

# Generalized Chiral Perturbation Theory

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## Abstract

The Generalized Chiral Perturbation Theory enlarges the framework of the standard  $\chi$ PT, relaxing certain assumptions which do not necessarily follow from QCD or from experiment, and which are crucial for the usual formulation of the low energy expansion. In this way, experimental tests of the foundations of the standard  $\chi$ PT become possible. Emphasis is put on physical aspects rather than on formal developments of  $G\chi$ PT

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# 1 Introduction

Up to now, very little is known about the chiral structure of the QCD ground state. The fact that in the limit of vanishing quark masses  $m_u = m_d = m_s = 0$ , the chiral symmetry of QCD is spontaneously broken down to  $SU_V(3)$  just ensures the existence of 8 Goldstone bosons coupled to 8 conserved axial currents. The strength of this coupling defines a mass scale  $F_0 = F_\pi|_{m_u=m_d=m_s=0} \simeq 90$  MeV, which is characteristic of spontaneous chiral symmetry breaking. Indeed,  $F_0$  is a long range order parameter which plays a fundamental role : a non-zero value of  $F_0$  not only signals a broken symmetry phase but, moreover, it is necessary for the spontaneous breakdown to occur. The fact that  $F_0$  is about ten times smaller than the typical mass-scale  $\Lambda_H \sim 1$  GeV of lightest massive bound states ( $\rho, N, \dots$ ) remains unexplained and it raises a question : What is the natural size of other order parameters, such as the quark condensate (further examples will be given shortly),

$$B_0 = -F_0^{-2} \langle \bar{u}u \rangle_0 = -F_0^{-2} \langle \bar{d}d \rangle_0 = -F_0^{-2} \langle \bar{s}s \rangle_0 , \quad (1)$$

which describe the chiral structure of the massless QCD vacuum ?

Generalized Chiral Perturbation Theory ( $G\chi$ PT) is motivated by the observation that nothing from our present theoretical or experimental knowledge allows us to answer the above question in one way or the other :  $B_0$  could be as large as the bound state scale  $\Lambda_H \sim 1$  GeV or as small as the fundamental order parameter  $F_0 \sim 90$  MeV. The main purpose of  $G\chi$ PT is to provide a sufficiently precise and broad theoretical framework which would allow to answer the above question experimentally.

It is being assumed for a long time that the chiral order in the QCD ground state resembles the magnetic order of a ferromagnet : The dominant effect is the quark condensate (1) - the analogue of the spontaneous magnetization. In particular, this order parameter dominates the (linear) response of the system to a small perturbation by quark masses [1] - the analogue of an external magnetic field. This picture, which underlies the standard  $\chi$ PT [2], is certainly appealing by its simplicity. However, before one takes it for granted, one should remember that Nature offers examples of magnetic systems, in which the broken symmetry phase is characterized by a magnetic order of entirely different kind, like antiferromagnets [3]. In this case, the average magnetization is not necessarily the relevant order parameter. Depending on structural details, the spontaneous magnetization of an antiferromagnet can be small or even vanish, to the extent that the ground state approaches the Néel-type magnetic order. The response of the system to an external magnetic field is then dominated by different order parameters, and it can become non-linear even for not too strong fields.

Phenomena similar to antiferromagnetism could be expected in QCD, if  $B_0$  (normalised at the scale  $\mu = \Lambda_H$ ) happened to be as small as  $100 \text{ MeV} \sim F_0$  [4]. Clearly, even a small but non-

vanishing condensate would still dominate the response to a perturbation by mathematically small quark masses. However, contrary to the case of magnetic systems, where the strength of external fields can be chosen at will, Nature has already made its choice of the size of quark masses. In the real world,  $m_u$ ,  $m_d$  and  $m_s$  are certainly small compared to the hadronic mass scale  $\Lambda_H$ . Yet, they need not be small enough to guarantee the linear response to the explicit symmetry breaking, which would be dominated by the condensate  $B_0$ .

G $\chi$ PT [4]-[6] provides a systematic extension of the standard  $\chi$ PT which allows to incorporate the above possibility into its formalism.

## 2 Expansion of Goldstone Boson masses

If quark masses are switched on, Goldstone bosons acquire a mass. Since in the real world  $m_u, m_d, m_s \ll \Lambda_H$ , the Goldstone boson masses can be expanded in powers of quark masses.

The coefficients of this expansion are order parameters defined in terms of correlation functions of the currents

$$S^a(x) = \bar{\psi}(x) \frac{\lambda^a}{2} \psi(x), \quad P^a(x) = \bar{\psi}(x) \frac{\lambda^a}{2} i\gamma_5 \psi(x). \quad (2)$$

If, for a given value of  $B_0 \neq 0$ , quark masses are sufficiently small, the linear term in this expansion dominates. Otherwise, the prominent role is played by a different order parameter  $A_0$ , defined by a two point correlator of the scalar and pseudoscalar densities (2) :

$$\delta^{ab} \Pi_{SP}(q^2) = \frac{i}{F_0^2} \int dx e^{iqx} \langle 0 | T \{ S^a(x) S^b(0) - P^a(x) P^b(0) \} | 0 \rangle \quad (3)$$

where  $a, b = 1 \dots 8$ , and  $|0\rangle$  denotes the ground state of the massless QCD. The small  $q^2$  behaviour of the two point function (3) is dictated by chiral Ward identities,

$$\Pi_{SP}(q^2) = \frac{B_0^2}{q^2} + \frac{5}{96\pi^2} \frac{B_0^2}{F_0^2} \ln \left( \frac{\mu^2}{-q^2} \right) + A_0(\mu) + O(q^2), \quad (4)$$

where the  $\mu$  dependence of the parameter  $A_0$  compensates the scale dependence of the logarithm.  $A_0$  is indeed an order parameter of chiral symmetry breaking : The operator in (3) does not contain the singlet representation of the chiral symmetry group. If, for a given value of quark masses,  $B_0$  turns out to be sufficiently small (i.e. if the ground state effectively behaves like an antiferromagnet), the pseudoscalar meson masses start to be dominated by the order parameter  $A_0$ .

In order to obtain the expansion of Goldstone boson masses explicitly, it is sufficient to consider for  $m_q \neq 0$  the two-point function of the divergence of the axial current at zero momentum transfer, where it is related by the well known Ward identity to the quark condensate. Separating in this Ward identity the pseudoscalar meson contribution and expanding the

remainder in powers of  $m_q$ , one obtains ( $m_u = m_d = \widehat{m}$ , for simplicity) :

$$\begin{aligned}
F_\pi^2 M_\pi^2 &= 2\widehat{m} F_0^2 B + 4\widehat{m}^2 F_0^2 A_0(\mu) + F_\pi^2 \delta M_\pi^2(\mu) \\
F_K^2 M_K^2 &= (m_s + \widehat{m}) F_0^2 B + (m_s + \widehat{m})^2 F_0^2 A_0(\mu) + F_K^2 \delta M_K^2(\mu) \\
F_\eta^2 M_\eta^2 &= \frac{2}{3}(2m_s + \widehat{m}) F_0^2 B + \frac{4}{3}(2m_s^2 + \widehat{m}^2) F_0^2 A_0(\mu) + \frac{8}{3}(m_s - \widehat{m})^2 F_0^2 Z_0^P + F_\eta^2 \delta M_\eta^2(\mu) .
\end{aligned} \tag{5}$$

The parameter  $B$  is related to the vacuum condensate  $B_0$  :

$$B = B_0 + 2(m_s + 2\widehat{m}) Z_0^S(\mu) . \tag{6}$$

The constants  $Z_0^S$  and  $Z_0^P$  are defined in terms of the low energy behaviour of the two point functions  $\langle S^0 S^0 - S^8 S^8 \rangle$  and  $\langle P^0 P^0 - P^8 P^8 \rangle$ , respectively. They both violate the Zweig rule :  $Z_0^S$  is expected to be small (compared to  $A_0$ ) since it is suppressed in the large  $N_c$ -limit.  $Z_0^P$ , on the other hand, receives a contribution from the axial anomaly and, in principle, does not need to be small. Both  $Z_0^P$  and  $Z_0^S$  are order parameters of chiral symmetry breaking. Finally,  $\delta M_\pi^2$ ,  $\delta M_K^2$  and  $\delta M_\eta^2$  contain chiral logarithms, which compensate for the scale dependences of  $A_0$  and  $Z_0^S$ , and higher powers of quark masses. A few comments are in order:

i) The overall convergence of the expansions (5) is controlled by the small parameter

$$\frac{m_q}{\Lambda_H} , \quad \Lambda_H \sim 1 \text{ GeV} . \tag{7}$$

This statement becomes particularly transparent in the large  $N_c$  limit, in which the chiral logarithms drop out. In general, the coefficient of the  $n$ -th power of quark masses is defined by means of the low momentum behaviour of a  $n$ -point function of quark bilinears (2). (An example is the coefficient  $A_0$ , c.f. Eq. (4)). These  $n$ -point functions satisfy superconvergent dispersion relations (they are order parameters) which can be saturated by the lowest massive bound states of mass  $\sim \Lambda_H \sim 1$  GeV. This leads to the estimate that the coefficient of  $m_q^n$  ( $n \geq 2$ ) in (5) should indeed be of the order  $\Lambda_H^{2-n}$  times a factor of order unity. Notice that quark confinement makes it difficult to extend the above estimate to the quark condensate  $B_0$  itself.

ii) The relative importance of the linear term in (5) is controlled by another parameter,

$$\frac{m_q}{m_0} , \quad m_0 = \frac{B_0}{2A_0} , \tag{8}$$

which indicates how small the quark masses should actually be such as to ensure the validity of the Gell-Mann-Oakes-Renner formula. If  $m_s \sim m_0$ , the contributions of the first and second order terms in the expansion of  $M_K^2$  and of  $M_\eta^2$  would become comparable, and if even  $m_u, m_d \sim m_0$ , the same would happen for the pion mass. There is no compelling reason why  $m_0$  should be as large as  $\Lambda_H$  and there is nothing unnatural in having  $m_q \sim m_0 \ll \Lambda_H$ . As already pointed out,  $B_0$  (at the 1 GeV scale) might be as small as  $100 \text{ MeV} \sim F_0$ , and  $A_0$  can

be estimated using the superconvergent dispersion relations for the two-point function (3) : the standard QCD sum-rule technique leads to an order of magnitude estimate

$$A_0|_{1 \text{ GeV}} \simeq 1 \div 5 . \quad (9)$$

(Notice that for small  $B_0$ , the  $\mu$ -dependence of  $A_0$  is very weak). Hence, the antiferromagnetic alternative for the QCD vacuum is perfectly consistent with a low value of  $m_0 = (10 \div 50)$  MeV, indicating the possibility of a strong violation of the Gell-Mann-Oakes-Renner formula, for  $M_\eta^2$ ,  $M_K^2$  and even for  $M_\pi^2$ .

iii) The Gell-Mann-Okubo mass formula is compatible with Eq. (5) even if the first order term does not dominate. From (5), one obtains

$$3F_\eta^2 M_\eta^2 + F_\pi^2 M_\pi^2 - 4F_K^2 M_K^2 = 4(m_s - \widehat{m})^2 F_0^2 (A_0 + 2Z_0^P) + \dots , \quad (10)$$

where the ellipsis represents higher order contributions arising from  $\delta M_P^2$ . Neglecting the latter (and the splitting of the decay constants  $F_P$ ), the GMO formula can even become exact, provided that  $A_0 + 2Z_0^P = 0$ . Notice, however, that a priori there is no reason for this relation to hold. Actually it can hardly hold exactly, since it is badly violated for  $N_c \rightarrow \infty$ : in this limit  $A_0 = O(1)$ , whereas  $Z_0^P = O(N_c)$ , reflecting the vanishing of the  $\eta'$  mass in the chiral limit as  $N_c \rightarrow \infty$  [7]. The fact that the GMO relation is well satisfied by experimental masses does not by itself imply the dominance of the quark condensate term in the expansion of Goldstone boson masses.

iv) The parameter which actually controls the relative size of the first and second order terms in Eqs. (5) is the quark mass ratio

$$r = \frac{m_s}{\widehat{m}} = \frac{2m_s}{m_u + m_d} . \quad (11)$$

It is well known that provided the first term dominates,  $r$  is given by the ratio of  $M_K^2$  and  $M_\pi^2$ ,  $r \simeq 26$  [1, 8, 9]. To the extent that  $B_0$  decreases, the ratio  $r$  becomes smaller, until it reaches a critical value corresponding to vanishing  $B_0$ . For  $r$  below this critical value, the vacuum would become unstable with respect to small perturbations produced by quark masses.

Eqs. (5) put the above statements under a quantitative control. From the first two equations, one obtains

$$\frac{2\widehat{m}B_0}{M_\pi^2} = \frac{1}{r^2 - 1} [(r - r_1^*)(r + r_1^* + 2) - 2(r_2^* - r)(r + 2)\zeta] \left( \frac{M_\pi^* F_\pi}{M_\pi F_0} \right)^2 \quad (12a)$$

$$\frac{\widehat{m}^2 A_0}{M_\pi^2} = 2 \frac{r_2^* - r}{r^2 - 1} \left( \frac{M_\pi^* F_\pi}{M_\pi F_0} \right)^2 . \quad (12b)$$

Here,

$$(M_\pi^*)^2 = M_\pi^2 - \delta M_\pi^2 , \quad (M_K^*)^2 = M_K^2 - \delta M_K^2 , \quad (13)$$

$$r_1^* = 2 \frac{F_K M_K^*}{F_\pi M_\pi^*} - 1 , \quad r_2^* = 2 \left( \frac{F_K M_K^*}{F_\pi M_\pi^*} \right)^2 - 1 ,$$

and  $\zeta$  stands for (a small) Zweig rule violating parameter,

$$\zeta = \frac{Z_0^S}{A_0} . \quad (14)$$

The QCD vacuum is likely to be in the phase in which the order parameters  $m_q B_0$ ,  $A_0$  and  $Z_0^S$  are all non-negative. This phase is characterized by

$$\tau_1^* \leq \tau \leq \tau_2^* , \quad 0 \leq \zeta \leq \frac{1}{2} \frac{\tau - \tau_1^*}{\tau_2^* - \tau} \frac{\tau + \tau_1^* + 2}{\tau + 2} . \quad (15)$$

For  $\tau = \tau_2^*$ , the order parameter  $A_0$  vanishes and the pion mass is given by the vacuum condensate  $B_0$ :  $\tau_2^*$  may be referred to as the **ferromagnetic critical point**. On the other hand, for  $\tau = \tau_1^*$ , the quark condensate  $B_0$  vanishes and the pion mass is entirely accounted for by the order parameter  $A_0$ :  $\tau_1^*$  is the **anti-ferromagnetic critical point**. The values of  $\tau_1^*$  and  $\tau_2^*$  may be expanded in powers of quark masses,

$$\tau_1^* = \tau_1 + O(m_q) , \quad \tau_2^* = \tau_2 + O(m_q) , \quad (16)$$

where the leading order values  $\tau_1$  and  $\tau_2$  are known :

$$\tau_1 = 2 \frac{M_K}{M_\pi} - 1 \simeq 6.3 , \quad \tau_2 = 2 \frac{M_K^2}{M_\pi^2} - 1 = 25.9 . \quad (17)$$

The corrections push both  $\tau_1^*$  and  $\tau_2^*$  upwards:  $\tau_1^*$  can reach the value  $8 \div 9$ , whereas  $\tau_2^*$  can actually be as large as 38 (this can be seen upon neglecting, in Eqs. (13),  $\delta M_\pi^2$  and  $\delta M_K^2$ , and by using the experimental value  $F_K/F_\pi=1.22$ ).

The question, whether the value of the quark mass ratio  $\tau$  is closer to the ferromagnetic or to the antiferromagnetic critical point, has to be decided experimentally.

### 3 Expansion of the effective Lagrangian

The above remarks do not affect the construction of the low energy effective Lagrangian [10]. Its form is merely dictated by the chiral symmetry [11] and by the transformation properties of the symmetry breaking quark mass term, and there is obviously no question to alter these fundamental properties of QCD.  $\mathcal{L}^{eff}$  is a function of 8 Goldstone boson fields (conventionally collected into an SU(3) element  $U$ ) and of external sources  $v_\mu$ ,  $a_\mu$ ,  $s$  and  $p$ , the scalar source  $s$  containing the quark mass matrix  $\mathcal{M}$ . The formalism and notation used here are standard [2], unless otherwise stated.  $\mathcal{L}^{eff}$  consists of an infinite tower of invariants

$$\mathcal{L}^{eff} = \sum_{(k,l)} \mathcal{L}_{(k,l)} , \quad (18)$$

where  $\mathcal{L}_{(k,l)}$  contains  $k$  powers of covariant derivatives and  $l$  powers of scalar or pseudoscalar sources. In the low energy limit,  $\mathcal{L}_{(k,l)}$  vanishes like the  $k$ -th power of external momenta  $p$  and the  $l$ -th power of the quark mass  $m_q$ ,

$$\mathcal{L}_{(k,l)} \sim p^k m_q^l. \quad (19)$$

Chiral perturbation theory is an expansion of  $\mathcal{L}^{eff}$  in powers of the pion (kaon) mass assuming that all external momenta are of that size. For sufficiently small quark masses, such that both  $m_q \ll \Lambda_H$  and  $m_q \ll m_0 = \frac{B_0}{2A_0}$  hold, one has  $\mathcal{L}_{(k,l)} = O(p^{k+2l})$ . In this case, one can write

$$\mathcal{L}^{eff} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \dots \quad (20)$$

where [2]

$$\mathcal{L}^{(d)} = \sum_{k+2l=d} \mathcal{L}_{(k,l)}. \quad (21)$$

This expansion defines the standard  $\chi$ PT. If, on the other hand, for actual values of quark masses one has  $m_q \sim m_0 = \frac{B_0}{2A_0} \ll \Lambda_H$ , both  $m_q$  and  $B_0$  should count as parameters of the size of the pion mass and, consequently,  $\mathcal{L}_{(k,l)} = O(p^{k+l})$ . This new counting yields a different expansion of  $\mathcal{L}^{eff}$ ,

$$\mathcal{L}^{eff} = \tilde{\mathcal{L}}^{(2)} + \tilde{\mathcal{L}}^{(3)} + \tilde{\mathcal{L}}^{(4)} + \tilde{\mathcal{L}}^{(5)} + \tilde{\mathcal{L}}^{(6)} + \dots \quad (22)$$

where [4]

$$\tilde{\mathcal{L}}^{(d)} = \sum_{k+l+n=d} B_0^n \mathcal{L}_{(k,l)}. \quad (23)$$

It should be stressed that Eqs. (20) and (22) represent two different expansions of the *same* effective Lagrangian. To all orders they are identical, at a given finite order they may differ.

It is straightforward to write down the most general expression [4] of  $\tilde{\mathcal{L}}^{(2)}$  which defines the leading  $O(p^2)$  order of  $G\chi$ PT :

$$\begin{aligned} \tilde{\mathcal{L}}^{(2)} = & \frac{1}{4} F_0^2 \{ \langle D_\mu U^\dagger D^\mu U \rangle + 2B_0 \langle U^\dagger \chi + \chi^\dagger U \rangle + \\ & + A_0 \langle (U^\dagger \chi)^2 + (\chi^\dagger U)^2 \rangle + Z_0^S \langle U^\dagger \chi + \chi^\dagger U \rangle^2 \\ & + Z_0^P \langle U^\dagger \chi - \chi^\dagger U \rangle^2 + H_0 \langle \chi^\dagger \chi \rangle \}. \end{aligned} \quad (24)$$

Here  $\chi$  collects the scalar and pseudoscalar sources,

$$\chi = s + ip = \mathcal{M} + \dots, \quad \mathcal{M} = \text{diag}(m_u, m_d, m_s). \quad (25)$$

Notice the absence of the factor  $2B_0$ , which appears in the standard definition of  $\chi$  [2]. The meaning of this difference will shortly become obvious. Compared to  $\mathcal{L}^{(2)}$ , Eq. (24) contains additional terms. The constants  $A_0$ ,  $Z_0^S$  and  $Z_0^P$  are the same as introduced in the previous section : Eq. (5), (with  $F_\pi = F_K = F_\eta = F_0$  and  $\delta M_P^2 = 0$ ) is indeed a straightforward

consequence of the Lagrangian (24). The fact that the  $A_0$ ,  $Z_0^S$  and  $Z_0^P$  terms now appear at the same order  $O(p^2)$  as the  $B_0$  term reflects the possibility that in Eqs. (5) the first order and second order terms are of comparable size. To this order  $O(p^2)$ , the low energy constants in (24) can be expressed in terms of physical masses  $M_\pi$ ,  $M_K$ ,  $M_\eta$ , of the quark mass ratio  $r = m_s/\bar{m}$  and of the Zweig rule violating parameter  $\zeta$  (14). Expanding Eqs. (12a), (12b) and (10), one gets

$$\frac{2\bar{m}B_0}{M_\pi^2} = \frac{1}{r^2 - 1} [(r - r_1)(r + r_1 + 2) - 2(r_2 - r)(r + 2)\zeta] \quad (26a)$$

$$\frac{4\bar{m}^2 A_0}{M_\pi^2} = 2 \frac{r_2 - r}{r^2 - 1} \quad (26b)$$

$$\frac{4\bar{m}^2 Z_0^S}{M_\pi^2} = 2 \frac{r_2 - r}{r^2 - 1} \zeta \quad (26c)$$

$$\frac{4\bar{m}^2 Z_0^P}{M_\pi^2} = \frac{1}{(r - 1)^2} \frac{\Delta_{GMO}}{2} - \frac{r_2 - r}{r^2 - 1}, \quad (26d)$$

where  $\Delta_{GMO} \equiv (3M_\eta^2 - 4M_K^2 + M_\pi^2)/M_\pi^2$ . It is seen that for the particular values of the parameters  $r = r_2$ ,  $\Delta_{GMO} = 0$ , Eqs. (26) imply  $A_0 = Z_0^S = Z_0^P = 0$  and  $2\bar{m}B_0 = M_\pi^2$ , i.e. one recovers the standard  $O(p^2)$  Lagrangian  $\mathcal{L}^{(2)}$ . Order by order,  $G\chi$ PT contains the standard  $\chi$ PT as a special case.

The next to the leading term of the expansion (22) is of odd order  $O(p^3)$ , which is a new feature, absent in the standard  $\chi$ PT. One has [6]

$$\begin{aligned} \tilde{\mathcal{L}}^{(3)} = & \frac{1}{4} F_0^2 \{ \xi \langle D_\mu U^+ D^\mu U (\chi^+ U + U^+ \chi) \rangle + \tilde{\xi} \langle D_\mu U^+ D^\mu U \rangle (\chi^+ U + U^+ \chi) \\ & + \rho_1 \langle (\chi^+ U)^3 + (U^+ \chi)^3 \rangle + \rho_2 \langle (\chi^+ U + U^+ \chi) \chi^+ \chi \rangle \\ & + \rho_3 \langle \chi^+ U - U^+ \chi \rangle \langle (\chi^+ U)^2 - (U^+ \chi)^2 \rangle \\ & + \rho_4 \langle (\chi^+ U)^2 + (U^+ \chi)^2 \rangle \langle \chi^+ U + U^+ \chi \rangle \\ & + \rho_5 \langle \chi^+ \chi \rangle \langle \chi^+ U + U^+ \chi \rangle \\ & + \rho_6 \langle \chi^+ U - U^+ \chi \rangle^2 \langle \chi^+ U + U^+ \chi \rangle + \rho_7 \langle \chi^+ U + U^+ \chi \rangle^3 \} . \end{aligned} \quad (27)$$

In writing down Eq. (27), we have adopted some conventions which are worth to be specified. The parameters of  $\tilde{\mathcal{L}}^{(2)} + \tilde{\mathcal{L}}^{(3)}$  are finite, (divergences only start at order  $O(p^4)$ ) and they may be viewed as independent variables. It is convenient to tag each of these variables by a QCD correlation function. An example is the  $O(p^2)$  parameter  $A_0$ , closely related to the two-point function (3). This relation may be further specified as

$$A_0(\mu) = A_0 + O(B_0^2 \ln \mu), \quad (28)$$

where  $A_0(\mu)$  is defined by Eq. (4). Here, the statement is that the expansion of  $A_0(\mu)$  in powers of  $B_0$  does not contain a linear term. This constrains the way one writes  $\tilde{\mathcal{L}}^{(3)}$ :

i) A term  $B_0 \mathcal{L}_{(0,2)}$  which, according to Eq. (23) could be present in  $\tilde{\mathcal{L}}^{(3)}$ , is obviously irrelevant. It can be absorbed into the covariant transformation of sources  $\chi$  and of parameters



of  $\mathcal{L}^{eff}$  which has been discovered some times ago by Kaplan and Manohar [12], (see also Ref. [13]). Indeed, Eq. (28) may be viewed as a physical condition fixing the reparametrization ambiguity of  $\mathcal{L}^{eff}$  pointed out in [12] and in [13]. In general, the  $B_0$  dependent terms in Eq. (23) will be introduced only if they are required by renormalization. This is not the case of  $\tilde{\mathcal{L}}^{(3)}$ .

ii) Similarly, a  $\mathcal{L}_{(2,1)}$  term like  $\langle D_\mu U^\dagger D^\mu \chi + D_\mu \chi^\dagger D^\mu U \rangle$  which would yield a contribution to Eq. (28) linear in  $B_0$ , can be transformed away by a source dependent redefinition of Goldstone boson fields. Notice that this convention has not been used in Ref. [4].

The main physical effect described by the Lagrangian  $\tilde{\mathcal{L}}^{(3)}$  is the splitting of the decay constants  $F_\pi$ ,  $F_K$ , and  $F_\eta$ . One easily finds

$$\begin{aligned}\frac{F_\pi^2}{F_0^2} &= 1 + 2\widehat{m}\xi + 2(2\widehat{m} + m_s)\tilde{\xi} \\ \frac{F_K^2}{F_0^2} &= 1 + (m_s + \widehat{m})\xi + 2(2\widehat{m} + m_s)\tilde{\xi} \\ \frac{F_\eta^2}{F_0^2} &= 1 + \frac{2}{3}(2\widehat{m} + m_s)\xi + 2(2\widehat{m} + m_s)\tilde{\xi}.\end{aligned}\quad (29)$$

This allows to express the constant  $\widehat{m}\xi$  as

$$\widehat{m}\xi = \frac{1}{r-1} \left( \frac{F_K^2}{F_\pi^2} - 1 \right), \quad (30)$$

whereas the Zweig rule violating parameter  $\tilde{\xi}$  remains at this stage undetermined. The  $\mathcal{L}_{(0,3)}$  part of  $\tilde{\mathcal{L}}^{(3)}$ , described by the constants  $\rho_1 \cdots \rho_7$  generates an  $O(m_q^3/\Lambda_H)$  contribution to the pseudoscalar masses. Notice that in the standard  $\chi$ PT these terms would count as  $O(p^6)$ .

$\tilde{\mathcal{L}}^{(3)}$  provides the simplest example of odd chiral orders characteristic of  $G\chi$ PT. They do not correspond to an increase in the number of loops, but to additional corrections in powers of the quark masses. In the standard  $\chi$ PT [2], the splitting in the decay constants is a  $O(M_\pi^2)$  effect arising from loops (tadpoles) and from the corresponding counterterms contained in  $\mathcal{L}^{(4)}$ . Here, the leading contribution (30) counts as  $O(M_\pi)$ , (actually, it can hardly be expressed in terms of the pion mass), and the loop effects only show up at the next,  $O(p^4)$ , order. Notice that for  $r \simeq 10$ , the constant (30) is of the order  $\widehat{m}\xi = (5 \div 6)\%$  - a typical size of other  $O(M_\pi)$  effects, such as the deviation from the Goldberger-Treiman relation.

The Lagrangian  $\tilde{\mathcal{L}}^{(4)}$  describing the next order,  $O(p^4)$ , consists of several components :

$$\tilde{\mathcal{L}}^{(4)} = \mathcal{L}_{(4,0)} + \mathcal{L}_{(2,2)} + \mathcal{L}_{(0,4)} + B_0^2 \mathcal{L}'_{(0,2)} + B_0 \mathcal{L}'_{(2,1)} + B_0 \mathcal{L}'_{(0,3)}. \quad (31)$$

$\mathcal{L}_{(4,0)}$  is the part of the standard  $\mathcal{L}^{(4)}$ , which consists of four derivatives and contains no  $\chi$ , i.e. no quark mass :  $\mathcal{L}_{(4,0)}$  is given by the standard five terms [2] described by the low energy constants  $L_1$ ,  $L_2$ ,  $L_3$ ,  $L_9$  and  $L_{10}$ .  $\mathcal{L}_{(2,2)}$  is a new term, which in the standard  $\chi$ PT would count as  $O(p^6)$ , whereas  $\mathcal{L}_{(0,4)}$  involves 4 insertions of a quark mass, and in the standard  $\chi$ PT

it would be relegated up to the order  $O(p^8)$ .  $\mathcal{L}_{(2,2)}$  and  $\mathcal{L}_{(0,4)}$  involve about 20 independent terms each. An experimental determination of all the corresponding low energy constants is obviously hard to imagine. However, a few particular combinations of these constants which contribute, for instance, to  $K_{I3}$  and  $K_{I4}$  decays, or to the  $\pi\pi$  scattering amplitude at the one loop level, can be estimated and included into the analysis.

The last three terms in Eq. (31) represent  $B_0$ -dependent counterterms of order  $O(p^4)$  which are needed to renormalize one loop divergences that arise from using the vertices of  $\tilde{\mathcal{L}}^{(2)}$  alone in the loop. They renormalize the constants  $A_0$ ,  $Z_0^S$  by higher order contributions, of order  $O(B_0^2)$ , and the constants  $\xi$ ,  $\tilde{\xi}$  and  $\rho_i$  by an amount  $O(B_0)$ . In generalized  $\chi$ PT, renormalization proceeds order by order in the expansion in powers of the constant  $B_0$ .

The standard  $O(p^4)$  order Lagrangian  $\mathcal{L}^{(4)}$  contains 10 low energy constants  $L_i$  [2]. They are all involved in  $\tilde{\mathcal{L}}^{(2)} + \tilde{\mathcal{L}}^{(3)} + \tilde{\mathcal{L}}^{(4)}$ . As already pointed out,  $L_1$ ,  $L_2$ ,  $L_3$ ,  $L_9$  and  $L_{10}$  - the constants of  $\mathcal{L}_{(4,0)}$  - play an identical role in both schemes.  $L_4$  and  $L_5$  are related to the constants  $\tilde{\xi}$  and  $\xi$  in  $\tilde{\mathcal{L}}^{(3)}$  :

$$\begin{aligned} L_4^r &= \frac{F_0^2}{8B_0} \left\{ \tilde{\xi} + O\left(\frac{B_0}{(4\pi F_0)^2} \ln \mu\right) \right\}, \\ L_5^r &= \frac{F_0^2}{8B_0} \left\{ \xi + O\left(\frac{B_0}{(4\pi F_0)^2} \ln \mu\right) \right\}. \end{aligned} \tag{32}$$

Finally, the standard  $O(p^4)$  constants  $L_6$ ,  $L_7$ ,  $L_8$  are related to the  $\tilde{\mathcal{L}}^{(2)}$  constants  $Z_0^S$ ,  $Z_0^P$  and  $A_0$ , respectively :

$$\begin{aligned} L_6^r &= \frac{F_0^2}{16B_0^2} \left\{ Z_0^S + O\left(\frac{B_0^2}{(4\pi F_0)^2} \ln \mu\right) \right\}, \\ L_7^r &= \frac{F_0^2}{16B_0^2} Z_0^P, \\ L_8^r &= \frac{F_0^2}{16B_0^2} \left\{ A_0 + O\left(\frac{B_0^2}{(4\pi F_0)^2} \ln \mu\right) \right\}. \end{aligned} \tag{33}$$

On the other hand,  $\tilde{\mathcal{L}}^{(2)} + \tilde{\mathcal{L}}^{(3)} + \tilde{\mathcal{L}}^{(4)}$  involves additional terms, not contained in  $\mathcal{L}^{(2)} + \mathcal{L}^{(4)}$ , and which the standard  $\chi$ PT relegates to higher orders  $d > 4$ . Setting the corresponding additional constants to zero, one recovers the standard  $\chi$ PT up to and including order  $O(p^4)$ . This phenomenon is general: Order by order, the standard expansion reappears as a special case of  $G\chi$ PT.

## 4 Examples of differences between standard and generalized $\chi$ PT

The difference between standard and generalized  $\chi$ PT merely concerns the symmetry breaking sector of the theory. In the chiral limit  $m_u = m_d = m_s = 0$ , observables described by purely

derivative terms of the effective Lagrangian  $\mathcal{L}_{(2,0)} + \mathcal{L}_{(4,0)} + \mathcal{L}_{(6,0)} + \dots$  are essentially given by soft pion theorems which are identical in both schemes. This concerns, in particular, the electromagnetic radius of the pion, the decay  $\pi \rightarrow e\nu\gamma$ , the  $\pi\pi$  scattering in the P-wave, the leading order  $K_{I3}$  and  $K_{e4}$  form-factors, etc...

The difficulty of disentangling both schemes resides in the fact that symmetry breaking effects are small and not well known experimentally. In this section we are going to give an overview of the main differences between the predictions of standard and generalized  $\chi$ PT for symmetry breaking effects which already appear at the leading order  $O(p^2)$ . Those have more chances to be observable, at least indirectly.

#### 4a - $m_d - m_u$ : Dismiss of the ellipse [14]

We have already stressed the importance of the quark mass ratio  $r = m_s/\widehat{m}$  as an independent parameter of  $G\chi$ PT. When isospin breaking is switched on,  $G\chi$ PT leads to a relation between the two quark mass ratios  $r$  and  $R$ , with

$$R = \frac{m_s - \widehat{m}}{m_d - m_u} . \quad (34)$$

Taking a suitable linear combination of the expansions of  $M_{\pi^+}^2$ ,  $M_{K^0}^2$  and  $M_{K^+}^2$ , the  $O(m_q)$  terms, the  $O(m_q^2)$  terms and even the  $O(m_q^2 \ln m_q)$  terms drop out and one obtains [4]

$$R\Delta M_K^2 - (M_K^2 - M_\pi^2) - \frac{1}{2} \frac{r-1}{r+1} (r_2 - r)M_\pi^2 = O\left(\frac{m_q^3}{\Lambda_H}\right) , \quad (35)$$

where  $\Delta M_K^2 = M_{K^0}^2 - M_{K^+}^2$  is the kaon mass difference in the absence of electromagnetism. Eq. (35) holds independently of the value of  $r$ , i.e. both in the standard and in the generalized  $\chi$ PT. In the standard case, Eq. (35) should be rather accurate, whereas in the generalized setting the neglected  $O(m_q^3)$  terms - arising from  $\mathcal{L}_{(0,3)}$  and given by the low energy constants  $\rho_i$  (27) - can easily represent a 30% correction.

To the best of our knowledge, Eq. (35) has never appeared in the literature on standard  $\chi$ PT. Instead, one rather expands ratios of pseudoscalar meson masses and one eliminates the  $O(m_q)$  terms from this expansion (see e.g. Eqs. (10.11) and (10.17) of the second of Refs. [2]). In this way, one arrives at the well known elliptic relation between  $\frac{m_s}{m_d}$  and  $\frac{m_u}{m_d}$ , extensively commented upon in the literature [14]. The difference between the two procedures is instructive : When expanding the *linear combinations* of pseudoscalar meson masses, one expands in powers of the small parameter  $\frac{m_q}{\Lambda_H}$ . On the other hand, when expanding mass ratios, one assumes, in addition, that the parameter  $\frac{m_q}{m_0}$ , ( $m_0 = \frac{B_0}{2A_0}$ ) is also small. Actually, the elliptic relation [14] should read

$$\frac{1}{Q^2} \left(\frac{m_s}{m_d}\right)^2 + \left(\frac{m_u}{m_d}\right)^2 = 1 + O\left[\left(\frac{m_s}{m_0}\right)^2\right] , \quad (36)$$

where

$$\frac{1}{Q^2} = \frac{M_\pi^2 \Delta M_K^2}{M_K^2 (M_K^2 - M_\pi^2)} . \quad (37)$$

The quark mass ratios lie on the ellipse only provided  $m_s \ll m_0$  or, equivalently, provided  $r = \frac{m_s}{m}$  is close to  $r_2 = 25.9$ , whereas Eq. (35) has a more general validity.

It is interesting to look at Eq. (35) in the light of recent discussions of a possible large violation of Dashen's theorem [15]. We can write

$$\Delta M_K^2 = (M_{K^0}^2 - M_{K^+}^2)_{exp} + \gamma(M_{\pi^+}^2 - M_{\pi^0}^2), \quad (38)$$

where the parameter  $\gamma$  describes the departure from Dashen's theorem :  $\gamma = 1$  if the theorem is exact. Various recent estimates [15] expect  $\gamma$  somewhere between 1 and 2. Taking in Eq. (35)  $r = 25.9$ , one obtains  $R = 43$  for  $\gamma = 1$  and  $R = 34.8$  for  $\gamma = 2$ . For  $r = 10$ , Eq. (35) predicts  $R = 66$  if  $\gamma = 1$ , whereas  $R = 53$  if  $\gamma = 2$ . Assuming  $R \sim 45$  (as expected from baryon masses and from  $\omega - \rho$  mixing [9]) we observe that by increasing  $\gamma$ , we are left with less room for  $O(\frac{m_s}{\Lambda_H})$  corrections to Eq. (35), which could provide a valuable information on the  $\tilde{L}^{(3)}$  parameters  $\rho_i$ .

#### 4b - Quark condensates for $m_q \neq 0$

The parameter  $B_0$  describes the quark condensate in the chiral limit  $m_u = m_d = m_s = 0$ . The leading order Lagrangian  $\tilde{L}^{(2)}$  allows to express  $\langle \Omega | \bar{q}q | \Omega \rangle$ , where  $|\Omega \rangle$  is the ground state for  $m_q \neq 0$ , beyond this limit. One gets

$$-F_0^{-2} \langle \Omega | \bar{q}q | \Omega \rangle = B_0 + m_q(A_0 + \frac{1}{2}H_0) + 4(\bar{m} + \frac{1}{2}m_s)Z_0^S + O(\frac{m_q^2}{\Lambda_H}). \quad (39)$$

where  $q = u, d, s$  denotes a given quark flavour. The point is that if  $B_0 \ll \Lambda_H$ , the  $O(m_q)$  contribution in Eq. (39) can be relatively important. Comparing for instance the term  $m_s A_0$  to  $B_0$  one finds, using Eqs. (26a) and (26b),  $m_s A_0 \sim 2.3 B_0$ , assuming  $r = 10$  and  $\zeta = 0$ . This should not be surprising : in G $\chi$ PT all contributions of  $\tilde{L}^{(2)}$  are supposed to be of the same order of magnitude. In practice, it implies that the  $m_q \neq 0$  condensates can exhibit a large flavour dependence. Unfortunately, there is no way to pin down the constant  $H_0$  which controls this dependence quantitatively :  $H_0$  is not a low energy order parameter, but rather a short distance counterterm (see Ref. [2] for a discussion of this point).

#### 4c - Large corrections to the soft pion theorems

Within generalized  $\chi$ PT the corrections to soft pion theorems can sometimes be rather important - of the same order as the soft pion result itself. Consider, for instance, the scalar form factor of the pion at vanishing momentum transfer or, equivalently, the pion  $\sigma$ -term

$$\delta^{ij} \sigma_\pi(0) = \langle p, \pi^i | \bar{m}(\bar{u}u + \bar{d}d) | p, \pi^j \rangle \quad (40)$$

The soft pion result for  $\sigma_\pi(0)$  is well known :

$$\sigma_\pi(0)|_{\text{soft pions}} = 2\bar{m}B_0. \quad (41)$$

Since in the G $\chi$ PT,  $2\widehat{m}B_0$  can be considerably smaller than  $M_\pi^2$ , one might be tempted to conclude, on the basis of the soft pion result (41), that in G $\chi$ PT  $\sigma_\pi(0)$  is smaller than in the standard theory. This conclusion does not take into account all  $O(p^2)$  contributions to  $\sigma_\pi(0)$  described by  $\tilde{\mathcal{L}}^{(2)}$ . Writing  $\sigma_\pi(0) = \widehat{m} \frac{\partial M_\pi^2}{\partial \widehat{m}}$ , using Eqs. (5) and (26), and neglecting the Zweig rule violating parameter  $\zeta$ , one obtains the correct  $O(p^2)$  result,

$$\sigma_\pi(0) = \widehat{m} \frac{\partial M_\pi^2}{\partial \widehat{m}} = \left(1 + 2 \frac{r_2 - r}{r^2 - 1}\right) M_\pi^2. \quad (42)$$

It is seen that, when  $r$  decreases from  $r = r_2$  to  $r = r_1 \simeq 6.3$ , the pion  $\sigma$ -term increases from  $M_\pi^2$  to  $2M_\pi^2$ . In G $\chi$ PT, the soft pion result will receive a large correction, whenever the soft pion theorem result is proportional to the quark condensate  $B_0$ . The reason is that both  $B_0$  and  $m_q$  count as quantities of order  $O(M_\pi)$ . The formalism of G $\chi$ PT automatically takes care of such large corrections. This phenomenon can in principle lead to modifications of the standard evaluation of the non-leptonic K-decay matrix elements in the large  $N_c$  limit, in particular, of the penguin-contribution to the ratio  $\frac{\epsilon'}{\epsilon}$  [16].

There is another relevant example of a similar nature : The  $\chi$ PT prediction for the low  $Q^2$  behavior of the spectral function associated with the divergence of the axial current,

$$\rho(Q^2) = \frac{1}{2\pi} \sum_n (2\pi)^4 \delta^4(Q - P_n) \left| \langle n | \partial^\mu (\bar{d} \gamma_\mu \gamma_5 u) | 0 \rangle \right|^2. \quad (43)$$

At low  $Q^2$ , the continuum part of  $\rho(Q^2)$  is dominated by the contribution of  $3\pi$  intermediate states. Using for the latter the soft pion theorem (and neglecting in the phase space integral (43) the pion mass) one gets the standard result [17]

$$\rho_{3\pi}(Q^2) = \frac{1}{768\pi^4} \frac{M_\pi^4}{F_\pi^2} Q^2 + \dots \quad (44)$$

Within the  $O(p^2)$  G $\chi$ PT, this result is considerably modified. Still neglecting the pion mass in the phase space integral (in order to facilitate a comparison with the standard result) one gets [18]

$$\rho_{3\pi}(Q^2) = \frac{1}{768\pi^4} \frac{M_\pi^4}{F_\pi^2} Q^2 \left\{ 1 + 10 \frac{r_2 - r}{r^2 - 1} + 30 \left( \frac{r_2 - r}{r^2 - 1} \right)^2 \right\} + \dots \quad (45)$$

As  $r$  decreases from  $r_2$  down to  $r_1$ , the enhancement factor in Eq. (45) increases from 1 to 13.5. This enhancement would considerably affect the existing estimates of  $\widehat{m}$  using the QCD sum rules [17].

#### 4d - $\pi\pi$ and $\pi K$ scattering

Our last example of a leading order difference between the standard and the generalized  $\chi$ PT is of a direct experimental relevance: It concerns the low energy  $\pi\pi$  [4] and  $\pi K$  [5] scattering. The  $\pi\pi$  amplitude predicted by the leading order G $\chi$ PT lagrangian  $\tilde{\mathcal{L}}^{(2)}$  reads

$$A_{lead}(s|t, u) = \frac{1}{F_0^2} (s - 2\widehat{m}\bar{B}), \quad (46)$$

where

$$\bar{B} = B_0 + 2m_s Z_0^S \quad (47)$$

is the non strange quark antiquark condensate in the  $SU(2) \times SU(2)$  chiral limit (see Eq. (39)). Eq. (46) holds both in the standard case and in the generalized  $\chi$ PT. In the standard case, however,  $2\bar{m}\bar{B} \sim M_\pi^2$ , and one recovers the well known formula first obtained by Weinberg [19]. In  $G\chi$ PT, this formula is modified already at the leading order  $O(p^2)$ :  $2\bar{m}\bar{B}$  can be considerably smaller than  $M_\pi^2$  by an amount which depends on  $\tau$ . For  $\tau$  decreasing from  $\tau_2=25.9$  to  $\tau=6.3$ ,  $2\bar{m}\bar{B}$  decreases from  $M_\pi^2$  to zero. The low energy  $\pi\pi$  scattering thus provides us with a quasi-unique experimental access to the order parameter  $\bar{m}B_0$ . The corresponding  $G\chi$ PT predictions, endowed with the necessary loop corrections, are presented in detail in the section on  $\pi\pi$  interactions of the present Handbook.

A similar conclusion holds in the case of  $\pi K$  scattering [5]. The latter gives a contribution to the  $K_{l4}$  form factor  $R$ , which is measurable in the  $K_{\mu 4}$  decay mode. Whereas the leading order predictions for the  $K_{l4}$  form factors  $F$ ,  $G$  and  $H$  are identical in both schemes, there is a detectable difference in the leading order expression for  $R$  [5]. Whether this difference can be observed in practice is presently under investigation [25].

## 5 Values of the low energy constants

It is not incorrect to state that  $G\chi$ PT is more general just because it admits a considerably wider range of values of certain low energy constants and of the current quark masses than the standard scheme. On the other hand, the standard  $\chi$ PT claims that the values of the  $O(p^4)$  constants  $L_1, \dots, L_{10}$  are well under control, both determining them from data [2] and estimating them via resonance saturation [20, 21]. Similar claims are often made about the values of the light quark masses [22]. The standard values of the low energy parameters can indeed be justified within the set of assumptions underlying the standard  $\chi$ PT. However, beyond this framework, the same experimental data, the same sum rules, etc..., can often yield rather different results. We start by discussing the constants  $L_i$  from the point of view of  $G\chi$ PT.

### 5a - $L_{10}$ and $L_9$

These two constants appear in  $\mathcal{L}_{(4,0)}$ , which is the part common to both  $\mathcal{L}^{(4)}$  and  $\bar{\mathcal{L}}^{(4)}$ . Their measurements via the electromagnetic radius of the pion and the radiative decay  $\pi \rightarrow e\nu\gamma$  [2] should not be altered in  $G\chi$ PT. Furthermore,  $L_{10}$  and  $L_9$  are, respectively, related to two and three point functions of vector and axial vector currents dominated by vector and axial vector meson poles [21, 23]. This makes the estimates of  $L_9$  and of  $L_{10}$  rather stable.

### 5b - $L_3$ , $L_2$ and $L_1$

These remaining three constants of  $\mathcal{L}_{(4,0)}$  are more difficult to measure. Although they are not directly related to explicit symmetry breaking effects, they enter the observables ( $K_{l4}$  form factors [24],  $\pi\pi$  D-waves [2]) together with  $\tau$ -dependent loop corrections. Consequently,

for lower values of  $r$ , the standard determination of  $L_1^r$ ,  $L_2^r$  and  $L_3$  is slightly modified [25], by not more than a factor of two.

The estimates of these constants via resonance saturation is also more involved since, in addition to vector and axial vector mesons,  $L_1$ ,  $L_2$  and  $L_3$  receive a contribution from scalar exchanges, which are not known so well [20]. The reason of this complication is the fact that, unlike  $L_9$  and  $L_{10}$ ,  $L_1$ ,  $L_2$  and  $L_3$  are related to four point functions of the vector and axial currents.

### 5c - $L_4$ and $L_5$

Let us concentrate on the  $\mathcal{L}_{(2,1)}$  constant  $L_5$  ( $L_4$  violates the Zweig rule) and, for simplicity, let us stick to the leading large  $N_c$  behaviour, denoting the leading part of  $L_5$  by  $\hat{L}_5$ . One has (Eq. (32)),

$$\hat{L}_5 = \frac{\xi F_\pi^2}{8B_0} \sim \frac{F_K^2 - F_\pi^2}{8(m_s - \bar{m})B_0}, \quad (48)$$

where we have used (30). The standard determination of  $\hat{L}_5$  [2] (or, equivalently, of  $L_5^r$  including the chiral logarithms) would replace  $(m_s - \bar{m})B_0$  by  $M_K^2 - M_\pi^2$ , leading to the value  $\hat{L}_5 \sim 2.2 \times 10^{-3}$ . Within  $G\chi$ PT, this last step could be misleading, provided  $r$  is well below  $r_2 \sim 25.9$ . Using instead the leading order formula (26 $r$ ) and neglecting the Zweig rule violating parameter  $\zeta$ , one obtains

$$\hat{L}_5 = \frac{F_K^2 - F_\pi^2}{4M_\pi^2} \frac{1}{r - r_1} \frac{r + 1}{r + r_1 + 2}. \quad (49)$$

Hence, the value of  $\hat{L}_5$  one extracts from Eq. (48) crucially depends on the quark mass ratio  $r$ : For  $r = r_2 \sim 25.9$  one gets the "standard value"  $\hat{L}_5 \sim 2.2 \times 10^{-3}$ , whereas for  $r$  decreasing down to  $r_1$ ,  $\hat{L}_5$  increases up to infinity. The estimates of  $\hat{L}_5$  via resonances merely concern scalar exchanges, whose description is considerably more ambiguous [26] than in the case of vector and axial vector mesons [21].

### 5d - $L_6$ , $L_7$ and $L_8$

The values of these  $\mathcal{L}_{(0,2)}$  constants are at the heart of our discussion of symmetry breaking effects.  $L_6^r$  violates the Zweig rule in the  $0^{++}$  channel and it will not be discussed here. The combination  $L_8^r + 2L_7$  can be related to the deviation  $\Delta_{GMO}$  from the Gell-Mann-Okubo mass formula [2] (see Eqs. (10) and (32)). The standard evaluation of  $L_8^r + 2L_7$  based on this relation suffers from a similar bias as in the case of  $\hat{L}_5^r$ : The expression for  $L_8^r + 2L_7$  involves the factor  $(m_s - \bar{m})^2 B_0^2$  in the denominator. As a result, the value of  $L_8^r + 2L_7$  is even more sensitive to  $r$  than  $\hat{L}_5$ , and it can actually be considerably larger than the standard value [2]  $L_8^r + 2L_7 \sim 0.1 \times 10^{-3}$ . A separate measurement of  $L_8$  is usually based on the relation with the isospin breaking quark mass ratio  $R$  (34), which is known from different sources [2, 9, 14]. In  $G\chi$ PT, the relation between  $L_8$  and  $R$  can easily break down, for the same reason which could invalidate the elliptic relation (36): The importance of unduly neglected  $O(m_s/m_0)$  contributions. As a consequence, the standard value [2] of  $L_8^r(M_\rho) = (0.9 \pm 0.3) \times 10^{-3}$  can be

underestimated by as much as two orders of magnitude. Writing

$$\hat{L}_8 = \frac{F_0^2}{16M_0^2}, \quad M_0^2 = 2m_0B_0 = \frac{B_0^2}{A_0}, \quad (50)$$

the renormalization group invariant mass parameter  $M_0$  can be estimated from Eqs. (26a,b). In the standard scheme ( $r \sim r_2$ ),  $M_0$  is expected to be at a GeV scale, whereas in G $\chi$ PT,  $M_0$  can be as small as  $M_\pi$ , or even smaller.

The estimate of  $\hat{L}_8$  from resonance contributions can be obtained from the two point function (3) which satisfies the superconvergent sum rule [26]

$$\int dq^2 \Im m \Pi_{SP}(q^2) = 0. \quad (51)$$

Saturating this sum rule by nothing but the pion and a single  $0^{++}$  state of mass  $M_S$ , one obtains  $M_0 = M_S \sim 1$  GeV, leading to a value for  $\hat{L}_8$  which is of the standard order of magnitude  $\sim 10^{-3}$ . Unfortunately, the above argument does not by itself support the standard picture, and can be turned around: If  $B_0 \ll \Lambda_H$ , the pion contribution ( $= B_0^2$ ) can hardly dominate the pseudoscalar component of the sum rule (51). In order to balance the scalar contribution, one has to include an excited  $\pi'$  state of mass  $M_{P'} > M_S$ ,  $M_{P'} \simeq (1200 \div 1300)$  MeV. It is then easy to see that the mass parameter  $M_0$  in Eq. (50) can take any value between zero and  $M_S$  [26].

Quite generally, the introduction of  $J = 0$  resonances, compatible with chiral symmetry and with the short distance properties of QCD correlation functions, into the effective lagrangian [20] is more tricky and more ambiguous than in the case of  $J = 1$  states [21]. The authors of Ref. [20] have, for instance, decided to disregard the contribution of the  $\pi'$  nonet. Doing so, they have a priori eliminated the low  $B_0$  alternative.

The standard  $\chi$ PT rewrites the expansion in quark masses as an expansion in powers of  $M_\pi^2 \sim 2\hat{m}B_0$ . In the vicinity of the antiferromagnetic critical point,  $r \sim r_1^+$ , the coefficients of the latter expansion, viz.  $L_4 \dots L_8$ , blow up, and the expansion has to be redefined. G $\chi$ PT is precisely such a redefinition. At the leading order, it is characterized by a single undetermined parameter not present in the standard scheme, the quark mass ratio  $r = m_s/\hat{m}$ . One may say that G $\chi$ PT parametrizes the deviations from standard predictions of  $\chi$ PT in terms of the deviation of  $r$  from its standard value  $r^{st} \sim r_2$  [2].

### 5e - Running light quark masses

In G $\chi$ PT, the decrease of the quark mass ratio  $r$  from  $r \sim 25$  to  $r \sim 10$  is likely to be interpreted as an increase in the value of the running quark mass  $\hat{m}(\mu)$  by a factor of  $2 \div 3$ . The order of magnitude of the mass differences  $m_s - \hat{m}$  and  $m_d - m_u$  should remain essentially unchanged. In this connection, it should be stressed that all existing estimates of  $\hat{m}(\mu)$  use in one way or the other the assumption  $r \sim 25$ . This is obvious for those approaches which deduce the value of  $\hat{m}(\mu)$  from the estimates of  $m_s - \hat{m}$  [8, 9]. It is however even true in the case of direct quantitative determinations of  $\hat{m}(\mu)$  from QCD sum rules [17]. The latter express the square of  $\hat{m}(\mu)$  as a weighted integral of the spectral function (43). Nothing is



known experimentally about  $\rho(Q^2)$  beyond the one pion intermediate state contribution. The only existing attempt to fill this gap makes use of the low  $Q^2$  behaviour (44) of  $\rho_{3\pi}(Q^2)$  as given by the standard  $\chi$ PT in order to normalize the whole  $\rho_{3\pi}(Q^2)$  contribution represented by a broad  $\pi'$  Breit-Wigner peak. In this way, the result  $\widehat{m}(1\text{GeV}) = (7 \pm 1)\text{MeV}$  is obtained [17]. If, instead, the  $3\pi$  contribution is normalized using the  $G\chi$ PT result (45), the above value of  $\widehat{m}(\mu)$  is increased by a factor  $2 \div 3$ . (Let us note in passing that within  $G\chi$ PT, the possibility of having  $m_u = 0$  appears even less likely than in the standard case [14].)

In general,  $G\chi$ PT admits and expects a larger absolute strength of the divergence of the  $\Delta S = 0$  axial current away from the pion pole than the standard scheme could possibly tolerate. The physical origin of this increase would be ascribed to the importance of the “ $\pi'$  contribution”, which can and should be checked experimentally: The  $\rho_{3\pi}(Q^2)$  component of the spectral function (43) can be measured in high statistics  $\tau \rightarrow 3\pi\nu_\tau$  decay experiments [18], and in this way, the determination of  $\widehat{m}(\mu)$  could be put on a solid experimental basis.

Concluding this section, it is worth emphasizing that nothing in the preceding discussion indicates that the standard determination of the  $L_i$ 's and of the light quark masses is internally inconsistent. The standard  $\chi$ PT together with the standard value of the low energy parameters is a *perfectly self-consistent scheme*. However, it is not the only possible consistent scheme and it does not contrain its proper justification.

## 6 Experimental tests

The framework of  $G\chi$ PT can be exploited in order to measure the quark mass ratio  $r = m_s/\widehat{m}$  together with a few other low energy constants in several independent experiments. Since for  $r = r^{\text{st}} \sim 25.9$ , the predictions of the standard  $\chi$ PT are always contained as a special case, measurements of  $r$  constitute a powerful test of the assumptions underlying the standard scheme ( $B_0 \sim \Lambda_H$ ). In order to control all relevant low energy parameters which are needed up to a given degree of precision, a simultaneous analysis of many processes may be necessary. In this way, one may also hope to achieve some control of higher orders of  $\chi$ PT not included into the analysis. At present, there are a few experimental issues which appear to be particularly relevant.

### 6a - Deviations from the Goldberger-Treiman relation [27]

The comparison of the quark mass expansion of the deviations from the Goldberger-Treiman relations in the three channels  $NP$ ,  $\Lambda P$ ,  $\Sigma N$ , yields a first order measurement of  $m_s/\widehat{m}$ . The result is very sensitive to the precise values of the strong coupling constants  $g_{\pi NN}$ ,  $g_{K\Lambda N}$  and  $g_{K\Sigma N}$ . Present values [28] lead to the bound  $r \leq 10.6 \pm 4.2$ , which requires a confirmation.  $\text{Da}\Phi\text{ne}$  could contribute to a new precise determination of the hyperon coupling constants from low energy  $KN$  and  $\bar{K}N$  scattering data.

### 6b - Low energy $\pi\pi$ scattering

Standard  $\chi$ PT leads to firm predictions [2] for scattering lengths and low energy phase

shifts. If  $\tau$  decreases, G $\chi$ PT reveals that the  $\pi\pi$  interaction in the I=0 S-wave becomes stronger [4], in agreement with existing (but not very accurate) data. The detailed discussion may be found in the  $\pi\pi$  Section of the present Handbook.

Da $\Phi$ ne can accurately measure the low energy phase shifts  $\delta_0^0 - \delta_1^1$  as a function of energy. This information, taken together with the direct determination of  $a_0^0 - a_0^2$  to 5% accuracy from the  $\pi^+\pi^-$  atom lifetime experiment which is planned at CERN [29], could well provide a decisive measurement of  $m_\pi/\widehat{m}$  and a crucial test of  $\chi$ PT.

#### 6c - $K_{l4}$ form factors [24, 25]

The  $\pi\pi$  phases and the  $K_{l4}$  form factors are closely tied together. A simultaneous analysis is necessary in order to determine the constants  $L_1$ ,  $L_2$  and  $L_3$  and the Zweig rule violating parameters  $\zeta$  and  $\bar{\xi}/\xi$ , and to reduce the theoretical uncertainty in the predictions for the  $\pi\pi$  scattering amplitude. In addition, the form factor  $R$  measurable in the  $K_{\mu 4}$  decay mode could provide an indirect access to the  $K\pi$  scattering amplitude [5, 25].

#### 6d - $\gamma\gamma \rightarrow \pi^0\pi^0$

The existing low energy data [30] are not very accurate. They can be reproduced by G $\chi$ PT to the one loop, provided  $\tau \leq 10$  [6]. Unfortunately, the stringent disagreement of the one loop result of standard  $\chi$ PT does not yet allow to conclude in favour of smaller values of  $\tau$ : At the two loop level, the standard  $\chi$ PT restores the agreement with the data without problems [31]. More accurate data in the threshold region could help to clarify this situation.

#### 6e - Azimuthal asymmetries in the decay $\tau \rightarrow 3\pi\nu_\tau$ [18]

The measurement of these angular asymmetries at a per cent level would provide a direct information about the magnitude of the divergence of the axial current in the  $\pi'$  region. When combined with QCD sum rules [17], this measurement yields a probe of the value of the running quark mass  $\widehat{m}(\mu)$  [18]. This test of G $\chi$ PT has the virtue of going beyond its own framework. Unfortunately, it also goes beyond the scope of Da $\Phi$ ne, requiring a tau-charm factory or the like.

The above list is not exhaustive. Additional  $K$ ,  $\eta$  or  $\eta'$  decays might be of interest.

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