly.

Theorem 5 is proved. Analyzing our construction, we also obtain the statement.

Theorem 6. *Let the assumptions of Theorem 5 hold. Then for any integer $M \geq 0$ there exist the functions*

$$\theta_M^{as} = \theta_0 + \sum_{j=1}^{M+1} \varepsilon^j(\theta_j + \rho_j V_j + U_j + Y_j) + \varepsilon^{M+2}(U_{M+2} + Y_{M+2}),$$
$$\varphi_M^{as} = \chi + \sum_{j=1}^{M+1} \varepsilon^j(\varphi_j + \gamma_j G_j + W_j + Z_j) + \varepsilon^{M+2}(W_{M+2} + Z_{M+2}) \tag{81}$$

such that

$$\varepsilon\frac{\partial}{\partial T}(\theta_M^{as} + \varphi_M^{as}) - \Delta\theta_M^{as} - f(x,T) = \varepsilon^{M+1}\mathcal{F}_M^\theta,$$
$$\kappa\varepsilon\frac{\partial \varphi_M^{as}}{\partial T} + \Delta\left(\varepsilon^2\Delta\varphi_M^{as} + \varphi_M^{as} - (\varphi_M^{as})^3 + \varepsilon\kappa_1\theta_M^{as}\right) = \varepsilon^{M+1}\mathcal{F}_M^\varphi, \tag{82}$$
$$\frac{\partial \theta_M^{as}}{\partial N}\Big|_\Sigma = \varepsilon^{M+2}F_M^\theta, \qquad \frac{\partial \varphi_M^{as}}{\partial N}\Big|_\Sigma = 0, \qquad \frac{\partial}{\partial N}\Delta\varphi_M^{as}\Big|_\Sigma = \varepsilon^M F_M^\varphi.$$

Here $\mathcal{F}_M^{\varphi,\theta}$, $F_M^{\varphi,\theta}$ are (smooth for $\varepsilon > 0$) functions such that

$$\|\mathcal{F}_M^\theta; C(\bar{Q})\| + \|\mathcal{F}_M^\varphi; C(\bar{Q})\| \leq c_1,$$
$$\|F_M^\theta; C(\Sigma)\| + \|F_M^\varphi; C(\Sigma)\| \leq c_2, \tag{83}$$
$$\|\mathcal{F}_M^\theta; L^2(\Omega)\| + \|\mathcal{F}_M^\varphi; L^2(\Omega)\| \leq c_3\sqrt{\varepsilon},$$

where the constants c_j are independent of ε.

5. Justification of the asymptotic solution for large time

Let us introduce the notation $\sigma = \theta - \theta_M^{as}$, $\omega = \varphi - \varphi_M^{as}$ and let the initial data θ^0, φ^0 exhibit a special behavior. Then, from (66), (67), (5) and (82), we get the following problem for the remainders σ, ω:

$$\frac{\partial}{\partial T}(\sigma + \omega) - \frac{1}{\varepsilon}\Delta\omega = -\varepsilon^M\mathcal{F}_M^\theta,$$
$$\kappa\frac{\partial\omega}{\partial T} + \Delta\Big\{\varepsilon\Delta\omega + \frac{1}{\varepsilon}\omega\big(1 - 3\varphi_M^2 - 3\varphi_M\omega - \omega^2\big) + \kappa_1\sigma\Big\} = -\varepsilon^M\mathcal{F}_M^\varphi, \tag{84}$$
$$\frac{\partial\sigma}{\partial N}\Big|_\Sigma = -\varepsilon^{M+2}F_M^\theta, \qquad \frac{\partial\omega}{\partial N}\Big|_\Sigma = 0, \qquad \frac{\partial}{\partial N}\Delta\omega\Big|_\Sigma = -\varepsilon^M F_M^\varphi,$$
$$\sigma|_{T=0} = -\varepsilon^{M+1/2}f_M^\theta, \qquad \omega|_{T=0} = -\varepsilon^{M+1/2}f_M^\varphi.$$

Here $\mathcal{F}_M^{\theta,\varphi}$, $F_M^{\theta,\varphi}$ are smooth functions satisfying (82), functions $f_M^{\theta,\varphi}$ are such that

$$\|f_M^\theta; L^2(\Omega)\| + \|f_M^\varphi; L^2(\Omega)\| \leq c\sqrt{\varepsilon} \tag{85}$$

with constant c independent of ε. To simplify the notation, we omit the superscript denoting asymptotic solutions.

The main result of this section is

Theorem 7. *Let there exist a sufficiently smooth solution of problem (66), (67), (5) during the time $[0, T_0]$, and let the quantity $T_0 > 0$ be independent of ε. Let also the assumptions of Theorem 5 be satisfied and $M \geq 2$. Then the estimates hold*

$$\|\omega; L^\infty((0,T_0); L^2(\Omega))\| + \|\sigma; L^\infty((0,T_0); L^2(\Omega))\| \leq c\,\varepsilon^{M+1},$$
$$\|\nabla\omega; L^2(Q)\| + \|\nabla\sigma; L^2(Q)\| \leq c\,\varepsilon^{M+1}, \qquad \|\Delta\omega; L^2(Q)\| \leq c\,\varepsilon^M \tag{86}$$

with constant c independent of ε.

Proof. Similarly to the proof of Theorem 4, after some transformations, we get the equality with an arbitrary constant $K > 0$:

$$\frac{1}{2}\frac{d}{dT}\Big\{(1 + \kappa K)\|\omega\|^2 + \|\sigma\|^2 + 2\int_\Omega \omega\sigma\,dx\Big\}$$
$$+ \frac{1}{\varepsilon}\|\nabla\sigma\|^2 + \varepsilon K\|\Delta\omega\|^2 + \frac{3K}{\varepsilon}\|\omega\nabla\omega\|^2$$
$$= \frac{K}{\varepsilon}\int_\Omega(\nabla\omega, \nabla\omega(1 - 3\varphi_M^2))dx - \frac{3K}{\varepsilon}\int_\Omega(\nabla\omega, \nabla\varphi_M\omega^2)dx$$
$$- \frac{1}{\varepsilon}(1 - \varepsilon\kappa_1 K)\int_\Omega(\nabla\omega, \nabla\sigma)dx - \varepsilon^M\int_\Omega\Big\{(\omega + \sigma)\mathcal{F}_M^\theta + \omega K\mathcal{F}_M^\varphi\Big\}dx$$
$$-\varepsilon^{M+1}\int_{\partial\Omega}\Big\{(\omega + \sigma)F_M^\theta - \omega K(\varepsilon\kappa_1 F_M^\theta + F_M^\varphi)\Big\}dx'. \tag{87}$$

Let us analyze the energy relation (87) by using the following version of Lemma 5.

Lemma 11. *Let φ_M be the asymptotic expansion* (81). *Then*

$$\frac{1}{\varepsilon}\int_\Omega(\nabla\omega,\nabla\omega(1-3\varphi_M^2))dx=-\frac{2}{\varepsilon}\int_\Omega|\nabla\omega|^2dx-\frac{3}{\varepsilon^2}\int_\Omega\frac{\omega^2}{\beta}\chi_\tau dx+3I,$$

$$\begin{aligned}I=&\frac{1}{\varepsilon}\int_\Omega|\nabla\omega|^2(1-\chi^2)dx+\frac{3}{2\varepsilon^2}\int_\Omega\frac{\omega^2}{\beta}\chi_\tau(1-\chi^2)dx\\&+\frac{1}{2}\int_\Omega\omega^2\chi_\tau\Delta\frac{1}{\beta}dx-\int_\Omega|\nabla\omega|^2(2\chi+\varepsilon\varphi_M^*)\varphi_M^*dx\\&+2\int_\Omega\omega\chi_\tau(\nabla\omega,\nabla\frac{1}{\beta})dx-\int_\Omega\omega(\nabla\omega,\nabla(2\chi\varphi_M^*+\varepsilon(\varphi_M^*)^2))dx\\&+\frac{1}{2\varepsilon}\int_\Omega\frac{\omega^2}{\beta}\frac{\partial}{\partial\tau}\left(\hat{\Pi}\frac{\partial}{\partial\eta}\chi(\eta,x)-\varepsilon\Delta\chi(\eta,x)\right)\Big|_{\eta=\frac{\tau-\psi}{\varepsilon}}dx.\end{aligned}$$

Here $\hat{\Pi}$ is the operator described in (16), *$\varphi_M^*=(\varphi_M-\chi)/\varepsilon$ denotes the lower terms of expansion* (81).

Proof of Lemma 11. It is easy to establish that

$$\int_\Omega(\nabla\omega,\nabla(\omega(1-3\varphi_M^2)))dx=-2\int_\Omega|\nabla\omega|^2dx+3\int_\Omega(\nabla\omega,\nabla(\omega(1-\varphi_M^2)))dx,$$

$$\begin{aligned}\frac{1}{\varepsilon}\int_\Omega(\nabla\omega,\nabla(\omega(1-\varphi_M^2)))dx=&\frac{1}{\varepsilon}\int_\Omega|\nabla\omega|^2(1-\chi^2)dx+\frac{1}{2\varepsilon}\int_\Omega\omega^2\Delta\chi^2dx\\&-\int_\Omega|\nabla\omega|^2(2\chi+\varepsilon\varphi_M^*)\varphi_M^*dx-\int_\Omega\omega(\nabla\omega,\nabla(2\chi\varphi_M^*+\varepsilon(\varphi_M^*)^2))dx.\end{aligned}$$

Since $1-\chi^2=\varepsilon\chi_\tau/\beta$, we have

$$J=\frac{1}{2\varepsilon}\int_\Omega\omega^2\Delta\chi^2dx=-\frac{1}{2}\int_\Omega\omega^2\frac{\partial}{\partial\tau}\Delta(\frac{1}{\beta}\chi)dx.$$

It is clear that

$$J=-\frac{1}{2}\int_\Omega\frac{\omega^2}{\beta}\frac{\partial}{\partial\tau}\Delta\chi dx+\frac{1}{2}\int_\Omega\omega^2\chi_\tau\Delta\frac{1}{\beta}dx+2\int_\Omega\omega\chi_\tau(\nabla\omega,\nabla\frac{1}{\beta})dx,$$

$$\varepsilon^2\Delta\chi\left(\frac{\tau-\psi}{\varepsilon},x\right)=\chi^3-\chi-\varepsilon\hat{\Pi}\frac{\partial}{\partial\eta}\chi(\eta,x)+\varepsilon^2\Delta_x\chi(\eta,x)\Big|_{\eta=\frac{\tau-\psi}{\varepsilon}}.$$

Thus

$$\begin{aligned}\int_\Omega\frac{\omega^2}{\beta}\frac{\partial}{\partial\tau}\Delta\chi dx=&\frac{2}{\varepsilon^2}\int_\Omega\frac{\omega^2}{\beta}\chi_\tau dx-\frac{3}{\varepsilon^2}\int_\Omega\frac{\omega^2}{\beta}\chi_\tau(1-\chi^2)dx\\&-\frac{1}{\varepsilon}\int_\Omega\frac{\omega^2}{\beta}\frac{\partial}{\partial\tau}\left(\hat{\Pi}\frac{\partial}{\partial\eta}-\varepsilon\Delta\right)\chi(\eta,x)\Big|_{\eta=\frac{\tau-\psi}{\varepsilon}}dx.\end{aligned}$$

This equality completes the proof of Lemma 11.

Now, using Lemma 11 and integrating (87) with respect to T, we get

$$\begin{aligned}&\frac{1}{2}\left\{(1+\kappa K)\|\omega\|^2+\|\sigma\|^2\right\}(\tau)+\int_0^T\Big\{\frac{2K}{\varepsilon}\|\nabla\omega\|^2\\&\quad+\frac{1}{\varepsilon}\|\nabla\sigma\|^2+\varepsilon K\|\Delta\omega\|^2+\frac{3K}{\varepsilon}\|\omega\nabla\omega\|^2+\frac{3K}{\varepsilon^2}\int_\Omega\frac{\omega^2}{\beta}\chi_\tau dx\Big\}d\tau'\\&=\frac{1}{2}\varepsilon^{2M+1}\left\{(1+\kappa K)\|f_M^\varphi\|^2+\|f_M^\theta\|^2+2\int_\Omega f_M^\varphi f_M^\theta dx\right\}-\int_\Omega\omega\sigma dx\\&\quad+\int_0^T\Big\{3KI-\frac{3K}{\varepsilon}\int_\Omega(\nabla\omega,\nabla\varphi_M\omega^2)dx\\&\quad-\frac{1}{\varepsilon}(1-\varepsilon\kappa_1K)\int_\Omega(\nabla\omega,\nabla\sigma)dx-\varepsilon^M\int_\Omega\left[(\omega+\sigma)\mathcal{F}_M^\theta+\omega K\mathcal{F}_M^\varphi\right]dx\\&\quad-\varepsilon^{M+1}\int_{\partial\Omega}\left[(\omega+\sigma)F_M^\theta-\omega K(\varepsilon\kappa_1F_M^\theta+F_M^\varphi)\right]dx'\Big\}d\tau'.\end{aligned}\qquad(88)$$

It is easy to see that

$$2\Big|\int_\Omega\omega\sigma dx\Big|\le\alpha\,\|\omega\|^2+\frac{1}{\alpha}\,\|\sigma\|^2,\qquad\alpha=\frac{1}{2}\left(\kappa K+\sqrt{(\kappa K)^2+4}\,\right),$$

$$2\Big|\int_\Omega(\nabla\omega,\nabla\sigma)dx\Big|\le\alpha\,\|\nabla\omega\|^2+\frac{1}{\alpha}\,\|\nabla\sigma\|^2.$$

Further, by the embedding theorem and (83), we get

$$\begin{aligned}&\varepsilon^{M+1}\Big|\int_{\partial\Omega}\left[(\omega+\sigma)F_M^\theta-\omega K(\kappa_2F_M^\theta+F_M^\varphi)\right]dx'\Big|\\&\le\ c\,\varepsilon^{M+1}(\|\omega;L^2(\partial\Omega)\|+\|\sigma;L^2(\partial\Omega)\|)\le c\,\varepsilon^{2M+2}+\frac{1}{4}\,\|\omega\|_1^2+\frac{1}{4}\,\|\sigma\|_1^2.\end{aligned}$$

Here, as usual, c denotes a universal constant and $\|f\|_k$ is the $H^k(\Omega)$ norm of f.

Therefore, choosing ε small enough, from (88) we obtain the following inequality

$$\begin{aligned}&\frac{\alpha-1}{2\alpha}\left\{\|\omega\|^2+\|\sigma\|^2\right\}(\tau)+\int_0^T\Big\{\frac{K}{\varepsilon}\,\|\nabla\omega\|^2+\frac{1}{2\varepsilon}\,\|\nabla\sigma\|^2\\&\quad+\varepsilon K\,\|\Delta\omega\|^2+\frac{3K}{\varepsilon}\,\|\omega\nabla\omega\|^2+\frac{3K}{\varepsilon^2}\int_\Omega\frac{\omega^2}{\beta}\chi_\tau\,dx\Big\}d\tau'\\&\le\ c\,\varepsilon^{2M+1}+\int_0^T\Big\{3K\,|I|+c(\|\omega\|^2+\|\sigma\|^2)\\&\qquad+\frac{3K}{\varepsilon}\,\Big|\int_\Omega(\nabla\omega,\nabla\varphi_M\omega^2)dx\Big|\Big\}d\tau'.\end{aligned}\qquad(89)$$

Further, using Lemma 6, it is not too difficult to prove the following analog of Lemma 7 (see also [35, 36])

Lemma 12. *Let ε be small enough. Then*

$$|I|\le\frac{\delta_1}{\varepsilon^2}\left\|\omega\sqrt{\Psi}\right\|^2+\frac{\delta_2}{\varepsilon}\|\nabla\omega\|^2+c\,\varepsilon^{1/2}\|\omega\|^2+c\,\varepsilon^{7/2}\|\Delta\omega\|^2,$$

where $\Psi=\chi_\tau/\beta$; δ_1, $\delta_2>0$ are arbitrary constants.

Now, using the Galliardo-Nirenberg inequality, we get the estimate for the last term in the right-hand side of (89).

$$
\begin{aligned}
&\frac{1}{\varepsilon}\left|\int_\Omega (\nabla\omega, \nabla\varphi_M \omega^2)dx\right| \\
&\quad\leq \frac{c}{\varepsilon}\int_\Omega |\omega|\,|\nabla\omega|^2 dx + \frac{c}{\varepsilon^2}\int_\Omega |\nabla\omega|\,\omega^2 dx \\
&\quad\leq \frac{c}{\varepsilon}\,\|\omega\|^{(8-n)/4}\,\|\omega\|_2^{(4+n)/4} + \frac{c}{\varepsilon^2}\,\|\nabla\omega\|\,\|\omega\|^{(8-n)/4}\,\|\omega\|_2^{n/4} \\
&\leq \frac{\delta_3}{\varepsilon}\,\|\nabla\omega\|^2 + \delta_4\varepsilon\,\|\omega\|_2^2 + c\,\varepsilon^{-(12+n)/(4-n)}\,\|\omega\|^{2(8-n)/(4-n)}.
\end{aligned}
$$

It is clear that, choosing reasonable constants δ_i, we can transform (63) as follows

$$
\begin{aligned}
&U(\tau) + c\int_0^T\Big\{\frac{1}{\varepsilon}\,\|\nabla\omega\|^2 + \frac{1}{\varepsilon}\|\nabla\sigma\|^2 + \varepsilon\,\|\Delta\omega\|^2 \\
&+\frac{1}{\varepsilon}\,\|\omega\nabla\omega\|^2 + \frac{1}{\varepsilon^2}\Big\|\sqrt{\Psi}\,\omega\Big\|^2\Big\}d\tau' \qquad (90)\\
&\leq c\,\varepsilon^{2M+1} + c\int_0^T\left\{U(\tau') + \varepsilon^{-r}(U(\tau'))^{1+\lambda}\right\}d\tau'.
\end{aligned}
$$

Here

$$
U(T) = \{\|\omega\|^2 + \|\sigma\|^2\}(\tau), \qquad \lambda = \frac{4}{4-n}, \quad r = \frac{12+n}{4-n}.
$$

According to the Gronuoll lemma and to Lemma 8, (90) yields

$$
\begin{aligned}
&\{\|\omega\| + \|\sigma\|\}(\tau) \leq c\varepsilon^{M+1/2}, \\
&\|\nabla\omega; L^2(Q)\| + \|\nabla\sigma; L^2(Q)\| \leq c\varepsilon^{M+1}, \quad \|\Delta\omega; L^2(Q)\| \leq c\varepsilon^M.
\end{aligned} \qquad (91)
$$

Finally, repeating the construction of Theorem 4, we establish that (91) and Theorem 6 yield the estimates (86). Theorem 7 is proved.

6. Proof of Lemma 9.

First of all we note that the weak limit of temperature must be a function smoother than the Heaviside function. Really, let $\overline{\theta}$ be of the form

$$
\overline{\theta} = a(x,\tau)H(\tau - \psi(x)) + b(x,\tau), \qquad (92)
$$

where b is a function smoother than H. Then (92) and the heat equation imply that the following relation must hold in the $\mathcal{D}'$ sense

$$
a\delta'(\tau - \psi) = (1+a)\delta(\tau - \psi) + \text{smoother terms}, \qquad (92')
$$

where δ, δ' denote the Dirac's δ-function and their derivative. Obviously, it is impossible, since $a \neq 0$.

Therefore, we must assume that the asymptotic expansion for the solution of problem (66), (67), (5) has the form

$$
\begin{aligned}
\theta(x,\tau,\varepsilon) &= \vartheta^M(x,\tau,\varepsilon) + \mathcal{V}^M\Big(\frac{S(x,\tau,\varepsilon)}{\varepsilon}, \frac{x_N}{\varepsilon}, x, \tau, \varepsilon\Big) + \mathcal{O}(\varepsilon^{M+1}), \\
\varphi(x,\tau,\varepsilon) &= \varepsilon\Phi^M(x,\tau,\varepsilon) + \mathcal{W}^M\Big(\frac{S(x,\tau,\varepsilon)}{\varepsilon}, \frac{x_N}{\varepsilon}, x, \tau, \varepsilon\Big) + \mathcal{O}(\varepsilon^{M+1}), \\
\vartheta^M(x,\tau,\varepsilon) &= \sum_{j=0}^M \varepsilon^j\theta_j(x,\tau), \qquad \Phi^M(x,t,\varepsilon) = \sum_{j=1}^M \varepsilon^{j-1}\varphi_j(x,\tau), \qquad (93)\\
\mathcal{V}^M(\eta,\tau,x,\tau,\varepsilon) &= \rho_0(x,t)V_0(\eta,x,\tau) \\
&+\sum_{j=1}^M \varepsilon^j\{\rho_j(x,\tau)V_j(\eta,x,\tau) + U_j(\eta,x,\tau) + Y_j(\tau,x',\tau)\}, \\
\mathcal{W}^M(\eta,\tau,x,\tau,\varepsilon) &= \chi(\eta,x,\tau) \\
&+\sum_{j=1}^M \varepsilon^j\{\gamma_j(x,\tau)G_j(\eta,x,\tau) + W_j(\eta,x,\tau) + Z_j(\tau,x',\tau)\}.
\end{aligned}
$$

Here V_0, $U_1 \in \mathcal{H}$, $\rho_0 \in C^\infty$, $\rho_0|_{\Gamma_T} = 0$, and we use the notation (73). It is clear that $\overline{\Delta\theta} \sim \delta(\tau - \psi)$, so, we avoid the contradiction (92').

Further, repeating the construction of Theorem 5 and using the notation (77), we get the following equations

$$
\Delta\theta_0^\pm = f, \qquad x \in \Omega_T^\pm, \quad \tau > 0, \qquad (94)
$$

$$
\frac{\partial^2 \tilde{U}_1}{\partial\eta^2} = 2\frac{(\nabla\psi, \nabla\rho_0)}{|\nabla\psi|^2}\frac{\partial\tilde{V}_0}{\partial\eta} - \eta\frac{\partial\rho_0}{\partial\tau}\frac{\partial^2\tilde{V}_0}{\partial\eta^2}\Big|_{\tau=\psi}. \qquad (95)
$$

The solvability condition for the model equation (95) yields the relation

$$
\left(2(\nabla\psi, \nabla\rho_0) + \rho_{0\tau}|\nabla\psi|^2\right)\Big|_{\tau=\psi}(\tilde{V}_0^+ - \tilde{V}_0^-) = 0. \qquad (96)
$$

Obviously, from (93) and (96) we get the matching conditions on the free interface

$$
[\theta_0^\pm]|_{\Gamma_T} = 0, \qquad \Big[\frac{\partial\theta_0^\pm}{\partial\nu}\Big]\Big|_{\Gamma_T} = 0. \qquad (97)
$$

Let us define the function θ_0 as the solution of the Neumann problem (68). Then we obtain the following problem for the discrepancy $\tilde{\theta}^\pm = \theta_0^\pm - \theta_0$

$$
\begin{aligned}
&\Delta\tilde{\theta}^\pm = 0, \qquad x \in \Omega_T^\pm, \quad \tau > 0, \\
&[\tilde{\theta}^\pm]|_{\Gamma_T} = 0, \qquad \Big[\frac{\partial\tilde{\theta}^\pm}{\partial\nu}\Big]\Big|_{\Gamma_T} = 0, \qquad (98)\\
&\frac{\partial\tilde{\theta}^-}{\partial N}\Big|_\Sigma = 0, \quad \int_{\Omega_T^+}\tilde{\theta}^+ dx + \int_{\Omega_T^-}\tilde{\theta}^- dx = 0.
\end{aligned}
$$

Since Γ_T is a smooth surface of codimension 1, the energy equality

$$
\int_{\Omega_T^+}|\nabla\tilde{\theta}^+|^2 dx + \int_{\Omega_T^-}|\nabla\tilde{\theta}^-|^2 dx = 0
$$

yields $\tilde{\theta}^{\pm} = 0$. Consequently, $U_1 = 0$ and we get the asymptotic expansion (73) for the solution of problem (66), (67), (5). According to Theorem 6, the exact solution θ can be written as follows:

$$\theta = \theta_M^{as} + \sigma, \quad \|\sigma; L^\infty(0, T_0; L^2(\Omega)) \cap L^2(0, T_0; H^1(\Omega))\| \le c\varepsilon^{M+1}$$

for all $M \geq 2$. It means that $\overline{\theta} = \overline{\theta_0^{as}} = \theta_0$ is a smooth function. Lemma 9 is proved.

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